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A differential inequality for the positive zeros of Bessel functions

E.K. Ifantis and P.D. Siafarikas
Department of Mathematics, University of Patras, Greece

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Abstract

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It is proved that the positive zeros $j_{\nu,k}$, $k=1,2,\dots$, of the Bessel function $J_\nu(x)$ of the first kind and order $\nu>-1$, satisfy the differential inequality $j_{\nu,k}\,\mathrm{d}j_{\nu,k}/\mathrm{d}\nu>1+(1+j_{\nu,k}^2)^{1/2}$, $\nu>-1$. This inequality improves the well-known inequality $\mathrm{d}j_{\nu,k}/\mathrm{d}\nu>1$, $\nu>-1$, which is the source of a large number of lower and upper bounds for the zeros $j_{\nu,k}$, $k=1,2,\dots$.

Keywords: Differential inequalities; bounds of zeros of Bessel functions.

1. Introduction

The differential inequality

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu}>1, \tag{1.1}$$

where $j_{\nu,k}$, $k=1,2,\dots$, is the kth positive zero of Bessel function $J_\nu(x)$ of the first kind and order ν, has attracted the attention of many authors. McCann and Love [8] have proved this in the interval $0<\nu<0{,}05$ and used this result to complete the proof of the inequality $j_{\nu,k}>j_{0,1}+\nu$, $\nu>0$. Elbert and Laforgia [1] have proved (1.1) for $\nu>0$ and used this result to prove the convexity with respect to ν of the function $j_{\nu,k}^2$ for $\nu>0$. The authors [4], among other results, have proved inequality (1.1) for $\nu>-1$ and used it to derive several upper and lower bounds for the zeros $j_{\nu,k}$, $k=1,2,\dots$. In this paper we prove the inequality

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu}>\frac{1}{j_{\nu,k}}+\left(1+\frac{1}{j_{\nu,k}^2}\right)^{1/2},\quad \nu>-1, \tag{1.2}$$

Correspondence to: Prof. E.K. Ifantis, Department of Mathematics, University of Patras, Patras, Greece.

which is more stringent than (1.1) for every $\nu > -1$. A consequence of (1.2) is that the functions

$$\sqrt{1+j_{\nu,k}^2} - \ln\left[1+\sqrt{1+j_{\nu,k}^2}\right] - \nu, \quad \nu > -1, \tag{1.3}$$

and

$$\sqrt{7+j_{\nu,k}^2} - \nu, \quad \nu > -1, \quad \text{provided } j_{\nu,k}^2 > 5.25, \tag{1.4}$$

increase with ν in the interval $(-1, +\infty)$. From the monotonicity of (1.3) and (1.4) we can find a number of lower and upper bounds for the zeros $j_{\nu,k}$, $k = 1, 2, \ldots$. We note the following two:

$$j_{\nu,k} > \left[\nu + \sqrt{1+j_{0,k}^2} + \ln\frac{1+\left(1+j_{0,k}^2+\nu^2+2\nu\sqrt{7+j_{0,k}^2}\right)^{1/2}}{1+\sqrt{1+j_{0,k}^2}}\right]^2 - 1, \quad \nu > 0, \tag{1.5}$$

and

$$j_{\nu,k}^2 > j_{0,k}^2 + \nu^2 + 2\nu\sqrt{7+j_{0,k}^2}, \quad \nu > 0, \tag{1.6}$$

which improve the lower bound $j_{\nu,k} > j_{0,k} + \nu$, $\nu > 0$, proved by different methods in [8] for $k = 1$ and in [4,7] for $k \geqslant 1$.

The inequalities (1.5) and (1.6) are reversed for $-1 < \nu < 0$. The upper bounds established now for $-1 < \nu < 0$ are more stringent than the upper bound $j_{\nu,k} < j_{0,k}$, $-1 < \nu < 0$, found in [4].

2. Preliminaries

In this section we present some notations, definitions and known results which are necessary for the proof of the inequality (1.1).

Consider an abstract separable Hilbert space H with the scalar product $(\cdot, \cdot)$ and with the orthonormal basis e_n, $n \geqslant 1$. The shift operator V is defined as follows: $Vf = \sum_{n=1}^{\infty}(f, e_n)e_{n+1}$, $f \in H$, and its adjoint V^* as: $V^*f = \sum_{n=1}^{\infty}(f, e_{n+1})e_n$. Usually we write $Ve_n = e_{n+1}$, $n \geqslant 1$, and $V^*e_n = e_{n-1}$, $V^*e_1 = 0$.

The operator V is an isometry, i.e., a linear operator with the property $\|Vf\| = \|f\|$, $f \in H$, while V^* is a partial isomety, i.e., linear with the property

$$\|V^*f\| = \sqrt{\|f\|^2 - |(f, e_1)|^2}, \quad f \in H, \tag{2.1}$$

where e_1 is the first element of the basis e_n, $n \geqslant 1$. This means that V^* is an isometry on the subspace $H\theta\{e_1\}$. It is well known [3] that the self-adjoint operator

$$T_0 = V + V^* \tag{2.2}$$

is bounded with $\|T_0\| = 2$. Its spectrum is purely continuous and covers the entire interval $[-2, +2]$. The diagonal operator

$$L_\nu e_n = \frac{1}{\nu+n} e_n, \quad n \geqslant 1, \tag{2.3}$$

can be defined for every $\nu \neq -n$. It is a compact operator because $\lim_{n\to\infty}(\nu+n)^{-1}=0$. In particular, for ν real and $\nu>-1$ it is self-adjoint and positive ($(L_\nu f, f)>0$, $f\in H$), so its square root $L^{1/2}$ exists and the self-adjoint and compact operator

$$S_\nu = L_\nu^{1/2} T_0 L_\nu^{1/2} \tag{2.4}$$

can be defined for every $\nu>-1$. One of the results proved in [6] is the following. The eigenvalues of S_ν are precisely the values

$$\pm\frac{2}{j_{\nu,k}},\quad k=1,2,\ldots, \tag{2.5}$$

where $j_{\nu,k}$, $k=1,2,\ldots$, are the positive zeros of the Bessel function $J_\nu(x)$, $\nu>-1$. We shall use here the differential equation

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} = j_{\nu,k}(L_\nu x_k(\nu), x_k(\nu)),\quad k=1,2,\ldots,\quad \nu>-1, \tag{2.6}$$

which has been proved in [4]. In (2.6), $x_k(\nu)$ is the normalized eigenvector ($\|x_k(\nu)\|=1$) which corresponds to the positive eigenvalue $2/j_{\nu,k}$ of S_ν. Finally, another result which we shall use here is the relation

$$|(e_1, x_k(\nu))|^2 = \frac{2(\nu+1)}{j_{\nu,k}^2},\quad k=1,2,\ldots,\quad \nu>-1. \tag{2.7}$$

(See [5] for the proof of (2.7).)

3. Proof of (1.1)

We set in (2.6) $x_k(\nu)=L_\nu^{-1/2}\,u_k(\nu)$ and obtain

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} = j_{\nu,k}\frac{(L_\nu x_k(\nu), x_k(\nu))}{(x_k(\nu), x_k(\nu))} = j_{\nu,k}\frac{(u_k(\nu), u_k(\nu))}{(L_\nu^{-1}u_k(\nu), u_k(\nu))}. \tag{3.1}$$

Since $L_\nu^{-1}u_k(\nu)=\frac{1}{2}j_{\nu,k}T_0u_k(\nu)$ (which follows easily from the eigenvalue equation $S_\nu x_k(\nu) = 2x_k(\nu)/j_{\nu,k}$), we find from (3.1):

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} = \frac{2(u_k(\nu), u_k(\nu))}{(T_0u_k(\nu), u_k(\nu))} = \frac{2(u_k(\nu), u_k(\nu))}{(Vu_k(\nu), u_k(\nu)) + (V^*u_k(\nu), u_k(\nu))},$$

and since $(Vu_k(\nu), u_k(\nu))$ is real, we have

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} = \frac{2(u_k(\nu), u_k(\nu))}{2(V^*u_k(\nu), u_k(\nu))} \geqslant \frac{\|u_k(\nu)\|}{\|V^*u_k(\nu)\|}. \tag{3.2}$$

But $V^*u_k(\nu)=\sum_{n=2}^{\infty}(u_k(\nu), e_n)e_{n-1}$, and due to (2.1):

$$\|V^*u_k(\nu)\|^2 = \sum_{n=2}^{\infty}|(u_k(\nu), e_n)|^2 = \|u_k(\nu)\|^2 - |(u_k(\nu), e_1)|^2. \tag{3.3}$$

From (3.2), because of (3.3), we have

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} > \frac{\|u_k(\nu)\|}{\left(\|u_k(\nu)\|^2 - |(u_k(\nu), e_1)|^2\right)^{1/2}} = \left(1 - \frac{|(u_k(\nu), e_1)|^2}{\|u_k(\nu)\|^2}\right)^{-1/2}. \tag{3.4}$$

Now we use the relation (2.7) and find

$$|(u_k(\nu), e_1)|^2 = |(L_\nu^{1/2} x_k(\nu), e_1)|^2 = \frac{1}{\nu+1} |(x_k(\nu), e_1)|^2 = \frac{2}{j_{\nu,k}^2}. \tag{3.5}$$

Finally,

$$\begin{aligned}\|u_k(\nu)\|^2 &= (u_k(\nu), u_k(\nu)) = (L_\nu^{1/2} x_k(\nu), L_\nu^{1/2} x_k(\nu)) = (L_\nu x_k(\nu), x_k(\nu)) \\ &= \frac{\mathrm{d}j_{\nu,k}/\mathrm{d}\nu}{j_{\nu,k}}.\end{aligned} \tag{3.6}$$

So, from (3.4), because of (3.5) and (3.6), we have

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} > \left(1 - \frac{2}{j_{\nu,k}\, \mathrm{d}j_{\nu,k}/\mathrm{d}\nu},\right)^{-1/2},$$

from which (1.2) follows.

Note that from the relation

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} = \frac{2\|u_k(\nu)\|^2}{(T_0 u_k(\nu), u_k(\nu))}$$

and the inequality

$$|(T_0 u_k(\nu), u_k(\nu))| \leqslant \|T_0 u_k(\nu)\| \cdot \|u_k(\nu)\| \leqslant \|T_0\|\, \|u_k(\nu)\|^2 = 2\|u_k(\nu)\|^2,$$

inequality (1.1), which follows from (1.2) for every $\nu > -1$, has been found in [4].

4. Monotonicity properties and bounds

From (1.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\sqrt{1 + j_{\nu,k}^2} - \ln\left(1 + \sqrt{1 + j_{\nu,k}^2}\right) - \nu\right] > 0, \quad \nu > -1. \tag{4.1}$$

Also since

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} > \frac{1}{j_{\nu,k}} + \frac{1}{j_{\nu,k}}\sqrt{1 + j_{\nu,k}^2} > \left(1 + \frac{4}{j_{\nu,k}}\right)^{1/2}, \qquad \nu > -1, \tag{4.2}$$

and

$$\frac{\mathrm{d}j_{\nu,k}}{\mathrm{d}\nu} > \frac{1}{j_{\nu,k}} + \frac{1}{j_{\nu,k}}\left(1 + \frac{7}{j_{\nu,k}^2}\right)^{1/2}, \qquad \nu > -1, \text{ provided } j_{\nu,k}^2 > 5.25, \tag{4.3}$$

we obtain from (4.2)

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\sqrt{4+j_{\nu,k}^2}-\nu\right]>0, \quad \nu>-1, \tag{4.4}$$

and from (4.3)

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\sqrt{j_{\nu,k}^2+7}-\nu\right]>0, \quad \nu>-1, \text{ provided } j_{\nu,k}^2>5.25. \tag{4.5}$$

From (4.1), (4.4) and (4.5) it follows that the functions

$$\sqrt{1+j_{\nu,k}^2}-\ln\left(1+\sqrt{1+j_{\nu,k}^2}\right)-\nu, \quad \nu>-1, \tag{4.6}$$

$$\sqrt{4+j_{\nu,k}^2}-\nu, \quad \nu>-1, \tag{4.7}$$

and

$$\sqrt{7+j_{\nu,k}^2}-\nu, \quad \nu>-1, \text{ provided } j_{\nu,k}^2>5.25, \tag{4.8}$$

increase as ν increases in the interval $(-1, \infty)$. Hence, we have the following inequalities for the zeros $j_{\nu,k}$ of the Bessel function $J_\nu(x)$:

$$\sqrt{1+j_{\nu,k}^2}-\ln\left(1+\sqrt{1+j_{\nu,k}^2}\right)-\nu>\sqrt{1+j_{\mu,k}^2}-\ln\left(1+\sqrt{1+j_{\mu,k}^2}\right)-\mu \quad \nu>\mu>-1, \tag{4.9}$$

$$\sqrt{j_{\nu,k}^2+4}-\nu>\sqrt{j_{\mu,k}^2+4}-\mu, \quad \nu>\mu>-1, \tag{4.10}$$

and

$$\sqrt{7+j_{\nu,k}^2}-\nu>\sqrt{7+j_{\mu,k}^2}-\mu, \quad \nu>\mu>-1, \text{ provided } j_{\nu,k}^2>5.25. \tag{4.11}$$

The above inequalities are quite sharp as $\nu \to \mu$, since in this limit they become equalities. Also from (4.9)–(4.11) we can obtain several lower and upper bounds for the zeros $j_{\nu,k}$, $k=1, 2, \ldots,$ of $J_\nu(z)$. For example, for $\mu=0$ we find from (4.9) and (4.10) the lower bounds

$$j_{\nu,k}>\left[\nu+\sqrt{1+j_{0,k}^2}+\ln\frac{1+\left(j_{\nu,k}^2\right)^{1/2}}{\left(1+j_{0,k}^2\right)^{1/2}}\right]^2-1, \quad \nu>0, \tag{4.12}$$

and

$$j_{\nu,k}^2>j_{0,k}^2+\nu^2+2\nu\sqrt{7+j_{0,k}^2}, \quad \nu>0, \ j_{0,k}^2\geqslant 5.25. \tag{4.13}$$

These lower bounds improve the lower bound

$$j_{\nu,k}^2>j_{0,k}^2+\nu^2+2\nu j_{0,k}, \quad \nu>0, \tag{4.14}$$

proved by different methods in [8] for $k=1$ and in [4,7] for $k\geqslant 1$.

Combining the bounds (4.12) and (4.13) we obtain the lower bound

$$j_{\nu,k}^2>\left[\nu+\sqrt{1+j_{0,k}^2}+\ln\frac{1+\left(1+j_{0,k}^2+\nu^2+2\nu\sqrt{7+j_{0,k}^2}\right)^{1/2}}{1+\sqrt{1+j_{0,k}^2}}\right]^2-1, \quad \nu>0, \tag{4.15}$$

which for $k=1$ take the form

$$j_{\nu,1}^2 > \left[\nu + 1.3222807 + \ln\left(1 + \sqrt{\nu^2 + 7.1507088\,\nu + 6.7831592}\,\right)\right]^2 - 1, \quad \nu > 0. \tag{4.16}$$

Numerical evidence indicates that (4.16) is sharper for $\nu \geqslant 0.99$ than the following lower bound:

$$j_{\nu,1} > j_{0,1} + 1.542889743\,\nu - 0.175493592\,\nu^2, \quad \nu > 0, \tag{4.17}$$

found in [2].

The inequalities (4.12), (4.13) and (4.15) are reversed for $-1<\mu<0$. The upper bounds established now for $-1<\mu<0$ are more stringent than the upper bound

$$j_{\mu,\kappa} < j_{0,k} + \mu, \quad -1<\mu<0, \tag{4.18}$$

found in [4].

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