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Discrete symmetry and CP phase of the quark mixing matrix

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Abstract

A simple specific pattern of the two 3×3 quark mass matrices is proposed, resulting in a prediction of the CP phase of the charged-current mixing matrix $V_{\rm CKM}$, i.e., $\sin 2\phi_1(\beta) = 0.733$, which is in remarkable agreement with data, i.e., $\sin 2\phi_1 = 0.728 \pm 0.056 \pm 0.023$ from Belle and $\sin 2\beta = 0.722 \pm 0.040 \pm 0.023$ from BaBar. This pattern can be maintained by a discrete family symmetry, an example of which is D_7, the symmetry group of the heptagon.

The three families of quarks have very different masses and mix with one another in the charged-current mixing matrix $V_{\rm CKM}$ in a nontrivial manner. This 3×3 matrix has three angles and one phase, the latter being the source of CP nonconservation in the Standard Model (SM) of particle interactions. In the context of the SM, this phase is now measured with some precision, i.e.,

$$\sin 2\phi_1 = 0.728 \pm 0.056 \pm 0.023 \tag{1}$$

from Belle [1], and

$$\sin 2\beta = 0.722 \pm 0.040 \pm 0.023 \tag{2}$$

from BaBar [2], where ϕ_1 (also known as β) is defined as the phase of the element V_{td}, i.e.,

$$V_{td} = |V_{td}| e^{-i\phi_1}. \tag{3}$$

Together with $|V_{us}|$, $|V_{cb}|$, and $|V_{ub}|$, the entire $V_{\rm CKM}$ matrix can now be fixed, up to sign and phase conventions. Given the experimentally measured values of these parameters, is there a pattern to be recognized? The answer is

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Table 1
Character table of D_7

Class	n	h	χ_1	χ_2	χ_3	χ_4	χ_5
C_1	1	1	1	1	2	2	2
C_2	7	2	1	−1	0	0	0
C_3	2	7	1	1	a_1	a_2	a_3
C_4	2	7	1	1	a_2	a_3	a_1
C_5	2	7	1	1	a_3	a_1	a_2

not obvious, because the relevant physics comes from the structure of the two 3×3 quark mass matrices

$$\mathcal{M}_u = V_L^u \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} (V_R^u)^\dagger, \tag{4}$$

$$\mathcal{M}_d = V_L^d \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} (V_R^u)^\dagger, \tag{5}$$

from which the observed quark mixing matrix is obtained

$$V_{\rm CKM} = (V_L^u)^\dagger V_L^d. \tag{6}$$

A theoretically consistent approach to understanding $\mathcal{M}_u$ and $\mathcal{M}_d$ is to extend the Lagrangian of the SM to support a family symmetry in such a way that the forms of these mass matrices are restricted with fewer parameters than are observed, thus making one or more predictions. Because of complex phases, this is often not a straightforward proposition. In this Letter, we advocate a simple specific pattern, i.e., $\mathcal{M}_u$ is diagonal, whereas $\mathcal{M}_d$ is of the form

$$\mathcal{M}_d = \begin{pmatrix} 0 & a & \xi b \\ a & 0 & b \\ \xi c & c & d \end{pmatrix}, \tag{7}$$

which was first proposed by one of us long ago [3]. The difference here is that whereas $|\xi|$ was fixed at m_u/m_c in that model, it is now a free parameter. The family symmetry used previously was $S_3 \times Z_3$, which still works, but with different Z_3 assignments and a larger Higgs sector. As a more elegant example for our discussion, we choose instead D_7, the symmetry group of the heptagon [4]. A recent proposal [5] based on Q_6 has both $\mathcal{M}_u$ and $\mathcal{M}_d$ of the form of Eq. (7), but with $\xi = 0$. To maintain this latter condition consistently, an extra Z_{12} symmetry has to be assumed. Here ξ is simply another parameter, equal to the ratio of two arbitrary vacuum expectation values.

The group D_7 has 14 elements, 5 equivalence classes, and 5 irreducible representations. Its character table is given by Table 1. Here n is the number of elements and h is the order of each element. The numbers a_k are given by $a_k = 2\cos(2k\pi/7)$. The character of each representation is its trace and must satisfy the following two orthogonality conditions:

$$\sum_{C_i} n_i \chi_{ai} \chi_{bi}^* = n\delta_{ab}, \qquad \sum_{\chi_a} n_i \chi_{ai} \chi_{aj}^* = n\delta_{ij}, \tag{8}$$

where $n = \sum_i n_i$ is the total number of elements. The number of irreducible representations must be equal to the number of equivalence classes.

The three irreducible two-dimensional representations of D_7 may be chosen as follows. For $\mathbf{2}_1$, let

$$C_1: \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_2: \quad \begin{pmatrix} 0 & \omega^k \\ \omega^{7-k} & 0 \end{pmatrix} \quad (k = 0, 1, 2, 3, 4, 5, 6),$$

$$C_3: \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \begin{pmatrix} \omega^6 & 0 \\ 0 & \omega \end{pmatrix}, \qquad C_4: \quad \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix}, \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix},$$
$$C_5: \quad \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix}, \tag{9}$$

where $\omega = \exp(2\pi i/7)$, then $\mathbf{2}_{2,3}$ are simply obtained by the cyclic permutation of $C_{3,4,5}$.

For D_n with n prime, there are $2n$ elements divided into $(n+3)/2$ equivalence classes: C_1 contains just the identity, C_2 has the n reflections, C_k from $k=3$ to $(n+3)/2$ has 2 elements each of order n. There are 2 one-dimensional representations and $(n-1)/2$ two-dimensional ones. For $D_3 = S_3$, the above reduces to the "complex" representation with $\omega = \exp(2\pi i/3)$ discussed in a recent review [6].

The group multiplication rules of D_7 are

$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1}, \qquad \mathbf{1}' \times \mathbf{2}_i = \mathbf{2}_i, \tag{10}$$
$$\mathbf{2}_i \times \mathbf{2}_i = \mathbf{1} + \mathbf{1}' + \mathbf{2}_{i+1}, \qquad \mathbf{2}_i \times \mathbf{2}_{i+1} = \mathbf{2}_i + \mathbf{2}_{i+2}, \tag{11}$$

where $\mathbf{2}_{4,5}$ means $\mathbf{2}_{1,2}$. In particular, let $(a_1, a_2), (b_1, b_2) \sim \mathbf{2}_1$, then

$$a_1 b_2 + a_2 b_1 \sim \mathbf{1}, \qquad a_1 b_2 - a_2 b_1 \sim \mathbf{1}', \qquad (a_1 b_1, a_2 b_2) \sim \mathbf{2}_2. \tag{12}$$

In the decomposition of $\mathbf{2}_1 \times \mathbf{2}_2$, we have instead

$$(a_2 b_1, a_1 b_2) \sim \mathbf{2}_1, \qquad (a_2 b_2, a_1 b_1) \sim \mathbf{2}_3. \tag{13}$$

To arrive at our proposed pattern, let

$$(u, d)_i \sim \mathbf{2}_1 + \mathbf{1}, \qquad d_i^c \sim \mathbf{2}_1 + \mathbf{1}, \qquad u_i^c \sim \mathbf{2}_2 + \mathbf{1}, \tag{14}$$
$$\phi_i^d \sim \mathbf{2}_1 + \mathbf{1}, \qquad \phi_i^u \sim \mathbf{2}_3 + \mathbf{1}, \tag{15}$$

where the scalar fields $\phi_i^{d,u}$ are distinguished by an extra symmetry such as supersymmetry so that they couple only to d^c, u^c, respectively. Using the multiplication rules listed above, we see that $\mathcal{M}_u$ is indeed diagonal, and $\mathcal{M}_d$ is of the form of Eq. (7), with a, d coming from $\langle \phi_3^d \rangle$ and $(b, \xi b)$, $(c, \xi c)$ from $\langle \phi_{1,2}^d \rangle$, respectively. These latter are distinct from $\langle \phi_{1,2}^u \rangle$, so that the constraint $|\xi| = m_u/m_c$ in Ref. [3] no longer applies.

As in Ref. [3], we can redefine the phases of $\mathcal{M}_d$ so that a, b, c, d are real, but ξ is complex. Assuming that $a^2 \ll b^2$ and $|\xi|^2 \ll 1$, then to a very good approximation,

$$m_b \simeq \sqrt{c^2 + d^2}, \qquad m_s \simeq \frac{bc}{\sqrt{c^2 + d^2}}, \qquad m_d \simeq \left| \frac{a^2 d}{bc} - 2\xi a \right|, \tag{16}$$
$$V_{cb} \simeq \frac{bd}{c^2 + d^2}, \qquad V_{us} \simeq -\frac{ad}{bc} + \xi, \qquad V_{ub} \simeq \frac{ac + \xi bd}{c^2 + d^2}. \tag{17}$$

Using the 6 experimental inputs on m_b, m_s, m_d, $|V_{cb}|$, $|V_{us}|$, and $|V_{ub}|$, the 6 parameters a, b, c, d, $\mathrm{Re}\,\xi$, and $\mathrm{Im}\,\xi$ are fixed, thereby predicting the CP phase of V_{CKM}. Numerical inputs of quark masses (in GeV) are taken from Ref. [7] evaluated at the scale M_W, i.e.,

$$m_d = 0.00473 \begin{pmatrix} +0.00061 \\ -0.00067 \end{pmatrix}, \qquad m_s = 0.0942 \begin{pmatrix} +0.0119 \\ -0.0131 \end{pmatrix}, \qquad m_b = 3.03 \pm 0.11. \tag{18}$$

Numerical inputs of mixing angles are taken from the 2004 Particle Data Group compilation [8], i.e.,

$$|V_{us}| = 0.2200 \pm 0.0026, \qquad |V_{cb}| = (41.3 \pm 1.5) \times 10^{-3}, \qquad |V_{ub}| = (3.67 \pm 0.47) \times 10^{-3}. \tag{19}$$

Taking the central values of the above 6 quantities, we find

$$a = 0.0142 \text{ GeV}, \qquad b = 0.1566 \text{ GeV}, \qquad c = 1.8223 \text{ GeV}, \qquad d = -2.4208 \text{ GeV}, \tag{20}$$

$$\mathrm{Re}\,\xi = 0.08124, \qquad \mathrm{Im}\,\xi = 0.08791. \tag{21}$$

After rotating the phase of V_{us} to make it real to conform to the standard convention, we then predict

$$\sin 2\phi_1 = 0.733, \tag{22}$$

in remarkable agreement with experiment, i.e., Eqs. (1) and (2). The three angles $\phi_1(\beta)$, $\phi_2(\alpha)$, $\phi_3(\gamma)$ of the unitarity triangle are then 23.6°, 98.4°, 58.0°, respectively.

We may also vary the 6 numerical inputs within their allowed ranges, taking into account the correlation between m_d and m_s (because m_d/m_s is tightly constrained). In that case,

$$\sin 2\phi_1 = 0.733 \begin{pmatrix} +0.107 \\ -0.152 \end{pmatrix}. \tag{23}$$

In the future, these input parameters will be determined with more precision and our model will be more severely tested.

Flavor-changing neutral-current interactions are mediated by the three neutral Higgs bosons in the d sector with Yukawa couplings given by

$$\mathcal{L}_Y = \frac{a}{v_3}\phi_3^0\bigl(q_1 q_2^c + q_2 q_1^c\bigr) + \frac{b}{v_1}\bigl(\phi_1^0 q_2 + \phi_2^0 q_1\bigr)q_3^c + \frac{c}{v_1}q_3\bigl(\phi_1^0 q_2^c + \phi_2^0 q_1^c\bigr) + \frac{d}{v_3}\phi_3^0 q_3 q_3^c + \text{h.c.}, \tag{24}$$

where q_i, q_j^c are the basis states of the mass matrix $\mathcal{M}_d$ of Eq. (7). Let

$$\mathcal{M}_d = V \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} (V^c)^\dagger, \tag{25}$$

then $V = V_{\rm CKM}$ and V^c is its analog for the charge-conjugate states. In this model, they are approximately given by

$$V \simeq \begin{pmatrix} 1 & -(ad/bc) + \xi & (ac + \xi bd)/(c^2 + d^2) \\ (ad/bc) - \xi^* & 1 & bd/(c^2 + d^2) \\ -a/c & -bd/(c^2 + d^2) & 1 \end{pmatrix}, \tag{26}$$

where $\xi = v_2/v_1$, and

$$V^c \simeq \begin{pmatrix} 1 & -(ad/bc) + \xi^* c^2/(c^2 + d^2) & a/b + \xi^* cd/(c^2 + d^2) \\ a\sqrt{c^2 + d^2}/bc & d/\sqrt{c^2 + d^2} & -c/\sqrt{c^2 + d^2} \\ -\xi c/\sqrt{c^2 + d^2} & c/\sqrt{c^2 + d^2} & d/\sqrt{c^2 + d^2} \end{pmatrix}. \tag{27}$$

Using $q_i = V_{i\alpha} d_\alpha$ and $q_j^c = V_{j\beta}^c d_\beta^c$, we can rewrite the couplings of $\phi_{1,2,3}^0$ in terms of the quark mass eigenstates and evaluate their contributions to flavor-changing processes such as K–$\bar{K}$ and B–$\bar{B}$ mixings, etc.

An important point to notice [9] is that if $\phi_{1,2}$ are replaced by ϕ_3 in the Yukawa sector, then there would be no flavor-changing interactions at all. Hence all such effects are contained in the terms

$$\left(\frac{\phi_1^0}{v_1} - \frac{\phi_3^0}{v_3}\right)\bigl(b q_2 q_3^c + c q_3 q_2^c\bigr) + \xi\left(\frac{\phi_2^0}{v_2} - \frac{\phi_3^0}{v_3}\right)\bigl(b q_1 q_3^c + c q_3 q_1^c\bigr). \tag{28}$$

Whereas the mass of the SM combination $(v_1\phi_1^0 + v_2\phi_2^0 + v_3\phi_3^0)/\sqrt{|v_1|^2 + |v_2|^2 + |v_3|^2}$ should be of order the electroweak breaking scale, the two orthogonal combinations contained in the above are allowed to be much heavier, say a few TeV.

The K_L–K_S mass difference gets its main contribution from the $(q_1 q_3^c)(q_3 q_1^c)^\dagger$ term through ϕ_2^0 exchange. Thus

$$\frac{\Delta m_K}{m_K} \simeq \frac{B_K f_K^2 b^2 c^2 d}{3(c^2 + d^2)^{3/2} m_2^2 v_1^2}. \tag{29}$$

Using $f_K = 114$ MeV, $B_K = 0.4$, $v_1 = 100$ GeV, and $m_2 = 7$ TeV, we find this contribution to be 2.5×10^{-17}, well below the experimental value of 7.0×10^{-15}. Similarly,

$$\frac{\Delta m_B}{m_B} \simeq \frac{B_B f_B^2 bcd}{3(c^2+d^2)^{1/2} m_2^2 v_1^2}, \tag{30}$$

and

$$\frac{\Delta m_{B_s}}{m_{B_s}} \simeq \frac{B_B f_B^2 bcd^2}{3(c^2+d^2) m_1^2} \left(\frac{1}{v_1^2} + \frac{1}{v_3^2} \right). \tag{31}$$

Using $f_B = 170$ MeV, $B_B = 1.0$, $v_{1,3} = 100$ GeV, and $m_{1,2} = 7$ TeV, we find these contributions to be 4.5×10^{-15} and 7.2×10^{-15} respectively, to be compared against the experimental value of 6.3×10^{-14} for the former and the experimental lower bound of 1.8×10^{-12} for the latter.

It is interesting to note that the form of Eq. (7) is easily adaptable to the Majorana neutrino mass matrix. By rearranging the two zeros, we can have

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix} \tag{32}$$

as advocated in Ref. [4], which is a successful description of neutrino oscillation phenomena. This hints at the intriguing possibility that despite their outward dissimilarities, both quark and lepton family structures may actually come from the same mold.

In conclusion, we have pointed out that the $\mathcal{M}_d$ of Eq. (7) predicts the correct value of the CP phase of the quark mixing matrix. Its form is derivable from a discrete family symmetry such as D_7, which also works for leptons as previously shown. Extra Higgs doublets are predicted, but their contributions to flavor-changing interactions are suitably suppressed if their masses are of order a few TeV.

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