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The Sizes of Compact Subsets of Hilbert Space and Continuity of Gaussian Processes

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1. THE SIZES OF COMPACT SETS

The first two sections of this paper are introductory and correspond to the two halves of the title.

As is well known, there is no complete analog of Lebesgue or Haar measure in an infinite-dimensional Hilbert space H, but there is a need for some measure of the sizes of subsets of H. In this paper we shall study subsets C of H which are closed, bounded, convex and symmetric ($-x \in C$ if $x \in C$). Such a set C will be called a *Banach ball*, since it is the unit ball of a complete Banach norm on its linear span. In most cases in this paper C will be compact.

We use three main measures of the size of C. One is as follows. Let $V_n = V_n(C)$ be the supremum of (n-dimensional Lebesgue) volumes of projections $P_n(C)$ where P_n is any orthogonal projection with n-dimensional range. Then we define the *exponent of volume* of C, $EV(C)$, by

$$EV(C) = \limsup_{n\to\infty} \frac{\log V_n}{n \log n}.$$

Another numerical measure of the size of C involves the notion of ϵ-entropy [*12*]. Let (S, d) be a metric space. The *diameter* of a set $T \subset S$ is defined as

$$\sup \{d(x, y) : x, y \in T\}.$$

Given $\epsilon > 0$, one defines $N(S, \epsilon)$ as the minimal number of sets of diameter at most 2ϵ which cover S. Then the *ϵ-entropy* of S, $H(S, \epsilon)$,

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is defined as $\log N(S, \epsilon)$. (The logarithm is taken to the base e. The ideas of information theory and thermodynamics play no explicit role in this paper.) Finally, we define the *exponent of entropy* r by

$$r = r(S) \equiv \limsup_{\epsilon \downarrow 0} \frac{\log H(S, \epsilon)}{\log (1/\epsilon)}.$$

(In case the lim sup is equal to a limit, r has been called the *metric order* of S—see [*12*], p. 22.)

We prove below (Proposition 5.8) that if $EV(C) \geqslant -1/2$, then $r(C) = +\infty$, while if $EV(C) < -1/2$, then

$$r(C) \geqslant -\frac{2}{1 + 2EV(C)}.$$

If the above inequality becomes an equality C will be called *volumetric.* In Section 6 we prove that ellipsoids, rectangular solids, certain "full approximation sets", and, if $EV(C) < -1$, octahedra, are volumetric. The question is left open for $-1 \leqslant EV(C) < -1/2$, but I conjecture (5.9) only that a Banach ball C with $EV(C) < -1$ is volumetric.

Our third general measure of the size of a Banach ball C involves the canonical "normal distribution" L on H ([*18*], pp. 116-119; [*9*]). L is a linear mapping of H into a set of Gaussian random variables with mean 0, which preserves inner products. Let A be a countable dense subset of C and

$$\underline{L}(C) = \sup \{| L(x) | : x \in A\}.$$

Then $\underline{L}(C)$ is a well-defined functionoid; i.e., a different choice of A will affect $\underline{L}(C)$ only on a set of zero probability.

For any $k > 0$, $r(kC) = r(C)$ and $EV(kC) = EV(C)$, but the random variable $\underline{L}(C)$ does not have this homothetic invariance. We call C a *GB-set* if $\underline{L}(C)$ is finite with probability one. This property is homothetically invariant, and for other reasons which will become clearer in the next section, we study mainly the GB-property rather than the entire random variable $\underline{L}(C)$. To relate this property to r and EV we have the following main results: if $r(C) < 2$ then C is a GB-set (V. Strassen (unpublished) and Corollary 3.2 below). If $r(C) = 2$, C need not be a GB-set (Section 6) and I conjecture (3.3) that if $r(C) > 2$ it never is. If $EV(C) > -1$, C is not a GB-set (Theorem 5.3); I conjecture (5.4) that C is a GB-set if $EV(C) < -1$, and prove this for $EV(C) < -3/2$ (Proposition 5.5). The conjectures are proved in all four classes of special cases considered in Section 6.

However, at $r = 2$ and $EV = -1$ there is some "overlap" and the GB-property is not a monotone function of the $H(S, \epsilon)$ as $\epsilon \downarrow 0$ nor of V_n as $n \to \infty$ (Proposition 6.10).

2. Continuity of Gaussian Processes

We shall study sample function continuity and boundedness of Gaussian processes from a general viewpoint. Let (S, d) be a metric space and let $\{x_t, t \in S\}$ be a real-valued Gaussian stochastic process over S (for definitions see, e.g., [5], p. 72). Then the x_t are all elements of a Hilbert space $H = L^2(\Omega, \mathscr{B}, \Pr)$ over some probability space $(\Omega, \mathscr{B}, \Pr)$. ($\Omega$ is a set, $\mathscr{B}$ a σ-algebra of subsets, and Pr a probability measure on $\mathscr{B}$). If two Gaussian processes over the same S have the same mean and covariance functions Ex_t and $Ex_s x_t$, then they have the same joint probability distributions for $\{x_t, t \in F\}$ for any finite or countable subset F of S ([5], p. 72, (3.3)). Such processes will be called "versions" of each other. We say that a process is *sample-continuous* if it has a version $\{x_t, t \in S\}$ such that for almost all ω in Ω, $t \to x_t(\omega)$ is continuous on S. (In case S is e.g. the real line it is well known that not all versions of a process will be continuous.)

Sequential convergence of functions on Ω almost everywhere implies convergence in measure and then, for the Gaussian case, convergence in H. Thus since S is metric, sample continuity implies that $t \to x_t$ is continuous from S into H and we can and will restrict ourselves to this case. Then, Ex_t is continuous on S, and x_t is sample-continuous if and only if $x_t - Ex_t$ is, so we may and do assume $Ex_t \equiv 0$.

A subset C of an abstract Hilbert space H_1 is realized as a Gaussian process $\{x_t, t \in C\}$ with $Ex_t \equiv 0$ and $Ex_s x_t = (s, t)$ by letting $x_t = L(t)$ where L is the "normal" random li ear functional or weak distribution mentioned in Section 1. We callnC a *GC-set* (Gaussian continuity set) if L is sample-continuous on C. Thus if $\{x_t, t \in S\}$ is a (Gaussian) process with $Ex_t \equiv 0$, the function $t \to x_t$ is continuous from S into H, and its range is a GC-set, then the process is sample-continuous.

For any set $A \subset H$ there is a sample-continuous process $\{x_t, t \in S\}$ whose range is A, letting S be A with discrete topology, but such examples are rather artificial and much of the study of sample-continuous Gaussian processes reduces to the study of GC-sets. (See e.g. the end of Section 4.)

A process $\{x_t, t \in S\}$ will be called *sample-bounded* if it has a version such that the sample functions $t \to x_t(\omega)$ are bounded uniformly on S, for each ω. Here we have a perfect correspondence: a Gaussian process is sample-bounded if and only if its range is a GB-set. The convex, closed, symmetric hull of a GB-set is a GB-set and is compact (Proposition 3.4 below). We shall on the whole restrict ourselves to compact sets, and a compact GC-set is a GB-set. Conversely, most, but not all, GB-sets are GC-sets. Sample continuity and boundedness are equivalent for ellipsoids and rectangular blocks (Propositions 6.3 and 6.6 below) and stationary processes on a finite interval ([*2*], Theorem 1). A narrow class of GB-sets which are not GC-sets appears among octahedra (Propositions 6.7 and 6.9 below), and other examples can be constructed by the law of the iterated logarithm. We shall prove severe narrowness of the class of GB-sets which are not GC-sets in general (Theorem 4.7).

V. Strassen proved (unpublished) in 1963 or 1964 that, if S is a set of Gaussian random variables with $r(S) < 2$, then (in our terminology) it is a GC-set. Strassen's result is sharpened somewhat (Theorem 3.1 below) to include some sets with $r(S) = 2$ and to yield a result of Fernique [*7*], [7a] for processes over the unit cube as a corollary (Theorem 7.1 below).

Conjecture 3.3 (if $r(S) > 2$ S is not a GB-set) is verified for certain random Fourier series with independent Gaussian coefficients, both those covered by a result of Kahane [*10*] and some others (Propositions 7.2 and 7.3 below).

In Section 4 we give some general results about $\bar{L}(C)$, convergence of series defining L, etc. Among other things, we establish an exact natural correspondence between GC-sets and the "measurable pseudo-norms" of L. Gross [*9*] (see Theorem 4.6 below).

Section 8 gives some brief comments on possible methods of attack in proving the conjectures.

3. Sample Continuity and ε-Entropy

Here is a sufficient condition for sample continuity in terms of ε-entropy:

Theorem 3.1. *Suppose S is a subset of a Hilbert space and*

$$\sum_{n=1}^{\infty} \frac{H(S, 1/2^n)^{1/2}}{2^n} < \infty. \tag{1}$$

Then S is a GC-*set.*

Proof. Given a positive integer n, we decompose S into $N(1/2^{n+4})$ sets of diameter at most $1/2^{n+3}$, and choose one point from each set, forming a set A_n. Let G_n be the set of all random variables $L(x - y)$ for x and y in $A_{n-1} \cup A_n$ and $\| x - y \| \leqslant 1/2^n$. Then the cardinality of G_n is at most $4N(2^{-n-4})^2$.

We shall use below the well-known estimate, for $a > 0$,

$$\int_a^\infty e^{-x^2/2}\,dx \leqslant \int_a^\infty \frac{xe^{-x^2/2}\,dx}{a} = \frac{e^{-a^2/2}}{a}.$$

Let $\{b_n\}$ be any sequence of positive real numbers. Let

$$P_n = \Pr(\max\{|(z)| : z \in G_n\} \geqslant b_n) \leqslant 4N(2^{-n-4})^2 (\exp[-4^n b_n^2/2])2^n/b_n.$$

Thus $P_n \leqslant b_n$ if $n \geqslant 2$ and $-4^n b_n^2/2 + 2H(2^{-n-4}) \leqslant 2 \log b_n$, or

$$[H(2^{-n-4}) - \log b_n]/4^{n-1} \leqslant b_n^2.$$

Let $a_n^2 = H(2^{-n-4})/4^{n-1}$. Then $\sum a_n < \infty$ by (1), and a_n is independent of b_n. But now we specify b_n, letting $b_n = \max(2a_n, 1/n^2)$. Then $a_n^2 \leqslant b_n^2/2$ and $\log b_n \geqslant -2 \log n$, so for n large enough

$$(-4 \log b_n)/4^n \leqslant 1/2n^4 \leqslant b_n^2/2,$$

and then $P_n \leqslant b_n$. Since $\sum b_n < \infty$, we have $\sum P_n < \infty$.

Thus for almost all ω there is an $n_0(\omega)$ such that $|z| < b_n$ for all $n \geqslant n_0(\omega)$ and all z in G_n.

Now let T be any countable dense subset of S. We shall show that on T, L is uniformly continuous with probability one. Its extension to S is then a version of L with continuous sample functions, as desired.

Given $\delta > 0$, we choose n_0 so that

$$3 \sum_{n=n_0}^{\infty} b_n < \delta \qquad \text{and} \qquad P(\Omega(n_0)) < \delta$$

where

$$\Omega(n_0) = \{\omega : n_0(\omega) > n_0\}$$

For any s in T, we choose points $A_n(s)$ in A_n such that $\| s - A_n(s) \| < 1/2^{n+3}$. Now if $n \geqslant n_0$, $s, t \in A$, and $\| s - t \| \leqslant 1/2^{n+3}$, then $\| A_n(s) - A_n(t) \| \leqslant 1/2^n$. Thus $L(A_n(s) - A_n(t)) \in G_n$. Also, $L(A_n(s) - A_{n+1}(s)) \in G_{n+1}$. Thus for $\omega \notin \Omega(n_0)$, $L(A_n(s))(\omega) \to L(s)(\omega)$

for all s in T, and for any t in T such that $d(s, t) \leqslant 1/2^{n_0+3}$, we have

$$|L(s)(\omega) - L(t)(\omega)| \leqslant \delta.$$

Letting $\delta \downarrow 0$, we see that L is uniformly continuous on T with probability 1. Q.E.D.

COROLLARY 3.2. *If S is a subset of a Hilbert space and $r(S) < 2$, then S is a* GC-*set.*

There are numerous examples of sets S with $r(S) = 2$ which are neither GC- nor GB-sets; see, e.g., Section 6 below. Moreover, Theorem 7.1 below and its partial converse, due to Fernique [7], indicate that, even when specialized to stochastic processes on the real line, Theorem 3.1 is essentially the best possible result of its kind.

However, we prove in Proposition 6.10(a) below that *no* sufficient condition for the GC-property of a Banach ball in terms of $H(S, \epsilon)$ is necessary, i.e., the GC-property is not a "monotone function" of the function $\epsilon \to H(S, \epsilon)$ as $\epsilon \downarrow 0$. Yet I make

Conjecture 3.3. If S is a GB-set (and hence if S is a compact GC-set), then $r(S) \leqslant 2$.

In Sections 6 and 7 below, Conjecture 3.3 is proved in a number of special cases. In the general case, I shall prove at present only the following:

PROPOSITION 3.4. *If S is a* GB-*set then S is totally bounded (i.e., its closure is compact).*

Proof. If S is a GB-set, it is certainly bounded. Suppose it is not totally bounded. Then for some $\epsilon > 0$ there is an infinite sequence $\{f_j\}_{j=1}^{\infty}$ in S such that the distance of f_{j+1} from the linear span F_j of $f_1, \ldots, f_j$ is at least ϵ for all j. Let

$$f_{n+1} = g_n + \sum_{j=1}^{n} a_{nj} f_j$$

where $\| g_n \| \geqslant \epsilon$ and $g_n \perp F_n$. Given $M > 0$, let

$$A_n = \{\omega : \max \{| L(f_j) | : 1 \leqslant j \leqslant n\} < M\}.$$

Then

$$\Pr (A_n \cap \{\omega : | L(f_{n+1}) | \geqslant M\})$$

$$\geqslant \Pr \left(A_n \text{ and } L(g_n) \geqslant M \text{ and } L\left(\sum_{j=1}^{n} a_{nj} f_j\right) \geqslant 0\right)$$

$$= \Pr (L(g_n) \geqslant M) \Pr (A_n)/2.$$

Now for some $\delta > 0$, we have, for all n,

$$\Pr(L(g_n) \geqslant M) \geqslant 2\delta; \qquad \text{so} \qquad \Pr(A_n) \leqslant (1 - \delta)^{n-1}$$

by induction. This contradicts the fact that S is a GB-set and completes the proof.

The method of proof just used will yield a stronger result. Using also (5.2) and Lemma 5.6 (cf. also Proposition 6.9), it can be shown that, if S is a GB-set, then for any $\delta > 0$

$$N(S, \epsilon) \leqslant \exp(\exp(1/\epsilon^{2+\delta}))$$

for ϵ sufficiently small. Since the examples in Section 6 indicate that this inequality has an unnecessary extra exponentiation, no further details will be given.

4. Pseudo-Norms

Let V be a real linear space and let W be a linear space of linear functionals on V. Then for any set $C \subset V$, the *polar* C^1 is defined by

$$C^1 = \{w \in W : w(x) \leqslant 1 \text{ for all } x \text{ in } C\}.$$

When C is symmetric,

$$C^1 = \{w \in W : | w(x) | \leqslant 1 \text{ for all } x \text{ in } C\}.$$

If A is a linear transformation of V into itself and W is closed under the adjoint A^* (i.e., composition with A), then for any $C \subset V$,

$$A(C)^1 = (A^*)^{-1}(C^1).$$

(Here $(A^*)^{-1}$ is a set mapping and A^* need not be invertible.) In particular V may be a Hilbert space and W its dual space, possibly identified with V.

On k-dimensional Euclidean space R^k, let λ or λ_k be Lebesgue measure and let G be the standard Gaussian probability measure;

$$dG = (2\pi)^{-k/2} \exp(-r^2/2)\, d\lambda,$$

where r is the distance from the origin.

Proposition 4.0 (Gross [9]). *Let A be a linear transformation* *from R^k into itself with norm $\| A \| \leqslant 1$ and let C be a convex symmetric set in R^k. Then*

$$G(A(C)^1) \geqslant G(C^1).$$

Proof. This follows directly from [9], Theorem 5, stated in different language. For A symmetric and invertible it is Lemma 5.2 of [9], and arguments to reduce to this case are given in the proof of Theorem 5. Q.E.D.

Now as usual, let H be a separable, infinite-dimensional Hilbert space. Every GB-set in H is included in some Banach ball which is still a GB-set.

For any subset C of H, let $s(C)$ be its linear span. Then if C is convex and symmetric, it is the unit ball of a norm $\| \cdot \|_C$ on $s(C)$. If C is a Banach ball, then $(s(C), \| \cdot \|_C)$ is a Banach space and its natural injection into H is continuous.

Let H^* be the dual space of H. (For clarity, we do not identify the two.) For each φ in H^*, let

$$\| \varphi \|_C' = \sup \{| \varphi(\psi) | : \psi \in C\}.$$

Then if C is a Banach ball in H, $\| \cdot \|_C'$ is the dual norm to $\| \cdot \|_C$ (composed with the natural map of H^* into the dual space $(s(C)', \| \cdot \|_C')$ of $(s(C), \| \cdot \|_C)$).

L is an assignment of random variables to elements of H, or equivalently to continuous linear functionals on H^*. The assignment can be extended to some nonlinear functionals in various ways. For example, if φ is a Borel measurable function on R^k and $f_1, ..., f_n \in H$, then $\varphi(f_1, ..., f_n)$ defines by composition a function on H^*. (Such a function is called "tame.") The assignment

$$L(\varphi(f_1, ..., f_n)) = \varphi(L(f_1), ..., L(f_n))$$

is well-defined, as is well known [9], [18]. Thus, e.g., we let $L(| f |) = | L(f) |$, $f \in H$.

Now in general, an assignment such as

$$L(\sup g_n) = \sup (L(g_n))$$

will not be well-defined, but if $g_n = | f_n |$, $f_n \in H = (H^*)^*$, then $\sup g_n = \| \cdot \|_C'$ where C is the closed symmetric convex hull of the f_n. Also

$$\sup L(g_n) = \sup (| L(f_n) |) = \tilde{L}(C),$$

and the assignment

$$L(\|\cdot\|_C) = \bar{L}(C)(\cdot)$$

is well-defined.

By f.d.p. (finite-dimensional projection) we shall mean an orthogonal projection of H onto a finite-dimensional subspace. For projections P and Q, one says $P \leqslant Q$ if the range of P is included in that of Q, and $P_n \uparrow I$ if $P_1 \leqslant P_2 \leqslant \cdots$ and $P_n(f) \to f$ in (Hilbert) norm for each f in H. Also $P \perp Q$ means the ranges of P and Q are orthogonal. If $\{f_n\}$ is an orthonormal basis of H, g_n are independent, normalized Gaussian random variables, and $L_n(f) = (f, f_n)\, g_n$, then the series

$$\sum_{n=1}^{\infty} L_n(\cdot) \tag{1'}$$

is a version of L. If $P_n \uparrow I$ (and the P_n are f.d.p.'s) then there is an orthonormal basis $\{f_j\}$ of H such that for each n, $\{f_1, ..., f_{k_n}\}$ is a basis of the range of P_n for some k_n, $k_n \uparrow \infty$. The convergence of $L \circ P_n$ to L is equivalent to convergence of a certain sequence of partial sums of (1′).

We shall need an infinite-dimensional form of Proposition 4.0.

PROPOSITION 4.1. *Let A be a linear operator from H into itself with $\| A \| \leqslant 1$, and let $C \subset H$. Then for any $t \geqslant 0$,*

$$\Pr(\bar{L}(AC) \leqslant t) \geqslant \Pr(\bar{L}(C) \leqslant t).$$

Proof. If C is finite, the result follows immediately from Proposition 4.0. In general, let C_n be finite sets which increase up to a dense set in C. Then

$$\Pr(\bar{L}(C) \leqslant t) = \lim_{n\to\infty} \Pr(\bar{L}(C_n) \leqslant t) \leqslant \lim_{n\to\infty} \Pr(\bar{L}(AC_n) \leqslant t)$$

$$= \Pr(\bar{L}(AC) \leqslant t), \qquad \text{Q.E.D.}$$

PROPOSITION 4.2. *If P_n are* f.d.p.'*s*, *$P_n \uparrow I$, $C \subset H$, and $t \geqslant 0$, then*

$$\Pr(\bar{L}(C) \leqslant t) = \lim_{n\to\infty} \Pr(\bar{L}(P_n C) \leqslant t).$$

Proof. Let A be countable and dense in C. Then $L(P_n f) \to L(f)$ as $n \to \infty$ for all f in A, with probability 1. Hence

$$\Pr(\bar{L}(C) \leqslant t) \leqslant \liminf_{n\to\infty} \Pr(\bar{L}(P_n C) \leqslant t).$$

On the other hand Proposition 4.1 yields

$$\limsup_{n\to\infty} \Pr(\bar{L}(P_n C) \leqslant t) \leqslant \Pr(\bar{L}(C) \leqslant t),$$

completing the proof.

The following definition is essentially that of Gross [9].

DEFINITION. A pseudo-norm $\|\cdot\|$ on H^* is *measurable* (for L) if for every $\epsilon > 0$ there is a f.d.p. P_0 such that, for every f.d.p. $P \perp P_0$,

$$\Pr(L(\|\cdot\| \circ P) > \epsilon) < \epsilon.$$

Note that $\|\cdot\| \circ P$ is a tame function on H^* so that L of it is defined. If $C \subset H$ then $\|\cdot\|_C'$ is measurable if and only if for every $\epsilon > 0$ there is a f.d.p. P_0 such that, for every f.d.p. $P \perp P_0$,

$$\Pr(\bar{L}(PC) > \epsilon) < \epsilon.$$

It then follows by Propositions 4.1 and 4.2 that

$$\Pr(\bar{L}P_0{}^{\perp}C) > \epsilon) \leqslant \epsilon.$$

(For any projection P, $P^{\perp} = I - P$ where I is the identity operator.)

THEOREM 4.3. *If C is a Banach ball in H, the following are equivalent:*

(a) *C is a* GB-*set*, i.e., $\Pr(\bar{L}C) < \infty) = 1$.

(b) $\Pr(\bar{L}(C) < \infty) > 0$

(c) [resp. (d)] $\bar{L}(P_n C)$ *converges in law for some* (resp. *every*) *sequence of* f.d.p.'*s* $P_n \uparrow I$.

(e) *L restricted to $s(C)$ has a version linear and continuous with probability 1 for $\|\cdot\|_C$.*

Proof. Let $\{f_n\}$ be an orthonormal basis of H. For each f in H, the series (1′) converges almost everywhere on Ω and in $L^2(\Omega)$. For any finite N,

$$\sum_{n=1}^{N} L_n(\cdot)(\omega)$$

is bounded on C for each ω in Ω, and finiteness of $\bar{L}(C)(\omega)$ thus depends on the g_n for $n > N$. Thus by the zero-one law ([13], B, p. 229), $\Pr(\bar{L}(C) < \infty) = 0$ or 1, and (a) is equivalent to (b).

(a) is equivalent to (c) and (d) by Proposition 4.2.

(a) is equivalent to (e) since a linear functional on a normed space is continuous if and only if it is bounded on the unit ball. The proof is complete.

Before treating GC-sets, we introduce some facts we need about function-valued random variables. Let S be a metric space with a countable dense subset $A = \{x_j\}_{j=1}^{\infty}$. Let $\mathscr{C}(S)$ be the Banach space of bounded continuous real-valued functions on S, with supremum norm $\|\cdot\|_{\infty}$. We say $X_1, X_2, \ldots$ are given as a set of $\mathscr{C}(S)$-valued random variables if probabilities

$$\Pr(X_i(t_j) \in A_{ij}, i, j = 1, 2, \ldots)$$

are defined for any points $t_1, t_2, \ldots,$ in S and Borel sets A_{ij} in the real line. Then the norms

$$\| X_i \|_{\infty} = \sup\{| X_i(t) | : t \in A\}$$

are measurable. Note, however, that $\mathscr{C}(S)$ will not be separable if S is not compact. Then, the distributions of the X_i will not be expected to be defined on all open sets in $\mathscr{C}(S)$ for the supremum norm topology (cf. [*6*]).

A random variable X in $\mathscr{C}(S)$ will be called *symmetric* if $-X$ has the same distribution as X. Independence of random variables X_i in $\mathscr{C}(S)$ is defined also, naturally, to mean that the sets of real random variables

$$A_i = \{X_i(t) : t \in S\}$$

are independent for different values of i.

Let X_i be independent and symmetric in $\mathscr{C}(S)$ and

$$S_n = X_1 + \cdots + X_n .$$

The following generalization of a Lemma of P. Lévy is proved much like the classical version (Loève [*13*], p. 247).

LEMMA 4.4. *For any* $\alpha > 0$,

$$\Pr(\max\{\| S_k \| : k = 1, \ldots, m\} > \alpha) \leqslant 2 \Pr(\| S_m \| > \alpha).$$

Proof. For each $k = 1, \ldots, m$, $j = 1, 2, \ldots,$ and $s = \pm 1$, let

$$A(k, j, s) = \{\omega : \| S_i \| \leqslant \alpha, i = 1, \ldots, k-1, | S_k(x_q) | \leqslant \alpha, \\ q = 1, \ldots, j-1, sS_k(x_j) > \alpha\}.$$

Then $\{\omega : \| S_k \| > \alpha$ for some k, $1 \leqslant k \leqslant m\}$ is the disjoint union of the $A(k, j, s)$. We also have for each k, j and s,

$$\Pr (A(k, j, s) \text{ and } \| S_m \| > \alpha) \geqslant \Pr (A(k, j, s) \text{ and } s(S_m - S_k)(x_j) \geqslant 0)$$
$$\geqslant \Pr (A(k, j, s))/2.$$

Hence

$$2 \Pr (\| S_m \| > \alpha) \geqslant \sum_{k,j,s} \Pr (A(k, j, s)) = \Pr (\max \{\| S_k \| : k = 1,..., m\} > \alpha);$$

Q.E.D.

Proposition 4.5. *The series $\sum_{n=1}^{\infty} X_n$ of independent symmetric $\mathscr{C}(S)$-valued random variable converges in $\mathscr{C}(S)$ (i.e., uniformly on S) with probability 1 if and only if it converges (uniformly) in probability.*

Proof. "Only if" is obvious. "If" is proved from Lemma 4.4 just as in the classical case where S has only one point: see [*13*], p. 249.

Theorem 4.6. *For any compact Banach ball C in H, the following are equivalent:*

(a) *for any $\epsilon > 0$, $\Pr (\bar{L}C) < \epsilon) > 0$;*

(b) *C is a* GC-*set;*

(c) [resp. (d)] *$L \circ P_n$ converges uniformly on C in probability for some* (resp. *all*) *sequences of* f.d.p.'*s $P_n \uparrow I$;*

(c′) [resp. (d′)] *replace "in probability" by "with probability 1" in* (c) [resp. (d)];

(e) *$\| \cdot \|_C$ is a measurable pseudo-norm on H^*.*

Proof. Throughout let A be a countable dense subset of C.

(a) $\Rightarrow$ (b): Let P_n be f.d.p.'s and $P_n \uparrow I$. Given $\epsilon > 0$, let

$$C_n(\epsilon) = \{\omega : \bar{L}(P_n{}^{\perp}C) < \epsilon/3\},$$
$$K(\epsilon) = \limsup C_n(\epsilon)$$
$$= \{\omega : C_n(\epsilon) \text{ holds for arbitrarily large } n\}.$$

Then $K(\epsilon)$ is a tail event, having a probability 0 or 1.

By (a) and Proposition 4.1,

$$0 < \Pr (\bar{L}(C) < \epsilon/3) \leqslant \Pr (\bar{L}P_n{}^{\perp}C < \epsilon/3)$$

for all n, where $P_n{}^{\perp} = I - P_n$. Thus $K(\epsilon)$ has positive probability, hence probability 1. Hence almost every ω belongs to $C_n(\epsilon)$ for some n. Then since $L \circ P_n$ is continuous, there is an $\alpha > 0$ such that if $x, y \in A$ and $\| x - y \| < \alpha$, then

$$| L(x) - L(y) | (\omega) \leqslant | L(P_n(x - y)) (\omega) | + 2\bar{L}P_n{}^{\perp}C < \epsilon.$$

Since ϵ was an arbitrary positive number, (b) is proved.

(b) $\Rightarrow$ (c): given $\epsilon > 0$ we use uniform continuity on C with probability 1 to infer that for some $\delta > 0$,

$$\Pr (\sup \{| L(x - y) | : x, y \in A, \| x - y \| < \delta\} \geqslant \epsilon) < \epsilon.$$

We choose a finite-dimensional subspace F such that $F \cap C$ is within δ of every point of C. Let P be the projection onto F. Then by Proposition 4.1

$$\begin{aligned} \epsilon &> \Pr (\sup \{| L(P^{\perp} (x - y)) | : x, y \in C, \| x - y \| \leqslant \delta\} \geqslant \epsilon) \\ &\geqslant \Pr (\sup \{| LP^{\perp}x | : x \in C\} \geqslant \epsilon) \end{aligned}$$

since, for any x in C, there is a y in $F \cap C$ with $\| x - y \| \leqslant \delta$ and $P^{\perp}y = 0$. Thus

$$\Pr (| (L - L \circ P)^{-} (C) | \geqslant \epsilon) < \epsilon$$

as desired, so (c) holds.

(c) $\Rightarrow$ (c′) by Proposition 4.5.

(c′) $\Rightarrow$ (d): let $L \circ Q_n \to L$ uniformly on C with probability 1 and $Q_n \uparrow I$, $P_m \uparrow I$, where P_m and Q_n are f.d.p.'s. Then given $\epsilon > 0$ there is an n such that

$$\Pr (\bar{L}(Q_n{}^{\perp}C) > \epsilon/2) < \epsilon/2.$$

Now the operator norm $\| P_m{}^{\perp}Q_n \| \to 0$ as $m \to \infty$ since Q_n has finite-dimensional range and $P_m{}^{\perp} \to 0$ pointwise. Hence $\bar{L}(P_m{}^{\perp}Q_nC) \to 0$ in probability as $m \to \infty$. Also

$$\begin{gathered} \Pr (\bar{L}(P_m{}^{\perp}Q_n{}^{\perp}C) \leqslant \epsilon/2) \geqslant \Pr (\bar{L}(Q_n{}^{\perp}C) \leqslant \epsilon/2) \geqslant 1 - \epsilon/2, \\ \bar{L}(P_m{}^{\perp}C) \leqslant \bar{L}(P_m{}^{\perp}Q_nC) + \bar{L}(P_m{}^{\perp}Q_n{}^{\perp}C); \end{gathered}$$

so, for m large enough,

$$\Pr (\bar{L}(P_m{}^{\perp}C) \geqslant \epsilon) \leqslant \epsilon$$

and (d) holds.

(d) $\Rightarrow$ (d′) by Proposition 4.5.

(d′) $\Rightarrow$ (e): clearly (d′) $\Rightarrow$ (c), and (c) $\Rightarrow$ (e) by Proposition 4.1.

(e) $\Rightarrow$ (a): given $\epsilon > 0$, we choose a f.d.p. P such that

$$\Pr(\underline{L}(P^{\perp}C) < \epsilon/2) > 0.$$

Then also

$$\Pr(\underline{L}(PC) < \epsilon/2) > 0$$

and since $\underline{L}(P^{\perp}C)$ and $\underline{L}(PC)$ are independent, we have

$$\Pr(\underline{L}(C) < \epsilon) \geqslant \Pr(\underline{L}(PC) < \epsilon/2 \text{ and } \underline{L}(P^{\perp}C) < \epsilon/2) > 0.$$

Q.E.D.

Not every GB-set is a GC-set, as we shall see below (Propositions 6.7 and 6.9). Thus all possible implications among the conditions listed in Theorems 4.3 and 4.6 are settled. However, these conditions suggest others, e.g., replacing "in law" in (c) and (d) of Theorem 4.3 by "in probability" or "with probability one". If $P_nC \subset C$ for all n, then $\underline{L}(P_nC)$ is nondecreasing, so the different forms of (c) are equivalent in this case. In Section 6 we present a GB-set (octahedron with axes $(\log n)^{-1/2}$), which is not a GC-set, and for which $P_nC \rightarrow C$ for certain natural projections $P_n \uparrow I$. Thus the stronger forms of (c) do not imply that C is a GC-set, but other possible implications are not settled.

We shall conclude this section with a result showing that the class of GB-sets which are not GC-sets is quite narrow.

If B and C are Banach balls in H, we shall say B is *C-compact* if $B \subset s(C)$ and B is compact for $\|\cdot\|_C$. If B is a GB-set, we call it *maximal* if whenever B is C-compact, C is not a GB-set. (No GB-set A is maximal in a strict set-theoretic sense since $2A$ includes A strictly and $2A$ is a GB-set.)

THEOREM 4.7. *Every* GB-*set is either maximal or a* GC-*set.*

Proof. Suppose B is C-compact where C is a Banach ball and a GB-set. Then L restricted to $s(C)$ has a version which is linear and continuous for $\|\cdot\|_C$. The $\|\cdot\|_C$ topology is stronger than the original Hilbert topology on $s(C)$ since C is bounded, hence these two topologies are equal on the compact set B ([*11*], Theorem 8, p. 141). Thus B is a GB-set. Q.E.D.

If $\|\cdot\|$ is a measurable pseudo-norm on H^*, then L is defined by a countably additive probability measure on the completion of H^* for $\|\cdot\|$.[1] At the moment the converse seems to be an open question.

Suppose $(x_t, t \in S)$ is a sample-continuous Gaussian process over a compact metric space (S, d). Then $t \to x_t$ is continuous from S into H, and

$$e(s, t) = (E(x_s - x_t)^2)^{1/2}$$

defines a pseudo-metric e on S which is continuous for d and hence defines a weaker topology. If (S, e) is Hausdorff, i.e., if $x_s \neq x_t$ for $s \neq t$, then the d and e topologies on S are equal, and hence the range of the process in H is a GC-set; its closed convex symmetric hull is a GB-set, which, by Theorem 4.7, is not much worse.

5. Sequences of Volumes

We shall need the volumes of certain simple sets in R^k. First, suppose A is a simplex, i.e., a convex hull of $(k + 1)$ points $x_0, \ldots, x_k$, having an interior. Let F be a *face* of A, i.e., a convex hull of k of its vertices. Let λ or λ_k be Lebesgue measure on R^k and S or S_{k-1} the $(k - 1)$-dimensional Lebesgue "surface" or "area" measure. Then

$$\lambda(A) = S(F)\, d/k,$$

where d is the distance from the vertex not in F to the hyperplane through F. Now suppose $x_0 = 0$ and let d_j be the distance from x_j to the linear span of $x_0, \ldots, x_{j-1}$, $j = 1, \ldots, k$. Then

$$\lambda(A) = \Big(\prod_{j=1}^{k} d_j\Big)\Big/k!.$$

Now, recalling the definitions of $V_n(C)$ and $EV(C)$ given is Section 1, we have the following fact (a stronger statement is given as Proposition 5.10 below):

Lemma 5.0. *If C is a convex set in H and $EV(C) < -1$, then C is totally bounded.*

[1] For the proof, see L. Gross, Abstract Wiener spaces, in *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability (1964)*. University of California Press, Berkeley, 1967.

Proof. If C is not totally bounded we make the same construction as in the proof of Proposition 2.4. Then for some $\epsilon > 0$, $V_n(C)$ is greater than or equal to the volume of the convex hull of $0, f_1, \ldots, f_n$, so

$$V_n(C) \geqslant \epsilon^n/n! \qquad \text{for all} \qquad n.$$

By Stirling's formula, this contradicts the hypothesis. Q.E.D.

Next, let $c_k = \lambda_k(B)$ where B is a ball of radius 1 in R^k. Then it can be shown by induction that, for any positive integer k,

$$c_{2k+1} = 2^{2k+1}\pi^k k!/(2k+1)!,$$
$$c_{2k} = \pi^k/k!.$$

Thus by Stirling's formula we have the following estimate:

$$\lim_{j\to\infty} c_j(\pi j)^{1/2}\,(j/2\pi e)^{j/2} = 1. \tag{5.1}$$

We shall also need the following fact. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n \downarrow 0$ as $n \to \infty$. For such a sequence and $\epsilon > 0$ we define

$$n(\epsilon) = n(\{a_n\}, \epsilon) = \max\,(n : a_n \geqslant \epsilon),$$

$$\lambda = \lambda(\{a_n\}) = \inf\left(\alpha : \sum_{n=1}^{\infty} a_n{}^{\alpha} < \infty\right).$$

Then it is known, and easy to prove, that

$$\lambda = \limsup_{\epsilon\downarrow 0} \log n(\epsilon)/\log(1/\epsilon). \tag{5.2}$$

Now let C be a convex symmetric set in H.

THEOREM 5.3. *If C is a* GB-*set, then*

$$\sup_n\,[n^{-1}\log V_n + \log n] < \infty.$$

Hence $EV(C) \leqslant -1$.

Proof. Since C is a GB-set there is an $M > 0$ such that

$$\Pr(\underline{L}(C) \leqslant M) > \gamma > 0.$$

C may be replaced in the above inequality by any orthogonal projection $P(C)$, according to Proposition 4.1. Multiplying C by a positive number leaves the relevant properties unchanged, so we may assume $M = 1$. Suppose the first conclusion is false. Then for any $K > 0$ there is an n such that $V_n \geqslant (K/n)^n$.

Let P_n be a projection with n-dimensional range F. Then

$$\gamma \leqslant \Pr(\bar{L}(P_n C) \leqslant 1) = G((P_n C)^1),$$

where the polar is taken in the dual of F and G is normalized Gaussian probability measure. We use the general inequality

$$\lambda_n(B)\, \lambda_n(B^1) \leqslant c_n{}^2$$

where B is any convex symmetric set in R^n (due to Santalò [17]). (Later work by Bambah [1] on a lower bound for $\lambda_n(B)\, \lambda_n(B^1)$ may also be noted.) For any $\beta > 0$ there is a P_n such that

$$\lambda_n(P_n C) \geqslant (n\beta)^{-n}, \qquad \text{so} \qquad \lambda_n((P_n C)^1) \leqslant c_n{}^2(n\beta)^n.$$

Using (5.1) we obtain for any $\alpha > 0$

$$\lambda_n((P_n C)^1) \leqslant c_n(\alpha n)^{n/2}$$

for certain arbitrarily large n. Now, given $\lambda_n(A)$ for a set A, $G(A)$ is clearly maximized when A is a ball $E(r)$ centered at 0, say of radius r. Hence

$$G((P_n C)^1) \leqslant G(E(r_n)),$$

where $r_n \leqslant (\alpha n)^{1/2}$. Then

$$G(E(r_n)) \leqslant \int_0^{(\alpha n)^{1/2}} r^{n-1} e^{-r^2/2}\, dr/I_n\,,$$

where

$$I_n = \int_0^\infty r^{n-1} e^{-r^2/2}\, dr.$$

The integrand increases for $0 < r < (n-1)^{1/2}$. But $(\alpha n/(n-1))^{1/2} \to 0$ as $n \to \infty$ and $\alpha \downarrow 0$, so $G(E(r_n)) \to 0$ as $n \to \infty$ through a suitable sequence, contradicting the fact that $G((P_n C)^1) \geqslant \gamma > 0$.

Thus for some $M > 0$,

$$\frac{\log V_n}{n \log n} \leqslant -1 + \frac{M}{\log n}$$

for all n, so $EV(C) \leqslant -1$. Q.E.D.

Conjecture. 5.4. If C is a Banach ball and $EV(C) < -1$, then C is a GC-set.

The above conjecture may be made plausible by a supporting conjecture (5.9 below) and proofs of both conjectures in four classes of special cases (Section 6). In the general case, I can prove the following.

PROPOSITION 5.5. *If C is a Banach ball and $EV(C) < -\frac{3}{2}$, then C is a* GC-*set.*

Before proving Proposition 5.5 we introduce another construction and some other facts. Given a compact Banach ball C in H and an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of H, let F_n be the linear span of $\varphi_1, \ldots, \varphi_n$, and

$$C_n = C \cap F_n \qquad (C_0 = F_0 = \{0\}).$$

Given two sets A and B in H we define their distance as usual,

$$e(A, B) = \sup_{x \in A} \inf_{y \in B} \| x - y \|,$$
$$d(A, B) = e(A, B) + e(B, A).$$

We shall say the basis $\{\varphi_j\}$ is *adapted* to C if

$$d(C_{n-1}, C) = d(C_{n-1}, C_n)$$

for $n = 1, 2, \ldots$. Since C is compact, a basis adapted to C always exists. Then the sequence $\{F_n\}$ of subspaces will also be called *adapted* to C.

If there is a sequence $G_0 \subset G_1 \subset \cdots$ of subspaces of H with each G_n n-dimensional and $d(C \cap G_n, C) \leqslant a_n$ for all n, $a_n \downarrow 0$, then the sequence $\{a_n\}$ will be called *adapted* to C (whether or not the G_n are).

In order to find an upper bound for ϵ-entropies of sets with a given adapted sequence $\{a_n\}$ we use the following result.

LEMMA 5.6. *Let $B(\{a_j\}_{j=1}^{n})$ be a rectangular n-dimensional block of sides $2a_j$, $0 < a_j \leqslant 1$,*

$$B \equiv B(\{a_j\}) \equiv \left\{\sum_{i=1}^{n} x_i \varphi_i : | x_i | \leqslant a_i , i = 1, \ldots, n\right\},$$

where the φ_i are orthonormal. Let $0 < \epsilon < 1$. Then

$$N(B, \epsilon) \leqslant \prod_{j=1}^{n} (2 + n^{1/2} a_j / \epsilon) \leqslant 3^n n^{n/2} / \epsilon^n.$$

Proof. We consider the cubes of side $2\epsilon/n^{1/2}$ whose vertices are of the form

$$\sum_{j=1}^{n} 2m_j\epsilon/n^{1/2}, \qquad |m_j| \leqslant 1 + n^{1/2}a_j/2\epsilon,$$

and the m_j are integers. B is included in the union of these cubes, their diameters are 2ϵ, and the number of them is bounded as indicated. Q.E.D.

The latter, cruder estimate in the above Lemma is sufficient for its applications below except for one rather delicate one (Proposition 6.10).

PROPOSITION 5.7. *Let C be a compact Banach ball in H and $\{a_n\}$ adapted to C. Then*

$$r(C) \leqslant \lambda(\{a_n\}).$$

Proof. Let $s = \lambda(\{a_n\})$ and let $F_0 \subset F_1 \subset F_2 \cdots$ be subspaces of H, F_n n-dimensional, such that for all n, $d(C_n, C) \leqslant a_n$ where $C_n = C \cap F_n$.

If $\beta > \alpha > s$ then for small enough $\epsilon > 0$,

$$n(\{a_n\}, \epsilon/2) \leqslant 1/\epsilon^\alpha$$

by (5.2). For such an $\epsilon < 1$ and $n = n(\epsilon/2)$,

$$N(C, \epsilon) \leqslant N(C_n, \epsilon/2).$$

Since r and s are homothetically invariant we can assume $a_1 \leqslant 1$. Clearly C_n is included in the block $B(\{a_j\}_{j=1}^n)$ of Lemma 5.6, so for ϵ small enough

$$N(C, \epsilon) \leqslant \exp(n(\log 3 + \tfrac{1}{2}\log n + \log(1/\epsilon))) \leqslant \exp(\epsilon^{-\beta}).$$

Thus $r(C) \leqslant \beta$ for all $\beta > s$ and $r(C) \leqslant s$. Q.E.D.

Proof of Proposition 5.5. By Lemma 5.0, C is compact. There is a $c > \frac{3}{2}$ such that $V_n(C) \leqslant n^{-nc}$ for n large enough. We choose a basis $\{\varphi_n\}$ adapted to C and v_n in C_{n+1} such that $e(v_n, C_n) = a_n = d(C, C_n)$, $n = 0, 1, \ldots$. Then C includes the symmetric convex hull of the v_n, so

$$V_n \geqslant \Big(\prod_{j=1}^{n} 2a_j\Big)\Big/ n!$$

Then by Stirling's formula there is a $\beta > \frac{1}{2}$ such that

$$a_n{}^n < n^{n-nc} = n^{-\beta n}$$

for n large enough, and $a_n \leqslant n^{-\beta}$. Thus 5.7 and 3.2 imply that C is a GC-set. Q.E.D.

Suppose given a Banach ball (= convex symmetric bounded closed set) C in H. Suppose also that $\{F_n\}$ is a sequence of subspaces adapted to C. Given $F_1, \ldots, F_{n-1}$, we assume F_n can be and is chosen among its possible values so as to minimize $\lambda_n(F_n \cap C)$. Then we define

$$W_n = \lambda_n(F_n \cap C),$$

$$EW(C) = \limsup_{n\to\infty} (\log W_n)/(n \log n).$$

For a sufficiently "smooth" set C, e.g., an ellipsoid, we shall have $EV(C) = EW(C)$ and even $V_n \equiv W_n$ (see Proposition 6.1 below). At the end of Section 6 we show that $EW(C) < EV(C)$ is possible.

Next we obtain a lower bound for $r(C)$ in terms of $EV(C)$. In each of the four classes of examples treated in Section 6, it becomes an equality at least for $EV(C) < -1$.

PROPOSITION 5.8. *For any convex symmetric set C in H,*

(a) $r(C) \geqslant -2/(2EV(C) + 1)$ *if* $EV(C) < -\frac{1}{2}$

(b) $r(C) = +\infty$ *if* $EV(C) \geqslant -\frac{1}{2}$.

Proof. If C is covered by m sets, each of diameter at most ϵ, then any n-dimensional projection P_nC is covered by m balls of radius ϵ, and

$$mc_n\epsilon^n \geqslant V_n, \qquad \text{so} \qquad N(C, \epsilon/2) \geqslant V_n/c_n\epsilon^n$$

for all n. Let $EV(C) = -b > -c$ and $c > \frac{1}{2}$. Then for n large enough

$$V_n/c_n\epsilon^n \geqslant n^{n/2}(\pi n)^{1/2}/[(2\pi e)^{n/2}\,\epsilon^n n^{nc}] = k_n,$$

say. The following paragraph gives motivation only.

To maximize k_n, we note that

$$k_{n+1}/k_n = ((n+1)/n)^{n[(1/2)-c]}\,(n+1)^{(1-c)}/\epsilon(2\pi e n)^{1/2},$$

which is asymptotic as $n \to \infty$ to

$$e^{-c}n^{(1/2)-c}/\epsilon(2\pi)^{1/2}.$$

At any rate, as $\epsilon \downarrow 0$ we choose $m = m(\epsilon)$ so that $m^{(1/2)-c}$ is asymptotic to $e^c\epsilon(2\pi)^{1/2}$, as is clearly possible. Then for any $\delta > 0$, and ϵ small enough,

$$\begin{aligned} k_m &= (m^{(1/2)-c}/\epsilon(2\pi e)^{1/2})^m (\pi m)^{1/2} \\ &\geqslant \exp(m(c - \tfrac{1}{2} - \delta)) \\ &\geqslant \exp((c - \tfrac{1}{2} - \delta)[(1 - \delta)/e^c\epsilon(2\pi)^{1/2}]^{2/(2c-1)}. \end{aligned}$$

Hence for some constant $\gamma > 0$,

$$N(C, \epsilon) \geqslant \exp(\gamma\epsilon^{2/(1-2c)})$$

for ϵ small enough. If $-b < -\frac{1}{2}$ we let c approach b and obtain (a). If $-b \geqslant -\frac{1}{2}$, we let c approach $\frac{1}{2}$ and obtain (b). Q.E.D.

Definition. A Banach ball C is *volumetric* if $EV(C) < -\frac{1}{2}$ and

$$r(C) = -2/(2EV(C) + 1).$$

Conjecture. 5.9. If C is a Banach ball and $EV(C) < -1$, then C is volumetric (hence $r(C) < 2$ and C is a GC-set).

A weaker inequality in the direction converse to 5.8 (a) is easily proved. Let $\{F_n\}$ be adapted to C and $T_n = \lambda_n(F_n \cap C)$. If $T_n \leqslant n^{-n(1+\delta)}$ for n large enough, $\delta > 0$, then since $a_1 \cdots a_n/n! \leqslant T_n$, we have $a_n \leqslant n^{-\delta}$ for n large enough; hence, by Proposition 5.7, $r(C) \leqslant 1/\delta$. Thus:

Proposition 5.10. *If* $\beta = EV(C)$ *or* $\beta = EW(C)$, $\beta < -1$, *then*

$$r(C) \leqslant -1/(\beta + 1).$$

Suppose given a compact Banach ball C in H for which $EW(C)$ is defined and equals

$$\lim_{n\to\infty} (\log W_n)/(n \log n)$$

(not just lim sup). Let $\{F_n\}$ and $\{a_n\}$ be adapted sequences of subspaces and numbers, respectively, and $\{\varphi_n\}$ an adapted orthonormal basis. Let A be the linear transformation such that

$$A(\varphi_n) = b_n\varphi_n \qquad \text{for all } n,\ b_n \downarrow 0.$$

Then the F_n and φ_n are adapted to $A(C)$, F_n now being uniquely determined, and

$$\begin{aligned} a_n(A(C)) &= b_n a_n, \\ W_n(A(C)) &= b_1 \cdots b_n W_n(C). \end{aligned}$$

Thus

$$EW(A(C)) = EW(C) + ew(\{b_j\}) \tag{5.11}$$

where

$$ew(\{b_j\}) = \limsup_{n\to\infty} \left(\sum_{j=1}^{n} \log b_j\right) \Big/ n \log n.$$

Thus the following is useful.

PROPOSITION 5.12. *If* $b_j \downarrow 0$,

$$ew(\{b_j\}) = -1/\lambda(\{b_j\}).$$

Proof. Given $\delta > 0$, we have by (5.2):

$$n(\epsilon) = n(\{b_j\}, \epsilon) \leqslant 1/\epsilon^{\lambda+\delta}$$

for ϵ small enough, and $n(\epsilon) \geqslant 1/\epsilon^{\lambda-\delta}$ for arbitrarily small $\epsilon > 0$. Now if $n = n(\epsilon)$,

$$\left(\sum_{j=1}^{n} \log b_j\right) \Big/ n \log n \geqslant (\log \epsilon)/(\log n).$$

When $n \geqslant 1/\epsilon^{\lambda-\delta}$ and $0 < \epsilon < 1$,

$$\log n \geqslant (\lambda - \delta) \log (1/\epsilon) \qquad \text{and} \qquad (\log \epsilon)/\log n \geqslant -1/(\lambda - \delta).$$

Thus letting $\delta \downarrow 0$ we have

$$ew(\{b_j\}) \geqslant -1/\lambda.$$

For the converse inequality, we can assume $b_1 < 1$. For any positive integer m let $\epsilon = \epsilon(m)$ satisfy $m = \epsilon^{-\lambda-2\delta}$. Then as $m \to \infty$, $\epsilon \downarrow 0$. Since

$$n(\epsilon) \leqslant 1/\epsilon^{\lambda+\delta} < 1/\epsilon^{\lambda+2\delta}$$

for ϵ small enough,

$$\begin{aligned}\left(\sum_{j=1}^{m} \log b_j\right) \Big/ m \log m &< (m - n(\epsilon(m)))\,(\log \epsilon)\, \epsilon^{\lambda+2\delta}/(\lambda + 2\delta) \log (1/\epsilon) \\ &\leqslant (1 - \epsilon^{\delta})\,(\log \epsilon)/(\lambda + 2\delta) \log (1/\epsilon) \\ &= (-1 + \epsilon^{\delta})/(\lambda + 2\delta) \to -1/(\lambda + 2\delta),\end{aligned}$$

where $\epsilon = \epsilon(m)$, $m \to \infty$. Thus, letting $\delta \downarrow 0$, we have

$$ew(\{b_j\}) \leqslant -1/\lambda.$$

Q.E.D.

6. Simple Subsets of Hilbert Space

In this section we study symmetric rectangular solids, ellipsoids, and "octahedra" and determine when they are GC- and GB-sets. We also study certain "full approximation sets" (see [*14*]), which are maximal sets with a given adapted sequence $\{a_n\}$, while octahedra are (among the) minimal sets.

For each class we shall have sequences $\{b_n\}$ of real numbers, $b_n \downarrow 0$, related to an orthonormal set $\{\varphi_n\}$ in H, usually complete. Let F_n be the subspace spanned by $\varphi_1, \ldots, \varphi_n$. For any orthonormal set $\{\varphi_n\}$ and any $b_n \geqslant 0$ we define the ellipsoid

$$E = E(\{b_n\}) = E(\{b_n\}, \{\varphi_n\}) = \left\{ \sum_{b_n > 0} x_n \varphi_n : \sum_{b_n > 0} x_n^2 / b_n^2 \leqslant 1 \right\}.$$

Clearly, E is compact if and only if the b_n for $b_n > 0$ can be arranged into a sequence $b_n \downarrow 0$. Then the $\{\varphi_n\}$, $\{F_n\}$ and $\{b_n\}$ are adapted to E. (The F_n are uniquely determined unless some positive b_n are equal, and the b_n are unique.)

More abstractly, we can define a compact ellipsoid as an image $A(B_1)$ of the unit ball $B_1 = \{x : \| x \| \leqslant 1\}$ in H under a compact operator A.[2]

It follows that if E is a compact ellipsoid and S is a bounded linear transformation from H into itself, then $S(E)$ is a compact ellipsoid.

Lemma 6.0. *If* $E = E(\{b_n\}, \{\varphi_n\})$ *is a compact ellipsoid and* P *is a* f.d.p.,

$$P(E) = E(\{\beta_n\}, \{\psi_n\}), \qquad b_n \downarrow 0, \qquad \beta_n \downarrow 0,$$

then $\beta_n \leqslant b_n$ *for all* n.

Proof. We may assume the $\{\varphi_n\}$ are complete. Given n let G_n be the linear span of $\psi_1, \ldots, \psi_n$. G_n has at least one-dimensional intersection with the set of vectors u orthogonal to $P(\varphi_j)$, $j = 1, \ldots, n-1$. If also $u \in P(E)$ then $\| u \| \leqslant \| Pv \|$ for some $v \in E(\{b_j\}, \{\varphi_j\}_{j \geqslant n})$, so $\| u \| \leqslant b_n$ and hence $\beta_n \leqslant b_n$. Q.E.D.

Now we find the exponents of volume of ellipsoids.

Proposition 6.1.

$$EV(E) = EW(E) = -\frac{1}{2} - \frac{1}{\lambda(\{b_n\})}.$$

[2] For the equivalence of the definitions, see R. T. Prosser. The ϵ-entropy and ϵ-capacity of certain time-varying channels. *J. Math. Anal. Appl.* **16** (1966), 553–573.

Proof. Lemma 6.0 implies that

$$V_n = W_n = c_n b_1 b_2 \cdots b_n \qquad \text{for all } n.$$

Let B be the unit ball $E(\{1\})$ in H. Then

$$V_n(B) \equiv W_n(B) \equiv c_n .$$

Using (5.11) and (5.12) the proof is complete.

PROPOSITION 6.2. *For any compact ellipsoid $E = E(\{b_n\})$,*

$$r(E) = \lambda(\{b_n\}).$$

Thus if $EV(E) < -\frac{1}{2}$, E is volumetric.

Proof.[3] We have $r \geqslant \lambda$ by Propositions 5.8 and 6.1, and $r \leqslant \lambda$ by Proposition 5.7. The second conclusion follows then from 6.1 and the definition of "volumetric" (just before Conjecture 5.9).

PROPOSITION 6.3. *The following are equivalent:*

(a) *$E = E(\{b_n\})$ is a* GC-*set*

(b) *E is a* GB-*set*

(c) *$\sum_{n=1}^{\infty} b_n^2 < \infty$ (E is a "Schmidt ellipsoid").*

Proof. (a) implies (b) clearly if E is compact; if not, both fail. If (b) holds, and A is the linear operator such that $A(\varphi_n) = b_n \varphi_n$, $L \circ A$ has a version continuous on H (Theorem 4.3(e) above). It is known that this is true if and only if A is a Hilbert-Schmidt operator (see [8], Lemma 4, p. 344). Thus (b) and (c) are equivalent. Next, assume (c). Then for some $k_n \uparrow \infty$, $\sum k_n^2 b_n^2 < \infty$. Let $E^1 = E\{k_n b_n\})$. Then E is E^1-compact and not maximal, so by Theorem 4.7, E is a GC-set. Q.E.D.

It follows immediately from the above results that Conjectures 3.3, 5.4, and 5.9 all hold for ellipsoids.

Now we turn to our second class of examples. Let $\{F_n\}$ be an increasing sequence of subspaces of H with F_n n-dimensional, $n = 0, 1, 2, \ldots$. Let $b_n \downarrow 0$. Specializing [14], we define the *full approximation set* $A = A(\{b_n\})$ as

$$\{x : \text{for all } n, \| x - y_n \| \leqslant b_n \text{ for some } y_n \text{ in } F_n\}.$$

[3] $r = \lambda$ is also proved by Prosser; see *op. cit.* in previous footnote.

It is easy to see that $A \supset E(\{b_n\})$. Also we can choose y_n in $A \cap F_n \equiv A_n$. Hence $\{b_n\}$ is adapted to A and A is simply a maximal set having $\{b_n\}$ as an adapted sequence.

PROPOSITION 6.4. *$r(A) = \lambda(\{b_n\})$. If $EV(A) < -\frac{1}{2}$ then A is volumetric.*

Proof. Since $A \supset E(\{b_n\})$, we have $r(A) \geqslant r(E) = \lambda(\{b_n\})$ by Proposition 6.2. $r \leqslant \lambda$ by Proposition 5.7, so $r = \lambda$.

If $EV(A) = -\frac{1}{2} - \delta$, $\delta > 0$, then $EV(E(\{b_n\})) \leqslant -\frac{1}{2} - \delta$ so

$$\begin{aligned} r(A) &= \lambda(\{b_n\}) = r(E) = -2/(1 + 2EV(E)) \leqslant 1/\delta \\ &= 1/(-\tfrac{1}{2} - EV(A)) = -2/(1 + 2EV(A)). \end{aligned}$$

The converse inequality holds by Proposition 5.8 (a), so A is volumetric. Q.E.D.

Note that the ellipsoid E with same parameters $\{b_n\}$, included in A, also has the same exponent of entropy and the same exponent of volume if that of either is less than $-\frac{1}{2}$. We have proved Conjectures 5.9 and (hence) 5.4 for A. Conjecture 3.3 also holds since if $r(A) > 2$ then $r(E) > 2$ and 3.3 holds for ellipsoids.

The condition $\sum b_n{}^2 < \infty$ is clearly necessary for $A(\{b_n\})$ to be a GB-set but I don't know whether it is sufficient for A to be a GC-set or GB-set.

Next we consider the rectangular solid or "block"

$$B = B(\{b_n\}) = \left\{\sum_{n=1}^{\infty} x_n\varphi_n : |\, x_n \,| \leqslant b_n\, , n = 1, 2,\ldots\right\}.$$

We assume as usual $b_n \downarrow 0$ and, to assure $B \subset H$, $\sum b_n{}^2 < \infty$. (Since $B(\{b_n\}) \supset E(\{b_n\})$, no GB-sets are lost here.) For blocks we shall not find adapted subspaces, but we shall characterize GC-blocks and GB-blocks and verify the three conjectures.

PROPOSITION 6.5. *If $\lambda = \lambda(\{b_n\})$, $t = EV(B(\{b_n\}))$, and $r = r(B)$, then $t = -1/\lambda = -\frac{1}{2} - 1/r$ if any of these terms is less than $-\frac{1}{2}$ (i.e., if $t < -\frac{1}{2}$, $\lambda < 2$ or $r < \infty$). Thus under these conditions B is volumetric.*

Proof. Given $\delta > 0$, we have for n large enough

$$b_n \leqslant V_n^{1/n} < n^{t+\delta},$$

so $\lambda \leqslant -1/t$ if $t < 0$. Thus by 5.8 (b), any of our hypotheses implies $\lambda < 2$.

Then by 5.2 we have for n large enough

$$b_n \leqslant n^{-1/(\lambda+\delta)}$$

so for $\delta < 2 - \lambda$ we have for k large

$$\left(\sum_{n=k}^{\infty} b_n^2\right)^{1/2} \leqslant k^{1/2-1/(\lambda+\delta)}$$

so letting $\delta \downarrow 0$ we have by Proposition 5.7

$$r \leqslant 1/(-\tfrac{1}{2} + 1/\lambda) < \infty$$

so by 5.8 $t < -\frac{1}{2}$ and $r \geqslant -2/(1+2t)$, so

$$t \leqslant -\frac{1}{2} - \frac{1}{r} \leqslant -\frac{1}{\lambda} \leqslant t. \qquad \text{Q.E.D.}$$

Thus Conjectures 5.4 and 5.9 hold for blocks.

PROPOSITION 6.6. *The following are equivalent:*

(a) $\sum b_n \mid L(\varphi_n) \mid$ *converges with probability 1;*

(b) $\sum b_n < \infty$*;*

(c) $B = B(\{b_n\})$ *is included in some* GC-*ellipsoid;*

(d) B *is a* GC-*set;*

(e) B *is a* GB-*set.*

Proof. (a) implies (b) by an application of the three-series theorem ([*13*], p. 237).

If $\sum b_n < \infty$, we let

$$a_n = \left(b_n \sum_{j=1}^{\infty} b_j\right)^{1/2}.$$

Then $E(\{a_n\})$ is a GC-ellipsoid by 6.3, and $B \subset E$, so (b) implies (c). Clearly (c) implies (d) which implies (e).

If B is a GB-set, then for almost every ω, there is an $M < \infty$ such that

$$\sum_{j=1}^{\infty} s_j b_j L(\varphi_j)(\omega) \leqslant M$$

for all possible choices of $s_j = \pm 1$. Hence (a) holds, and the proof is complete.

Now if a block B is a GB-set, then $r(B) \leqslant 2$ by (c) so Conjecture 3.3 holds for blocks.

If $r(B) < 2$ and E is the ellipsoid of (c), then it is easily shown that $r(B) < r(E) < 2$.

Next we discuss some other classes of subsets of H: orthogonal sets $S(\{b_n\})$ and their closed symmetric convex hulls, octahedra $\mathrm{Oc}\,(\{b_n\})$. These sets refute a number of conjectures which up to now might have seemed plausible (cf. Propositions 6.7, 6.9, 6.10, and the remarks between and after them) while satisfying Conjectures 3.3, 5.4, and 5.9.

Given $b_n \downarrow 0$ and $\{\varphi_n\}$ an orthonormal basis let

$$S = S(\{b_n\}) = \{0\} \cup \{b_n \varphi_n\}_{n=1}^{\infty},$$

$$\mathrm{Oc} = \mathrm{Oc}\,(\{b_n\}) = \text{symmetric closed convex hull of } S$$

$$= \left\{ \sum_{n=1}^{\infty} b_n x_n \varphi_n : \sum_{n=1}^{\infty} |\, x_n \,| \leqslant 1 \right\}.$$

In this case, as for ellipsoids but not blocks, the $\{\varphi_n\}$ and $\{b_n\}$ are adapted to $\mathrm{Oc}\,(\{b_n\})$. It is easy to see that

$$N\,(\mathrm{Oc}, \epsilon) \geqslant N(S, \epsilon) = n(\{b_n\}, \sqrt{2}\epsilon) \pm \tfrac{1}{2} + \tfrac{1}{2}$$

for all $\epsilon > 0$ such that $b_n = \epsilon$ for at most one value of n.

PROPOSITION 6.7. *The following are equivalent:*

(a) $\mathrm{Oc}\,(\{b_n\})$ *is a* GC-*set;*

(b) $S(\{b_n\})$ *is a* GC-*set;*

(c) $b_n = o\,(\log n)^{-1/2}$.

Proof. Clearly (a) $\Rightarrow$ (b). To prove the converse, note that (b) is equivalent to $b_n L(\varphi_n) \to 0$ as $n \to \infty$ with probability one. Given $\epsilon > 0$, for almost all ω there is an N such that for all $n > N$,

$$|\, b_n L(\varphi_n)\,(\omega) \,| < \epsilon/4,$$

and there is a $\delta > 0$ such that whenever $x, y \in \mathrm{Oc}\,(\{b_n\})$ and $\|\, x - y \,\| < \delta$,

$$\left| L\left(\sum_{n=1}^{N} (x_n - y_n)\, \varphi_n \right) \right| < \epsilon/2,$$

and we infer (a).

Now (b) is equivalent by the zero-one law ([13] A, p. 228) to the following: for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \Pr\left(b_n \,|\, L(\varphi_n) \,|\, \geqslant \epsilon\right) < \infty,$$

or

$$\sum_{n=1}^{\infty} \int_{\epsilon/b_n}^{\infty} e^{-x^2/2}\, dx < \infty.$$

As is well known, an integration by parts shows that as $M \to \infty$,

$$\int_{M}^{\infty} e^{-x^2/2}\, dx \qquad \text{is asymptotic to} \qquad e^{-M^2/2}/M.$$

Thus (b) is equivalent to

$$\sum_{n=1}^{\infty} b_n \exp\left(-\,\epsilon^2/2b_n^{\,2}\right) < \infty.$$

Letting $b_n = \alpha_n (\log n)^{-1/2}$, $n \geqslant 2$, we obtain the series

$$\sum_{n=2}^{\infty} \alpha_n (\log n)^{-1/2}\, n^{-\epsilon^2/2\alpha_n^{\,2}}. \tag{6.8}$$

If (c) holds, i.e., $\alpha_n \to 0$, then the terms of (6.8) become less than n^{-2} for n large, so (b) holds.

Conversely suppose (c) is false, so that for some $\delta > 0$, $\alpha_n \geqslant \delta$ for arbitrarily large values of n. For such an n and $n^{1/2} \leqslant j \leqslant n$, we have

$$\begin{aligned} \alpha_j &= b_j (\log j)^{1/2} \geqslant b_n (\log j)^{1/2} \\ &= \alpha_n (\log j/\log n)^{1/2} \geqslant \alpha_n/2. \end{aligned}$$

Letting $\epsilon = \delta/2$ we then have

$$\begin{aligned} &\epsilon^2/2\alpha_j^{\,2} \leqslant 4\epsilon^2/2\alpha_n^{\,2} \leqslant \tfrac{1}{2}, \\ \sum_{n^{1/2} \leqslant k \leqslant n} b_j\, j^{-\epsilon^2/2\alpha_j^{\,2}} &\geqslant \sum_{n^{1/2} \leqslant j \leqslant n} \delta (j \log n)^{-1/2}/2 \\ &\geqslant \delta (n - n^{1/2} - 1)/2n^{1/2} \log n \to \infty \end{aligned}$$

as $n \to \infty$ (recall that δ is independent of n). Thus (6.8) diverges and (b) fails, so (b) $\Rightarrow$ (c). Q.E.D.

PROPOSITION 6.9. *The following are equivalent:*

(a) $\mathrm{Oc}(\{b_n\})$ *is a* GB-*set;*

(b) $S(\{b_n\})$ *is a* GB-*set;*

(c) $b_n = O((\log n)^{-1/2})$.

Proof. We use some notation and results of the previous proof. (By the way, note that Theorem 4.7 and either of 6.7 and 6.9 make the other at least very plausible.) Here the equivalence of (a) and (b) is obvious. (b) is equivalent to the statement that for some $M > 0$, $b_n \mid L(\varphi_n) \mid < M$ for n sufficiently large, with probability 1, or that (6.8) converges for $\epsilon = M$. If (c) holds, i.e., if for some $N > 0$, $\mid \alpha_n \mid \leqslant N$ for all n, we can let $M = 2N$ and infer (b). If $\{\alpha_n\}$ is unbounded, then given M we choose n so that $\alpha_n \geqslant 2M$. Then $\alpha_j \geqslant M$ for $n^{1/2} \leqslant j \leqslant n$,

$$\sum_{n^{1/2} \leqslant j \leqslant n} M(n \log n)^{-1/2} \geqslant (n - n^{1/2} - 1)\, M(n \log n)^{-1/2}.$$

Since n can be chosen arbitrarily large, (6.8) diverges for $\epsilon = M$ for all $M > 0$. Thus (b) implies (c). Q.E.D.

We infer from Propositions 6.3 and 6.7 that a GC-set, $\mathrm{Oc}(\{1/\log n\})$, is not included in any GB-ellipsoid, since

$$\sum_{n=2}^{\infty} 1/(\log n)^2 = +\infty$$

(see [*15*], Lemma 2).

We next show that the GC- and GB-properties are not monotone functions of the "size" of a set as measured by volumes V_n or by ϵ-entropy.

PROPOSITION 6.10. *There exist a* GC-*set* $\mathrm{Oc} = \mathrm{Oc}(\{a_n\})$ *and a non*-GB-*ellipsoid* $E = E(\{b_n\})$ *such that*

(a) $H(E, \epsilon)/H(\mathrm{Oc}, \epsilon) \to 0$ *as* $\epsilon \downarrow 0$,

(b) $V_n(E)/V_n(\mathrm{Oc}) \to 0$ *as* $n \to \infty$.

Proof. We let $a_n = \alpha_n(\log n)^{-1/2}$, $n \geqslant 2$, where $\alpha_n \downarrow 0$ sufficiently slowly; for definiteness we can let $\alpha_n = (\log \log n)^{-1/4}$, $n \geqslant 3$. Let

$$b_n = (n \log n \log \log n)^{-1/2}, \qquad n \geqslant 3.$$

Then Oc is a GC-set and E is not a GB-set. $V_n(E)$ is asymptotic to a constant times

$$\left(\frac{2\pi e}{n}\right)^{n/2} n^{-1/2} \left(\frac{e}{n}\right)^{n/2} n^{-1/4} \prod_{j=3}^{n} (\log j \log \log j)^{-1/2},$$

and for n large,

$$V_n(\mathrm{Oc}) \geqslant \left(\frac{e}{n}\right)^{n} (3\pi n)^{-1/2} \prod_{j=2}^{n} \alpha_j (\log j)^{-1/2},$$

Thus there is a $K > 0$ such that for n large,

$$V_n(E)/V_n(\mathrm{Oc}) \leqslant K \prod_{j=3}^{n} (4\pi^2/\log\log j)^{1/4},$$

which implies (b).

To prove (a) it suffices to show that

$$H(E(\{(n \log n)^{-1/2}\}), \epsilon)/H(S(\{a_n\}), \epsilon) \to 0$$

as $\epsilon \downarrow 0$. Let $S = S(\{a_n\})$.

Given $\epsilon > 0$, let

$$N(S, \epsilon) = n = n(\{a_j\}, \sqrt{2}\epsilon) + \tfrac{1}{2} \pm \tfrac{1}{2}\,.$$

Because of the slow growth of the logarithms, this implies that, for ϵ small enough,

$$\begin{gathered} 1/9\epsilon^4 \leqslant (\log n)^2 \log\log n \leqslant 1/\epsilon^4, \\ \log n \leqslant 1/\epsilon^2, \qquad \log\log n \leqslant 2 \log(1/\epsilon), \\ H(S, \epsilon) = \log n \geqslant 1/5\epsilon^2(\log(1/\epsilon))^{1/2}. \end{gathered}$$

To estimate $N(E, \epsilon)$ from above we take the smallest integer n such that

$$(n \log n)^{-1/2} \leqslant \epsilon/2, \qquad \text{i.e.,} \qquad n \log n \geqslant 4/\epsilon^2.$$

For ϵ small enough this implies $n \log n < 5/\epsilon^2$. Now

$$N(E, \epsilon) \leqslant N(E_n\,, \epsilon/2) \leqslant N(B_n\,, \epsilon/2)$$

where

$$E_n = E(\{\beta_j\}), \qquad B_n = B(\{\beta_j\}),$$

$$\beta_j = (j \log j)^{-1/2}, \qquad j = 2,..., n, \qquad \beta_j = 0, \qquad j > n.$$

By Lemma 5.6, for ϵ small and hence for n large enough,

$$N(B_n, \epsilon/2) \leqslant \prod_{j=2}^{n} (2 + n^{1/2}(j \log j)^{-1/2}/\epsilon)$$

$$\leqslant 3^n n^{n/2} \epsilon^{-n} (n!)^{-1/2}.$$

(*Note:* the logarithms have served to make n smaller, but they are no longer needed.)
For n large we have $n! \geqslant (n/e)^n$, so

$$N(E, \epsilon) \leqslant (3e/\epsilon n^{1/2})^n = \exp\{n[\log 3 + 1 + \log(1/\epsilon) - \tfrac{1}{2}\log n]\}.$$

Since $n < 5/\epsilon^2$, we have, for ϵ small enough,

$$\log n < \log 5 + 2 \log(1/\epsilon) < 3 \log(1/\epsilon),$$
$$n \geqslant 4/\epsilon^2 \log n > 4/3\epsilon^2 \log(1/\epsilon),$$
$$\log n > \log(4/3) + 2 \log(1/\epsilon) - \log\log(1/\epsilon),$$
$$H(E, \epsilon) \leqslant 5[3 + \log\log(1/\epsilon)]/\epsilon^2 \log(1/\epsilon).$$

Thus $H(E, \epsilon)/H(S, \epsilon) \to 0$ as $\epsilon \downarrow 0$. Q.E.D.

Suppose given a sufficient condition that a set C be a GC-set, asserting that $H(C, \epsilon)$ is sufficiently small (e.g., Theorem 2.1) or that the $V_n(C)$ are sufficiently small (e.g., Proposition 5.5, Conjecture 5.4). Then the GC-octahedron of Proposition 6.10 will never satisfy such a condition since the ellipsoid does not. Hence no such sufficient conditon can be necessary.

In the converse direction, likewise, a sufficient condition for a Banach ball *not* to be a GB-set such as Theorem 5.3 or Conjecture 3.3 cannot be necessary.

One may, however, seek "best possible" conditions of the given kinds. In the four cases, Theorem 3.1 has a fairly strong claim to be best (see the next section). Theorem 5.3 has a weaker claim. Conjectures 3.3 and 5.4, if they are true, could probably be improved upon.

The volume of the n-dimensional octahedron

$$\left\{\sum_{j=1}^{n} x_j \varphi_j : \sum_{j=1}^{n} |x_j| \leqslant 1\right\}$$

is $2^n/n!$, which is asymptotic to $(2e)^n/n^n(2\pi n)^{1/2}$ by Stirling's formula. Thus by 5.11 and 5.12

$$EW(\mathrm{Oc}(\{b_n\})) = -1 - 1/\lambda(\{b_n\}). \tag{6.11}$$

A sequence $b_n \downarrow 0$ such that $\lambda(\{b_n\}) < \infty$ is $o((\log n)^{-1/2})$ (cf. end of the proof of Proposition 6.7). Thus conjecture 5.4 holds for octahedra. The next proposition implies that conjecture 5.9 also holds for octahedra.

PROPOSITION 6.12. *Let $\lambda = \lambda(\{b_n\})$, $s = EV(\mathrm{Oc}\,(\{b_n\}))$, $r = r(\mathrm{Oc})$. Then $r = -2/(2s+1) = 2\lambda/(2+\lambda)$ if any of these terms is less than 2 (i.e., if $r < 2$, $s < -1$, or $\lambda < \infty$). Thus under these conditions* Oc *is volumetric.*

Proof. $r \geqslant -2/(2s+1)$ in general by 5.8 (a). If $s < 1$, then $\lambda < \infty$ and $s \geqslant -1 - 1/\lambda$ by 6.11. Thus any of the hypotheses implies $\lambda < \infty$, and then $1/\lambda \geqslant -1 - s$,

$$2\lambda/(2+\lambda) = 2/((2/\lambda)+1) \leqslant -2/(2s+1)$$

if $s < -\frac{1}{2}$. It will now suffice to show that if $\lambda < \infty$,

$$r \leqslant 2/((2/\lambda)+1)$$

(since then $r < 2$ and $s < -1 < -\frac{1}{2}$).

Let $0 < \gamma < 1/\lambda$. Then for n large enough, $b_n < 1/n^\gamma$ by 5.2. Thus for some $K > 0$, $\mathrm{Oc}\,(\{b_n\}) \subset KC_\gamma$ where $C_\gamma = \mathrm{Oc}\,(\{1/n^\gamma\})$, and $r(\mathrm{Oc}) \leqslant r(KC_\gamma) = r(C_\gamma)$. Thus it is enough to prove that $r(C_\gamma) \leqslant 2/(1+2\gamma)$.

For x in C_γ, we have

$$x = \sum x_j \varphi_j / j^\gamma, \qquad \sum |x_j| \leqslant 1.$$

Given $\epsilon > 0$, let $A(x)$ be the set of all j such that

$$|x_j| > j^{2\gamma}\epsilon^2/4.$$

Then the number m of integers in $A(x)$ satisfies

$$m^{1+2\gamma}/(1+2\gamma) = \int_0^m x^{2\gamma}\,dx \leqslant \sum_{j=1}^m j^{2\gamma} \leqslant 4/\epsilon^2.$$

Let $\alpha < \beta < \gamma$. Then for some $c(\gamma)$,

$$m \leqslant c(\gamma)/\epsilon^{2/(1+2\gamma)} < \epsilon^{-2/(1+2\beta)}$$

for ϵ small enough. (Of course m depends on γ and ϵ).

The largest integer N in $A(x)$ is at most $(2/\epsilon)^{1/\gamma}$. Thus the number of possible choices of $A(x)$ is at most

$$\binom{N}{m} < N^m < \exp\left(c(\gamma)\log(2/\epsilon)/\gamma\epsilon^{2/(1+2\gamma)}\right)$$
$$< \exp\left(\epsilon^{-2/(1+2\beta)}\right)$$

for ϵ small enough. For any x in C_γ,

$$\Big\| x - \sum_{j\in A(x)} x_j\varphi_j/j^\gamma \Big\| = \Big(\sum_{j\notin A(x)} x_j^2/j^{2\gamma}\Big)^{1/2} \leqslant \max\{|x_j|/j^{2\gamma} : j\notin A(x)\}^{1/2} \leqslant \epsilon/2.$$

Thus

$$N(C_\gamma, \epsilon) \leqslant \sum_A N(C_\gamma(A), \epsilon/2)$$

where the sum is over the possible sets $A = A(x)$ and $C_\gamma(A)$ is the set of all sums

$$\sum_{j\in A(x)} x_j\varphi_j/j^\gamma, \qquad \sum |x_j| \leqslant 1.$$

Here we use a crude estimate from Lemma 5.6 to obtain for ϵ small enough

$$N(C_\gamma, \epsilon) \leqslant \exp\left(\epsilon^{-2/(1+2\beta)}\right)(3m^{1/2}/\epsilon)^m$$
$$\leqslant \exp\left(\epsilon^{-2/(1+2\alpha)}\right).$$

Thus $r(C_\gamma) \leqslant 2/(1+2\alpha)$. Letting $\alpha \uparrow \beta \uparrow \gamma$ we infer $r(C_\gamma) \leqslant 2/(1+2\gamma)$. Q.E.D.

By Proposition 6.9, to prove Conjecture 3.3 for octahedra its uffices to prove the following, where $a_n = (\log n)^{-1/2}$, $n \geqslant 2$.

PROPOSITION 6.13. $r(\mathrm{Oc}(\{a_n\})) = 2$.

Proof. $r \geqslant 2$ since this Oc is not a GC-set (Corollary 3.2, Proposition 6.7), or by volumes (5.8 (a) and 6.11).

To prove $r \leqslant 2$ we shall use the method of the previous proof with some additional complications. Let $\epsilon > 0$ and $\delta > 0$. Given x in Oc let $A(x)$ be the set of all j such that

$$|x_j| > \epsilon^2/4a_j^2 = (\epsilon^2 \log j)/4, \qquad j \geqslant 2.$$

Then (for ϵ small enough) $A(x)$ has at most $4/\epsilon^2$ elements. The largest possible integer n in $A(x)$ satisfies $n \leqslant \exp(4/\epsilon^2)$. For any x in Oc

$$\Big\| x - \sum_{j \in A(x)} x_j a_j \varphi_j \Big\| \leqslant (\max_{j \notin A(x)} |x_j| a_j^2)^{1/2} \leqslant \epsilon/2.$$

Let $Q(\epsilon)$ be the number of possible sets $A(x)$ for a given $\epsilon > 0$. Then by Lemma 5.6,

$$N(\mathrm{Oc}, \epsilon) \leqslant Q(\epsilon)\,(6/\epsilon^2)^{4/\epsilon^2} < Q(\epsilon) \exp(\epsilon^{-2-\delta})$$

for ϵ small enough.

(The estimate $Q(\epsilon) \leqslant n^{4/\epsilon^2} \leqslant \exp(16/\epsilon^4)$ is clearly inadequate.) Let s be a positive integer such that $1/s < \delta$. For $r = 0, 1, \ldots, s-1$, let

$$Z_{rs} = \{ j : 4\epsilon^{-2r/s} \leqslant \log j < 4\epsilon^{-2(r+1)/s} \}.$$

If $j \in A(x) \cap Z_{rs}$, then

$$|x_j| \geqslant \epsilon^{2(s-r)/s},$$

so the number of elements of $A(x) \cap Z_{rs}$ is at most $\epsilon^{2(r-s)/s}$. Thus the number of ways of choosing $A(x) \cap Z_{rs}$ is at most

$$[\exp(4\epsilon^{-2(r+1)/s})]^{\epsilon^{2(r-s)/s}} = \exp[4\epsilon^{-2(r+1)/s}\,\epsilon^{2(r-s)/s}] \leqslant \exp(\epsilon^{-2(1+\delta)}).$$

Thus for ϵ small enough

$$Q(\epsilon) \leqslant 2e^4 \exp(s\epsilon^{-2(1+\delta)}) \leqslant \exp(\epsilon^{-2-3\delta}),$$

and

$$N(\mathrm{Oc}, \epsilon) \leqslant \exp(\epsilon^{-2-5\delta}).$$

Letting $\delta \downarrow 0$ we get $r(\mathrm{Oc}) \leqslant 2$. Q.E.D.

Next we show that $EW(C)$ may be strictly smaller than $EV(C)$. Let

$$C = \mathrm{Oc}(\{2/(2n+1)\}) \times E(\{1/n\}),$$

a Banach ball in $H \times H$ which of course is a separable Hilbert space. Then subspaces adapted to C are uniquely determined, with

$$a_{2n} = 2/(2n+1), \qquad a_{2n+1} = 1/(n+1).$$

It follows easily that $EW(C) = -7/4$. Taking projections of the ellipsoid only we get $EV(C) \geqslant -3/2$. By 5.8 (a), 6.2, and 6.12 we obtain $r(C) = 1$, $EV(C) = -3/2$. Thus in measuring volumes it seems better to use EV primarily, as we have done, rather than EW, since, e.g., Conjecture 5.9 is false if EV is replaced by EW, and r and EW are no functions of each other over a reasonable range.

We have not evaluated $EV(\mathrm{Oc}\{b_n\})$ if $\lambda(\{b_n\}) = +\infty$, although then for $\{b_n\}$ bounded we have $EW(\mathrm{Oc}) = -1$. Thus it is conceivable that Conjecture 5.9 could hold even for $EV < -\frac{1}{2}$, but it seems unlikely.

7. Processes on Euclidean Spaces

In this section we apply Theorem 3.1 to Gaussian processes over a finite-dimensional Euclidean parameter set, e.g., the usual one dimensional "time". Conjecture 3.3 is also verified in certain cases. Since any compact Banach ball is a continuous image of the unit interval,[4] our hypotheses in general do not restrict the geometry of the Banach balls in H which arise, and we do not try to evaluate their volumes.

Theorem 7.1 (Fernique [7], [7a] for $T =$ cube). *Suppose $\{x_t, t \in T\}$ is a Gaussian process where T is a bounded subset of R^k. Suppose φ is a nonnegative real-valued function such that*

(a) $E\,|\,x_s - x_t\,|^2 \leqslant \varphi(|\,s - t\,|)^2$ *for all* $s, t \in T$,

(b) $\varphi(u)$ *is monotone-increasing on some interval* $0 < u < \alpha$,

(c) $\int_M^\infty \varphi(e^{-x^2})\,dx < \infty$ *for some* $M < \infty$.

Then x_t is sample-continuous.

Proof. Let C be the set of all x_t, $t \in T$, $C \subset H$. We shall prove that C is a GC-set and hence, since x_t is continuous from T into H by (a) and (c), that x_t has a continuous version.

Since T is bounded, there is an $A > 0$ such that

$$N(T, \delta) \leqslant A/\delta^k \qquad \text{for all} \qquad \delta > 0$$

(see [12], Section 3, I, p. 20; cf. also Lemma 5.6 above). (b) and (c) imply $\varphi(\delta) \downarrow 0$ as $\delta \downarrow 0$.

For any $\epsilon > 0$ let

$$\delta \equiv \Psi(\epsilon) \equiv \sup\{t : \varphi(t) < \epsilon\}.$$

(If φ is continuous and ϵ is small enough, $\delta = \varphi^{-1}(\epsilon)$.)

[4] See, for example, K. Kuratowski, "Topologie" (3rd ed.), Vol. II, Chapter VII, Section 45. Warszawa, 1961.

Let $\delta_n = \Psi(1/2^n)$, defined and positive for n large enough (unless $\varphi \equiv 0$, in which case the conclusion is trivial). Then $\delta_n \downarrow 0$ as $n \to \infty$. Now

$$N(C, 1/2^n) \leqslant A/\delta_n{}^k,$$

so

$$H(C, 1/2^n) \leqslant \log A + k \log (1/\delta_n).$$

Let $x_n = (\log (1/\delta_n))^{1/2}$. By Theorem 3.1 it suffices to prove that

$$\sum_{n=1}^{\infty} x_n/2^n < \infty.$$

(Note how the dimension becomes irrelevant.)
Now

$$\varphi(e^{-x^2}) \geqslant 1/2^n \qquad \text{for} \qquad x_{n-1} \leqslant x < x_n ,$$

so

$$\int_{x_N}^{\infty} \varphi(e^{-x^2})\, dx \geqslant \sum_{n=N}^{\infty} (x_{n+1} - x_n)/2^{n+1} = \sum_{n=N+1}^{\infty} x_n/2^n - \sum_{m=N}^{\infty} x_m/2^{m+1}$$

$$= \tfrac{1}{2} \sum_{n=N+1}^{\infty} x_n/2^n - x_N/2^{N+1},$$

so the required series converges. Q.E.D.

Fernique [7] shows that Theorem 7.1 is optimal of its kind in a sense, even for $k = 1$, since if

$$\int^{\infty} \varphi(e^{-x^2})\, dx = + \infty$$

and φ satisfies some additional mild monotonicity assumptions, then counterexamples to sample continuity exist. However, note that we may take a process x_t on $T = [0, 1]$ satisfying the hypotheses of Theorem 7.1 and transform it by a "steep" homeomorphism f of T, e.g. $f(t) = 1/\log (1/t)$, into a process $x_{f(t)}$ which may no longer satisfy 7.1. (c) but of course is still sample-continuous. The ϵ-entropy of the range is unchanged, so Theorem 3.1 applies to $x_{f(t)}$ and has a broader range of applications. Note however that such a transformation destroys stationarity of the process, and for stationary processes Theorem 7.1 may be essentially the best possible.

It has been shown [*4*] that for T an interval, hypothesis (c) of Theorem 7.1 can be replaced by any of several conditions, of which the best ([*4*], p. 186, 3°) seems to be

$$\sum_{k=1}^{\infty} 2^{k/2}[\varphi(1/2^{2^k})]^{1/2} < \infty.$$

But this condition is easily shown to imply Fernique's.

Next we discuss random Fourier series and the work of Kahane [*10*]. Let $\{x_t, t \in R\}$ be a Gaussian process, stationary and periodic of period 2π. [*Note:* Fernique's counterexamples showing that Theorem 7.1 (c) cannot be improved are all of this type, so the additional hypotheses do not change that situation.) We assume x_t is continuous in probability and that $Ex_t \equiv 0$. It is then well known and not hard to prove that a version of x_t is given by

$$x_t(\omega) = \frac{\beta_0 \xi_0}{2} + \sum_{n=1}^{\infty} \beta_n(\xi_n \sin nt + \eta_n \cos nt), \tag{1''}$$

where the $\xi_i(\omega)$ and $\eta_j(\omega)$ are all independent, normalized Gaussian random variables and the β_n are nonnegative constants, $\sum \beta_n{}^2 < \infty$. (Conversely, any such series (1″) defines a process of the given type.) Kahane [*10*] assumes $\beta_0 = 0$, which does not affect the sample continuity.

Let

$$t_i{}^2 = \sum_{n=2^i+1}^{n=2^{i+1}} \beta_n{}^2.$$

(*Note:* t_i are not values of t!) Kahane ([*10*], p. 2, Théorèmes 3, 4) proves the

THEOREM. *The condition* $\sum_{i=1}^{\infty} t_i < \infty$ *is necessary for sample continuity or boundedness of* x_t *and, if the* t_i *are decreasing, also sufficient (even for almost sure uniform convergence of* (1″)).

Neither half of the above theorem will be proved here, and I doubt that the methods of this paper would give such a complete result. However, it will be shown that Conjecture 3.3 holds to the extent that Kahane's rather sharp result applies. Also we shall treat some additional cases where Kahane's theorem does not apply but the conjecture still holds.

PROPOSITION 7.2. *Suppose* $t_1 \geqslant t_2 \geqslant \cdots$ *and* $\sum t_i < \infty$. *Let* S *be the set of all* x_t *in* H. *Then* $r(S) \leqslant 2$.

Proof. We can restrict ourselves to $0 \leqslant t < 2\pi$. For any s and t in $(0, 2\pi)$,

$$E((x_s - x_t)^2) = E\left\{\left[\sum_{n=1}^{\infty} \beta_n\xi_n(\cos ns - \cos nt) + \beta_n\eta_n(\sin ns - \sin nt)\right]^2\right\}$$

$$= 2\sum_{n=1}^{\infty} \beta_n^2(1 - \cos(n(s - t))).$$

Let

$$\beta^2 = \sum_{n=1}^{\infty} \beta_n^2 = \sum_{i=1}^{\infty} t_i^2, \qquad b = \sum_{i=1}^{\infty} t_i .$$

Given $\epsilon > 0$, we choose a minimal $M(\epsilon)$ such that

$$\sum_{n=M+1}^{\infty} \beta_n^2 \leqslant \epsilon^2/8.$$

For all x, $1 - \cos x \leqslant x^2$, so if

$$| s - t | \leqslant \epsilon/2\sqrt{2}\beta M,$$

then

$$2\sum_{n=1}^{M} \beta_n^2(1 - \cos(n(s - t))) \leqslant \epsilon^2/4$$

Hence

$$N(S, \epsilon) \leqslant 2\sqrt{2}\beta M/\epsilon + 1.$$

Now $M(\epsilon) \leqslant 2^i$ for the least i such that

$$\sum_{j=1}^{\infty} t_j^2 \leqslant t_i b \leqslant \epsilon^2/8.$$

For any $\delta > 0$,

$$n(\{t_i\}, \epsilon^2/8b) < (1/\epsilon^2)^{1+\delta}$$

for ϵ small enough by (5.2). Thus

$$M(\epsilon) \leqslant 2\epsilon^{-2(1+\delta)}$$

for ϵ small enough. Hence $r(S) \leqslant 2$. Q.E.D.

PROPOSITION 7.3. *If* $\sum \beta_n < \infty$, *then series* (1″) *converges uni-*

formly in t with probability 1, so x_t is sample-continuous. Then if X is the set of all x_t in H, $r(X) \leqslant 2$.

Proof. $\sum \beta_n(|\xi_n| + |\eta_n|)$ and hence (1″) converge uniformly in t by the three-series theorem. We represent X in H as follows: let $\{\varphi_n\}$ be an orthonormal basis, and

$$X = \left\{ \sum_{n=1}^{\infty} \beta_n[(\cos nt)\,\varphi_{2n} + (\sin nt)\,\varphi_{2n-1}] : 0 \leqslant t < 2\pi \right\}.$$

Then $X \subset B(\{b_n\})$ where $\beta_n = b_{2n-1} = b_{2n}$, and $\sum b_n < \infty$. As remarked after Proposition 6.6, $r(B) \leqslant 2$, so $r(X) \leqslant 2$. Q.E.D.

For "lacunary" random Fourier series of the form

$$x_t(\omega) = \sum_{k=1}^{\infty} \beta_k \xi_k(\omega) \cos(n_k t),$$

where $n_{k+1}/n_k \geqslant \gamma > 1$ for all k, $\beta_k \geqslant 0$ and the ξ_k are independent normalized Gaussian random variables, it is easy to see that $\sum t_i < \infty$ implies $\sum \beta_k < \infty$. Thus Kahane's theorem and Proposition 7.3 together imply that Conjecture 3.3 holds for lacunary series (without any further monotonicity assumptions).

8. Comments on the Conjectures

Of the three Conjectures 3.3, 5.4, and 5.9, Conjecture 5.4 is supported by 5.9 which has nothing *a priori* to do with Gaussian processes. One might seek similar support for 3.3. But $r(C) > 2$ does not imply (for octahedra) that the $W_n(C)$ are too large to satisfy Theorem 5.3. However, Theorem 5.3 may not be the best possible, and the largest $V_n(C)$ and $W_n(C)$ I know for a GB-set C are those of $\mathrm{Oc}(\{K(\log n)^{-1/2}\})$, $K > 0$, namely

$$V_n(C) \geqslant W_n(C) = \frac{K^n}{n!} \prod_{j=2}^{n} (\log j)^{-1/2}.$$

The following approach to some of our problems might seem natural. Given $\epsilon > 0$ and $C_n \subset R^n$, let $C_n{}^\epsilon$ be the set of points within ϵ of C_n. Then

$$N(C_n, 2\epsilon) \leqslant \lambda_n(C_n{}^\epsilon)/c_n\epsilon^n.$$

For C_n convex, $\lambda_n(C_n{}^\epsilon)$ can be expressed in terms of "mixed volumes" of C_n (Bonnesen and Fenchel [*3*], Paragraph 29, p. 38; Paragraph 32, p. 49; Paragraph 38, p. 61). I believe some estimates of Santalò [*16*] are of this sort.

However, such estimates do not seem adequate for our purposes. Consider for example Oc $(\{\log n)^{-1/2}\})$. To estimate $N(\mathrm{Oc}, \epsilon)$ is more or less equivalent to estimating $N(\mathrm{Oc}_n, \epsilon/2)$ where Oc_n is the intersection of Oc with the span of $\varphi_1, ..., \varphi_n$, and n is approximately e^{4/ϵ^2}. Then every point of the boundary of Oc_n is at least $(n \log n)^{-1/2}$ from the origin, so

$$\lambda_n(\mathrm{Oc}_n^{\epsilon}) \geqslant c_n \epsilon^n (1 + 1/2n^{1/2})^n,$$
$$\lambda_n(\mathrm{Oc}_n^{\epsilon})/c_n \epsilon^n \geqslant \exp(\gamma \exp(2/\epsilon^2))$$

if $\gamma < \frac{1}{2}$ and n is large enough. Thus this method seems quite inferior to that used to prove Proposition 6.13, in this case, since it produces an extra exponentiation.

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Another main result, Theorem 5.3, is proved using L.A. Santalò's theorem [*17*] on volumes of convex symmetric sets and their polars. I thank G. D. Chakerian for telling me of Santalò's result via a network of mutual friends.

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