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On homomorphism spaces of metrizable groups

Gábor Lukács
Department of Mathematics & Statistics, York University, 4700 Keele Street, Toronto, Ont., M3J 1P3, Canada

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Abstract

For two not necessarily commutative topological groups G and K, let $\mathscr{H}(G,K)$ denote the space of all continuous homomorphisms from G to K with the compact-open topology. We prove that if G is metrizable and K is compact then $\mathscr{H}(G,K)$ is a k-space. As a consequence we obtain that if D is a dense subgroup of G then $\mathscr{H}(D,K)$ is homeomorphic to $\mathscr{H}(G,K)$, and if G is separable h-complete, then the natural map $G \to \mathscr{C}(\mathscr{H}(G,K),K)$ is open onto its image.

MSC: 22A05

The aim of the present paper is to generalize the result of Chasco [2] that for every abelian metrizable group G, its dual group $\hat{G}$ (i.e. the group of homomorphisms into the unit circle, $\mathbb{T}$) is a k-space under the compact-open topology. We prove that the space of homomorphisms $\mathscr{H}(G,K)$ is a k-space whenever G is a (not necessarily commutative) metrizable topological group and K is a compact topological group which satisfies assumptions that we call "radical-based" below.

Definition 1. A topological group K is *radical-based*, if it has a countable base $\{\Lambda_n\}$ at e, such that each Λ_n is symmetric, and for all $n \in \mathbb{N}$:

(1) $(\Lambda_n)^n \subset \Lambda_1$;
(2) $a^1, a^2 \ldots, a^n \in \Lambda_1$ implies $a \in \Lambda_n$.

Any topological subgroup K of the unitary group of a C^*-algebra is radical-based: one can define $\Lambda_n = \{u \in K \mid \|u - e\| < \varepsilon_n\}$ for a suitably chosen sequence $\{\varepsilon_n\}$.

E-mail address: lukacs@mathstat.yorku.ca (G. Lukács).

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Recall that a Hausdorff topological space X is called a *k-space* if $F \subset X$ is closed if and only if $F \cap C$ is closed for every closed compact subset C of X.

Theorem 1. *For a metrizable topological group G and a radical-based compact group K, $\mathscr{H}(G,K)$ is a k-space.*

In order to prove Theorem 1 we will need the two results below. To shorten notations, for $\alpha \in \mathscr{H}(G,K)$ we put $S_\alpha(A,B) = S(A,B)\alpha \cap \mathscr{H}(G,K)$ where $S(A,B) = \{\gamma \mid \gamma(A) \subset B\}$ ($A \subset G$ and $B \subset K$).

Lemma 1. *If K is radical-based then its base $\{\Lambda_n\}$ at e satisfies*:

(a) $\Lambda_{2k}\Lambda_{2k} \subset \Lambda_k$ *for all* $k \in \mathbb{N}$;
(b) $\overline{\Lambda_{2k}} \subset \Lambda_k$ *for all* $k \in \mathbb{N}$.

Lemma 2. *Suppose that G is metrizable and K is compact and radical-based. Let $\alpha \in \mathscr{H}(G,K)$ and U be a neighborhood of e in G. Then $S_\alpha(U,\Lambda_2)$ is precompact, i.e. $\overline{S_\alpha(U,\Lambda_2)}$ is compact.*

Proof of Theorem 1. Let $\Phi \subset \mathscr{H}(G,K)$ be a set such that for any compact subset Ξ of $\mathscr{H}(G,K)$, $\Phi \cap \Xi$ is closed. We have to prove that Φ is closed. To that end let $\zeta \in \mathscr{H}(G,K)$ such that $\zeta \notin \Phi$. It suffices to find a compact subset C of G and $l \geqslant 2$ such that $S_\zeta(C,\Lambda_{2l}) \cap \Phi = \emptyset$.

The group G is first countable, so let $\{U_n\}_{n=1}^{\infty}$ be a base at e. We may assume that U_n is decreasing. Set $U_0 = G$. We are going to find $l \geqslant 2$ and construct inductively a family $\{F_n\}_{n=0}^{\infty}$ of finite subsets of G such that for all $n \geqslant 0$

(1) $F_n \subset U_n$,
(2) $\bigcap_{k=1}^{n} S_\zeta(F_k,\overline{\Lambda_{2l}}) \cap \overline{S_\zeta(U_{n+1},\Lambda_2)} \cap \Phi = \emptyset$.

First we have to construct F_0. By Lemma 2, $\overline{S_\zeta(U_1,\Lambda_2)}$ is compact, thus by the assumption $\overline{S_\zeta(U_1,\Lambda_2)} \cap \Phi$ is closed. On compact subsets of $\mathscr{C}(G,K)$ the compact-open topology coincides with the topology of pointwise convergence. But $\zeta \notin \overline{S_\zeta(U_1,\Lambda_2)} \cap \Phi$, so there exists a neighborhood of ζ in the pointwise topology which is disjoint from $\overline{S_\zeta(U_1,\Lambda_2)} \cap \Phi$. It is clear that sets of the form $S_\zeta(F,\Lambda_l)$ where $F \subset G$ is finite form a base at ζ for the pointwise topology on $\mathscr{H}(G,K)$. So there exists F_0 such that

$$S_\zeta(F_0,\Lambda_l) \cap \overline{S_\zeta(U_1,\Lambda_2)} \cap \Phi = \emptyset. \tag{1}$$

(Without loss of generality we may assume $l \geqslant 2$.) By Lemma 1, $\overline{\Lambda_{2l}} \subseteq \Lambda_l$, thus $S_\zeta(F_0,\overline{\Lambda_{2l}}) \subset S_\zeta(F_0,\Lambda_l)$. In particular:

$$S_\zeta(F_0,\overline{\Lambda_{2l}}) \cap \overline{S_\zeta(U_1,\Lambda_2)} \cap \Phi = \emptyset. \tag{2}$$

Suppose that we have already constructed $F_0,\dots,F_{n-1}$ such that (1) and (2) hold. For all $x\in U_n$ we define

$$\Delta_x=\bigcap_{k=0}^{n-1} S_\zeta(F_k,\overline{\Lambda_{2l}})\cap S_\zeta(\{x\},\overline{\Lambda_{2l}})\cap\overline{S_\zeta(U_{n+1},\Lambda_2)}\cap\Phi. \tag{3}$$

Notice, that the sets Δ_x are closed, because each $S_\zeta(F_k,\overline{\Lambda_{2l}})$ is closed even in the pointwise topology. But then

$$\bigcap_{x\in U_n}\Delta_x=\bigcap_{k=0}^{n-1} S_\zeta(F_k,\overline{\Lambda_{2l}})\cap S_\zeta(U_n,\overline{\Lambda_{2l}})\cap\overline{S_\zeta(U_{n+1},\Lambda_2)}\cap\Phi. \tag{4}$$

Since $S_\zeta(U_n,\overline{\Lambda_{2l}})\subset S_\zeta(U_n,\Lambda_2)$, this means (using assumption (2)) that

$$\bigcap_{x\in U_n}\Delta_x\subset\bigcap_{k=0}^{n-1} S_\zeta(F_k,\overline{\Lambda_{2l}})\cap\overline{S_\zeta(U_n,\Lambda_2)}\cap\Phi=\emptyset. \tag{5}$$

Δ_x are closed subsets of $\overline{S_\zeta(U_{n+1},\Lambda_2)}$, which is compact by Lemma 2. Therefore, there must be a finite set $F_n\subset U_n$ such that $\bigcap_{x\in F_n}\Delta_x=\emptyset$, in other words:

$$\bigcap_{k=0}^{n-1} S_\zeta(F_k,\overline{\Lambda_{2l}})\cap S_\zeta(F_n,\overline{\Lambda_{2l}})\cap\overline{S_\zeta(U_{n+1},\Lambda_2)}\cap\Phi=\emptyset, \tag{6}$$

as desired.

Let $C=\bigcup_{n=0}^{\infty}F_n\cup\{e\}$. We have $F_n\subset U_n$, so C is a set of elements converging to e. Thus C is sequentially compact, but since G is metrizable, it means that C is compact. It is clear that $S_\zeta(C,\overline{\Lambda_{2l}})\cap\overline{S_\zeta(U_n,\Lambda_2)}\cap\Phi=\emptyset$. Since $\mathscr{H}(G,K)=\bigcup_{n=1}^{\infty}S_\zeta(U_n,\Lambda_2)$, this means that $S_\zeta(C,\overline{\Lambda_{2l}})\cap\Phi=\emptyset$. Therefore $S_\zeta(C,\Lambda_{2l})\cap\Phi=\emptyset$. □

A topological space X is *hemicompact* if X is the countable union of compact subspaces X_n, such that every compact subset of X is contained in a finite union of the sets X_n.

Corollary 1. *For a metrizable topological group G and a radical-based compact group K, $\mathscr{C}(\mathscr{H}(G,K),K)$ is completely metrizable.*

Proof. Once metrizability has been shown the completeness is obvious, because $\mathscr{H}(G,K)$ is a k-space, and K is complete (because it is compact). Since in [1] it was shown that if X is hemicompact then $\mathscr{C}(X,K)$ is metrizable, it suffices to show that $\mathscr{H}(G,K)$ is hemicompact.

Let Ξ be a compact subset of $\mathscr{H}(G,K)$. By the Ascoli Theorem Ξ is equicontinuous, in particular there exists a neighborhood U of e such that $\xi(U) \subset \Lambda_2$ for all $\xi \in \Xi$. In other words, $\Xi \subset S_e(U,\Lambda_2)$. Let $\{U_n\}$ be a base at $e \in G$. For some $n \in \mathbb{N}$, $U_n \subset U$, thus $\Xi \subset S_e(U,\Lambda_2) \subset S_e(U_n,\Lambda_2)$. By Lemma 2, $\overline{S_e(U_n,\Lambda_2)}$ is compact, and clearly $\mathscr{H}(G,K) = \bigcup_{n=1}^{\infty} \overline{S_e(U_n,\Lambda_2)}$, hence $\mathscr{H}(G,K)$ is hemicompact, as desired. □

Definition 2. G is *h-complete* if for any continuous homomorphism $f : G \to H$ the subgroup $f(G)$ is closed in H.

The Corollary below generalizes a theorem by Chasco [2] stating that for a separable metrizable complete abelian group G, the natural map $G \to \hat{\hat{G}}$ is an isomorphism of topological groups if and only if it is bijective.

Corollary 2. *Suppose that G is separable metrizable and h-complete, and suppose further that K is compact and radical-based. Then, the natural map $N : G \to \mathscr{C}(\mathscr{H}(G,K),K)$ is continuous and open onto the image. (In particular it is an embedding if and only if it is one-to-one.)*

Proof. Since G is metrizable it is clear that N is continuous, because the natural map $G \to \mathscr{C}(\mathscr{C}(G,K),K)$ is continuous. To see that it is open onto its image, we notice that since G is h-complete, $N(G)$ is closed in $\mathscr{C}(\mathscr{H}(G,K),K)$, and therefore it is complete metric. Hence $N(G)$ is a Baire space. Applying an open map type of theorem ([4, Corollary 32.4]) one obtains that N is open onto the image. □

Theorem 2. *Let K be a compact radical-based group. If D is a dense subgroup of the metrizable group G then $\mathscr{H}(D,K) \cong \mathscr{H}(G,K)$.*

Proof. Clearly, $\mathscr{H}(D,K) = \mathscr{H}(G,K)$ as sets, and we have an induced map $\imath : \mathscr{H}(G,K) \to \mathscr{H}(D,K)$ by restriction. Since $\imath$ is continuous, and bijective, we only have to show that $\imath$ is open. To that end we will show that the inverse image of a compact set is compact. Since $\mathscr{H}(D,K)$ is a k-space it will imply that $\imath$ is open.

Take a compact subset Φ of $\mathscr{H}(D,K)$. Then, by the Ascoli Theorem Φ is equicontinuous; in particular there exists a neighborhood U of e in G, such that $\zeta(U \cap D) \subset \Lambda_4$ for all $\zeta \in \Phi$, and so $\zeta(\overline{U \cap D}) \subset \bar{\Lambda}_4 \subset \Lambda_2$ for all $\zeta \in \Phi$. Thus $\Phi \subset S_e(\overline{U \cap D},\Lambda_2)$.

Let V be a symmetric neighborhood of e in G such that $V^2 \subset U$, and let $x \in V$. There exists a sequence $\{x_n\} \subset D$ such that $x_n \to x$, and thus for $n \geqslant n_0$, $x_n \in xV \subset VV \subset U$. Since $x_n \in D$, $x_n \in U \cap D$ for $n \geqslant n_0$, thus $x \in \overline{U \cap D}$, and hence $V \subset \overline{U \cap D}$. Therefore $\Phi \subset S_e(\overline{U \cap D},\Lambda_2) \subset S_e(V,\Lambda_2)$. By Lemma 2, $S_e(V,\Lambda_2)$ is precompact, hence Φ is compact in $\mathscr{H}(G,K)$ as it is closed there. □

We note that Theorem 2 is a generalization to the non-abelian case of a similar result by Chasco [2].

Uncited but important references

The following books are not quoted above explicitly, but they were a great help to the author in preparing the present paper: General Topology [3], Uniform Spaces [5], General Topology [6] and Topological Groups [7].

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References

[1] R. Arens, A topology for spaces of transformations, Ann. Math. 47 (1946) 480–495.
[2] M.J. Chasco, Pontryagin duality for metrizable groups, Arch. Math. 70 (1998) 22–28.
[3] Ryszard Engelking, General Topology, Polish Scientific Publishers, Poland, 1977.
[4] T. Husain, Introduction to Topological Groups, Saunders, Philadelphia, 1966.
[5] J.R. Isbell, Uniform Spaces, Mathematical Surveys, Number 12, American Mathematical Society, Providence, RI, 1964.
[6] J.L. Kelley, General Topology, Van Nostrand, Princeton, 1955.
[7] L.S. Pontryagin, Topological Groups, Vol. 2, L. S. Pontryagin Selected Works, Gordon and Breach Science Publishers, New York, 1986.