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Universal covers and 3-manifolds

Francisco F. Lasheras

Departamento de Geometría y Topología, Universidad de Sevilla, Apdo 1160, 41080-Sevilla, Spain

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Abstract

In this paper, we show that if a finitely presented group G is the fundamental group of a finite fake surface in which the link of any vertex is not homeomorphic to the 1-skeleton of a tetrahedron, then there is a finite 2-complex K with $\pi_1(K) \cong G$ and whose universal cover $\tilde{K}$ has the proper homotopy type of a 3-manifold. As a consequence, the cohomology group $H^2(G;\mathbf{Z}G)$ is free abelian. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 57M07; 57M10; 57M20

1. Introduction

For a finitely generated group G, it is well known that its first cohomology group $H^1(G;\mathbf{Z}G)$ is free abelian and it "measures" the number of ends of G [2]. On the other hand, if G is finitely presented the second cohomology group $H^2(G;\mathbf{Z}G)$ is known to be torsion-free (see [4]), and the freeness of this group is related to the *semistability of G at infinity*, i.e., whether or not any two proper rays in the universal cover $\tilde{X}$ defining the same end are properly homotopic, where X is a finite 2-complex with $\pi_1(X) \cong G$. More precisely, Geoghegan and Mihalik [4] showed that if G is semistable at infinity then $H^2(G;\mathbf{Z}G)$ is free abelian. Regarding this cohomology group, Farrell [3] proved that if G contains an element of infinite order then $H^2(G;\mathbf{Z}G)$ is either $0, \mathbf{Z}$ or is not

E-mail address: fjfer@cica.es (F.F. Lasheras).

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finitely generated. The following question is well known:

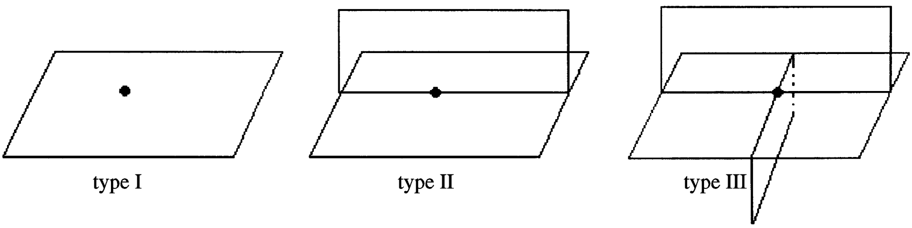
(Q1) *Is the cohomology group* $H^2(G;\mathbf{Z}G)$ *free abelian for any finitely presented group* G?

No counterexample is known at the time of writing. One could ask a stronger question for a finitely presented group G:

(Q2) *Does there exist a finite* 2*-complex* K *having* G *as fundamental group and whose universal cover* $\tilde{K}$ *has the proper homotopy type of a* 3*-manifold* M (*with boundary*)?

Indeed, an affirmative answer to (Q2) would also give us an affirmative answer to (Q1), since the cohomology group with compact supports $H^2_c(\tilde{K};\mathbf{Z})$ is isomorphic to a direct sum $H^2(G;\mathbf{Z}G) \oplus (\textit{free abelian})$ (see [4]), and Lefschetz duality tells us that $H^2_c(\tilde{K};\mathbf{Z}) \cong H_1(M,\partial M;\mathbf{Z})$ is free abelian, since simply connected manifolds are orientable.

As an application of the results in [7] we get an affirmative answer to (Q2) for a certain class of groups. For this, we need to recall that a two-dimensional locally finite simplicial complex X is a *fake surface* if each vertex of X has a neighborhood of one of the following types:



It is worth mentioning that each compact two-dimensional polyhedron has the homotopy type of a finite fake surface [10]. This way, we can think of a finitely presented group G in terms of finite fake surfaces having G as fundamental group. We denote by $\mathscr{C}$ the class of those finitely presented groups which are the fundamental group of a finite fake surface with no vertices of type III. The class $\mathscr{C}$ contains groups which are not 3-manifold groups (Remark 2.11). On the other hand, we have:

Proposition 1.1. *The class* $\mathscr{C}$ *does not contain non-trivial perfect groups.*

Therefore, the fundamental group of a homology 3-sphere is not in $\mathscr{C}$. Our main results are

Theorem 1.2. *If* G *is in* $\mathscr{C}$ *then there is a finite* 2*-complex* K *with* $\pi_1(K)\cong G$ *and whose universal cover* $\tilde{K}$ *has the proper homotopy type of a* 3*-manifold.*

Corollary 1.3. *Let* X *be a simply connected fake surface with no vertices of type III. If* X *is the universal cover of a finite complex, then the cohomology group with compact supports* $H^2_c(X;\mathbf{Z})$ *is free abelian. In particular, for any group* G *as in* $\mathscr{C}$, $H^2(G;\mathbf{Z}G)$ *is free abelian.*

2. Fake surfaces and the class $\mathscr{C}$

In this section, we show that there are groups in $\mathscr{C}$ which are not 3-manifold groups (see Remark 2.11), and we prove Proposition 1.1. For this, we need to recall some definitions and results from [5,11], some of which will be used in the next section to prove Theorem 1.2.

Definition 2.1. A *T-bundle* is a bundle whose fiber T consists of three segments sharing a vertex. A T-bundle over a circle is obtained from $T\times I$ by identifying $T\times\{0\}$ with $T\times\{1\}$ via a homeomorphism of T.

Proposition 2.2 (Ikeda [5, Lemma 5]). *By numbering the three segments of T, the homeomorphisms of T are classified* (*up to isotopy*) *by the elements of the symmetric group S_3. Thus, three types of T-bundles over a circle occur, induced by the identity permutation, the 3-cycle* (231), *and the 2-cycle* (23), *which we will denote by types* (i), (ii) *and* (iii), *respectively.*

Given a fake surface X, we denote by $\Gamma\subset X$ the graph consisting of all the 1-simplexes of order 3, i.e., those which are the face of three 1-simplexes.

Proposition 2.3 (Wright [11, Section 3]). *Let X be a fake surface. Let C be a simple closed curve in Γ and N be a regular neighborhood of C in X. Then, there is embedded in N a T-bundle over C whose type is uniquely determined by C.*

Remark 2.4. If Γ contains only vertices of type II, then a T-bundle embedded in the regular neighborhood N is itself a regular neighborhood of C in X. Thus, since a regular neighborhood is a mapping cylinder [1], the T-bundles over a circle $C\subset\Gamma$ may be classified as follows:

(a) The mapping cylinder of a map $f:S_1\cup S_2\cup S_3\to C$ with $S_i=S^1$ and $f|S_i$ of degree ±1, if the T-bundle is of type (i).

(b) The mapping cylinder of a map $f:S^1\to C$ of degree ±3, if the T-bundle is of type (ii).

(c) The mapping cylinder of a map $f:S_1\cup S_2\to C$ with $S_i=S^1$, $f|S_1$ of degree ±1 and $f|S_2$ of degree ±2, if the T-bundle is of type (iii).

Lemma 2.5. *Let X be a fake surface with no vertices of type III. Let $C\subset\Gamma$ be a simple closed curve and $N\subset X$ be a regular neighborhood of C in X. If there is embedded in N a T-bundle over C of type* (ii), *then $H_1(X;\mathbf{Z})\neq0$.*

Proof. Let us denote by N' such a T-bundle of type (ii) in N. Then, by (2.4), N' is a regular neighborhood of C in X with a single boundary component $S\cong S^1$. Take $X'=X\cup v*S$, where v is a conning point over S. From a Mayer–Vietoris sequence one gets

$$H_1(X';\mathbf{Z})\cong H_1(N'\cup v*S;\mathbf{Z})\oplus H_1(\overline{X-N'}\cup v*S;\mathbf{Z})$$

with $H_1(N' \cup v * S; \mathbf{Z}) \cong \mathbf{Z}_3$. Thus, $H_1(X'; \mathbf{Z}) \neq 0$ and hence $H_1(X; \mathbf{Z}) \neq 0$, since otherwise X' would be 1-acyclic, by construction. □

Definition 2.6. Let X be a fake surface and $N(\Gamma)$ be a regular neighborhood of Γ in X. The connected components of $X - N(\Gamma)$ are called 2-*components* of X. Notice that each 2-component is an open surface.

Although [5, Lemma 13] is for finite fake surfaces, it also shows the following:

Lemma 2.7. *Let X be a* 1*-acyclic connected fake surface, and let F be a* 2*-component of X with compact closure $\bar{F}$. If $\bar{F}$ has j boundary components, then $X - F$ has j connected components.*

Lemma 2.8 (Ikeda [5, Lemma 12]). *Let X be a finite fake surface. If X is* 1*-acyclic, then each* 2*-component of X is a* 2*-sphere with a finite number of disks removed.*

Notation. Given a fake surface X, the number of connected components of $\Gamma \subset X$ is denoted by $\mu(X)$.

Lemma 2.9 (Ikeda [5, Proposition 4]). *Let X be a connected* 1*-acyclic finite fake surface, and let F be one of its* 2*-components. Assume that $\mu(X) = 1$. Then, the closure $\bar{F}$ has at most one boundary component.*

Lemma 2.10 (Ikeda [5, Lemma 14a]). *Let X be a connected finite fake surface with $\mu(X) \geq 2$. Then, there is a* 2*-component F of X such that the closure $\bar{F}$ has more than one boundary component.*

Next, we consider the class $\mathscr{C}$ of all finitely presented groups which are the fundamental group of a finite fake surface with no vertices of type III.

Remark 2.11. The class $\mathscr{C}$ contains groups which are not 3-manifold groups. To see this, we may consider the Baumslag–Solitar group G given by the presentation $\langle a, b; ab^{-1}a^{-1}b^2 \rangle$ which is not the fundamental group of any 3-manifold (see [6]). The standard 2-complex X associated to this group presentation is a finite fake surface with no vertices of type III. Moreover, the graph $\Gamma \subset X$ of triple edges consists of a single circle (represented by the generator b) which has a T-bundle over it of type (c) in X. It is worth mentioning that Baumslag–Solitar groups, and in general all one-relator groups, have been shown to be semistable at infinity [8,9].

Proof of Proposition 1.1. For the proof of this proposition we will admit some kind of boundary points in a fake surface as a fourth type of vertices. These new vertices will have neighborhoods just as if they were on the boundary of a surface. Let then X be a finite fake surface (possibly with boundary) with no vertices of type III, and assume X is 1-acyclic. We are to show that X is simply connected. We will proceed by induction on $\mu(X)$. For if $\mu(X) = 0$, the conclusion follows, since $X = S^2$ or D^2.

If $\mu(X)=1$ and $\partial X=\emptyset$ then X consists of a T-bundle over a circle and a finite number of disks attached to it along the boundary, by Lemmas 2.8 and 2.9. Since $\mu(X)=1$, the number of disks attached is no greater than three. Moreover, the T-bundle cannot be of type (b) in (2.4), by Lemma 2.5. If the T-bundle is of type (c), then the fake surface X consists of a projective plane to which a disk has been attached along a generator of its fundamental group, and hence X would be simply connected. Finally, if the T-bundle is of type (a), then X is a 2-sphere with a disk attached along the equator, which is also simply connected.

On the other hand, if $\mu(X)=1$ and $\partial X\neq\emptyset$ then one can check that the fake surface X is obtained from a fake surface X' with $\mu(X')=1$ and $\partial X'=\emptyset$, as above, by removing a finite number of disks in $X'-\Gamma'$, since Lemmas 2.5 and 2.8 both hold in the case $\partial X\neq\emptyset$. Thus, it is easy to check that again X 1-acyclic implies simply connected.

Assume the result true for 1-acyclic fake surfaces Y (possibly with boundary) with no vertices of type III and $\mu(Y)<\mu(X)$. We construct a 1-acyclic fake surface X' with $\partial X'=\emptyset$ by gluing a disk along each component of ∂X, if any. By Lemma 2.10, there is a 2-component F of X' whose closure has more than one boundary component ($\mu(X')\geq 2$). Let C be one of those boundary components, and B be a circle in F parallel to C not meeting ∂X. Then, by Lemma 2.7, B disconnects X' into two connected components X_1' and X_2' with the same boundary B. Let $X_1\subset\bar{X}_1'$ and $X_2\subset\bar{X}_2'$ be the closures of the corresponding connected components of $X-B$. From the Mayer–Vietoris sequence

$$\cdots\longrightarrow H_1(B;\mathbf{Z})\longrightarrow H_1(X_1;\mathbf{Z})\oplus H_1(X_2;\mathbf{Z})\longrightarrow H_1(X;\mathbf{Z})=0,$$

we get that $H_1(X_i;\mathbf{Z})$ $(i=1,2)$ is either trivial or a 1-generator group. We want to conclude that one of the X_i's, say X_1, is 1-acyclic, and the other satisfies $H_1(X_2;\mathbf{Z})\cong\langle B\rangle/\langle mB\rangle$, for some $m\geq 0$. For this, we observe that X_i is a "tree" of spaces, i.e. a union of spheres with holes and conjunction spaces where a conjunction space is a T-bundle over a circle of type (a) or (c). Call a conjunction space type (a) or type (c), accordingly. These spaces are unioned together in a tree like fashion to give X_i, where the spheres with holes are separated by the conjunction spaces. Hence one can compute a general type of presentation for $\pi_1(X_i)$ using Van Kampen's theorem. The surface pieces have free (or trivial) fundamental group. The generators of these free groups provide generators for $\pi_1(X_i)$. The connecting spaces of type (a) either introduce relations equating a generator in one free group with a generator in each of two other free groups (in the case none of the three surfaces bounding this conjunction space is a disk) or (if one or two of these surfaces is a disk) a generator in one or two of the free groups is killed. In any case, it is easy to see that components of the union of all of the surface pieces together with all of the conjunction pieces of type (a) have free fundamental group.

Observe that a conjunction space of type (c) is simply a Möbius band with an annulus attached to its center circle. Hence, the conjunction spaces of type (c) introduce relations of the form: the square of a generator in one free group is either trivial or equal to a generator in another free group. It is easy to see that if one abelianizes such a presentation (to get $H_1(X_i)$) and one gets a 1-generator group, then the group must be Z_{2n} for $n \geq 0$. Now the conclusion follows. Thus, by the induction process, X_1 is simply connected, since $\mu(X_1) < \mu(X)$. Next, we take $Y_2 = X_2 \cup v * B$, where v is a conning point over B. Then, Y_2 is a 1-acyclic fake surface with $\mu(Y_2) < \mu(X)$, whence Y_2 is also simply connected. We conclude that $X \cup v * B = X_1 \cup Y_2$ is then simply connected, by Van Kampen's theorem, and since B was already nullhomotopic in X_1, $X \cup v * B$ is homotopy equivalent to the wedge $X \vee S^2$, which implies that X is simply connected. □

3. Universal covers and 3-manifolds

The purpose of this section is to prove the main results of this paper. For this, we need to recall some definitions and one of the main theorems of [7].

Definition 3.1. Let X be a two-dimensional locally finite simplicial complex, and assume the link $lk(v,X)$ is planar, for every vertex v of X. Let (v,w) be a 1-simplex of X. Given an embedding $\phi_v : lk(v,X) \to \mathbf{R}^2$, we denote by $\theta_{\phi_v}(w)$ the cyclic ordering determined by ϕ_v on $lk((v,w),X)$ as we go around $\phi_v(w)$ following the orientation in $\mathbf{R}^2$. Note that if the cardinality $|lk((v,w),X)|$ is ≤ 2, then there is only one cyclic ordering $\theta_{\phi_v}(w)$.

We keep denoting by $\Gamma \subset X$ the graph consisting of all the 1-simplexes of X which are of order > 2 (i.e., those which are the face of at least three 2-simplexes of X). Consider the cochain complex of Γ over $\mathbf{Z}_2$

$$0 \to C^0(\Gamma;\mathbf{Z}_2) \xrightarrow{\delta} C^1(\Gamma;\mathbf{Z}_2) \to 0.$$

Given a family $\Phi = \{\phi_v\colon lk(v,X) \to \mathbf{R}^2, v \in X^0\}$ of embeddings, we can associate it a cochain (cocycle)

$$\omega_\Phi = \sum_{\sigma \subset \Gamma} \omega_\Phi(\sigma) \cdot \sigma \in C^1(\Gamma;\mathbf{Z}_2),$$

where $\omega_\Phi(\sigma) = 0$ if $\theta_{\phi_{o(\sigma)}}(t(\sigma))$ and $\theta_{\phi_{t(\sigma)}}(o(\sigma))$ are opposite, and $\omega_\Phi(\sigma) = 1$ otherwise. Here $o(\sigma)$ and $t(\sigma)$ are the vertices of σ. By extension, we define $\omega_\Phi(\sigma) = 0$ for every 1-simplex σ of order ≤ 2.

We say that a simplicial complex X "thickens" to a polyhedron Y if Y admits a CW-structure containing, as a subcomplex, a copy of a subdivision of X onto which Y collapses. The following theorem is proved in [7]:

Theorem 3.2. *Let X be a two-dimensional connected locally finite simplicial complex. Then, X thickens to an orientable 3-manifold if and only if*

(i) *$lk(v,X)$ is planar, for every vertex v of X, and*

(ii) *there exists a family of embeddings $\Phi = \{\phi_v\colon lk(v,X) \to \mathbf{R}^2, v \in X^0\}$ so that the associated cochain ω_Φ is trivial.*

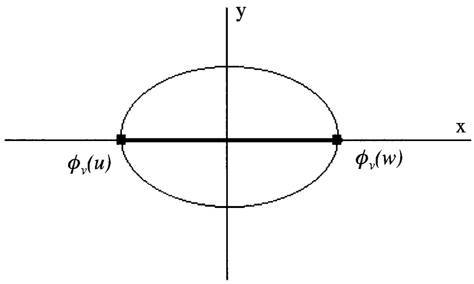


Fig. 1.

Proof of Theorem 1.2. Let X be a finite fake surface with no vertices of type III and having the group G as fundamental group, and let $\tilde{X}$ be its universal cover with covering map $p:\tilde{X} \to X$. Note that the fake surface $\tilde{X}$ does not have vertices of type III either. Let $\Gamma \subset X$ be the graph of all triple edges, as in Section 2. We are going to produce a finite 2-complex $K \supset X$ with the same fundamental group G, and a 2-complex W with planar links which thickens to a 3-manifold and is proper homotopy equivalent to the universal cover $\tilde{K}$ of K. We will show this thickening property of W by using the criterion given in Theorem 3.2, that is, we are to find a family Φ consisting of an embedding of the link of each vertex of W in $\mathbf{R}^2$ so that the associated cochain ω_Φ is trivial.

We start with the proof choosing embeddings for the link of some of the vertices of $\tilde{X}$ as follows. Observe that Γ consists of a disjoint union of circles C_i, $i \in I$, since X is finite and each vertex of Γ has valence 2 in Γ. Let us fix an index $i \in I$. If C_i lifts to a disjoint union of lines in $\tilde{X}$, we can take embeddings ϕ_v for the link of each vertex v on those lines so that the cochain ω_Φ vanishes on them, i.e., $\omega_\Phi(\sigma) = 0$ for every 1-simplex σ on any of those lines. To achieve this, we pick a vertex on each line and start going away from that vertex choosing embeddings for the link of the vertices we encounter on the corresponding line, so that the required condition is being satisfied at every step.

Assume now that C_i lifts to a disjoint union of circles $\tilde{C}_j$, $j \in J$. Let us fix an index $j \in J$, and denote by $\tilde{N}_{i,j}$ a T-bundle over $\tilde{C}_j$ in $\tilde{X}$. Note that $\tilde{N}_{i,j}$ is a regular neighborhood of $\tilde{C}_j$ in $\tilde{X}$, since there are no vertices of type III, and so it can be regarded as one of the mapping cylinders described in Remark 2.4. Observe that $\tilde{N}_{i,j}$ cannot be of type (b) in Remark 2.4, by Lemma 2.5. Thus, there are only two cases to consider depending on whether $\tilde{N}_{i,j}$ is of type (a) or (c).

Case 1: $\tilde{N}_{i,j}$ is of type (a) in Remark 2.4, i.e., $\tilde{N}_{i,j} - \tilde{C}_j$ has three connected components F_1, F_2 and F_3. We can choose embeddings for the link of the vertices on $\tilde{C}_j$ so that the cochain ω_Φ vanishes on $\tilde{C}_j$. For this, we first choose an orientation on the circle $\tilde{C}_j$. For each vertex $v \in \tilde{C}_j$, we choose an embedding $\phi_v : lk(v,\tilde{X}) \to \mathbf{R}^2$ whose image is as indicated in Fig. 1 and so that $\phi_v(F_1 \cap lk(v,\tilde{X})) \subset \{(x,y), y > 0\}$, $\phi_v(F_2 \cap lk(v,\tilde{X})) \subset \{(x,y), y = 0\}$, $\phi_v(F_3 \cap lk(v,\tilde{X})) \subset \{(x,y), y < 0\}$, and the orientation on $\tilde{C}_j$ determined by the sequence of vertices u, v, w agrees with the chosen orientation (see Fig. 1). One can readily check that if $\sigma = (v,w)$ is a 1-simplex of $\tilde{C}_j$, then the orderings $\theta_{\phi_v}(w)$ and $\theta_{\phi_w}(v)$ are opposite. With this, we conclude Case 1 without modifying our original complex X.

Next, if $\tilde{N}_{i,j}$ is of type (c), we will add some cells to X to obtain the finite complex K, and we will modify its universal cover $\tilde{K}$ within its proper homotopy type to construct our complex W. We do it as follows.

Case 2: $\tilde{N}_{i,j}$ is of type (c) in Remark 2.4, i.e., $\tilde{N}_{i,j} - \tilde{C}_j$ has two connected components F_1 and F_2 whose closures $\bar{F}_1$ and $\bar{F}_2$ are an annulus and a Möbius band, respectively. Let us denote by $\tilde{S}_2$ the boundary of $\bar{F}_2$, and by $\tilde{S}_1$ the boundary component of $\bar{F}_1$ which is not $\tilde{C}_j$. Notice that both $\tilde{S}_1$ and $\tilde{S}_2$ consist of double edges, and the same is true for their images $S_k = p(\tilde{S}_k) \subset X$, $k = 1,2$. We recall that $p(\tilde{C}_j) = C_i$. Thus, $N_{i,j} = p(\tilde{N}_{i,j})$ is a T-bundle over C_i (and a regular neighborhood) with boundary $S_1 \cup S_2$. Therefore, $N_{i,j}$ is not of type (a). On the other hand, $N_{i,j}$ is not of type (b) either, i.e., $S_1 \neq S_2$, since in that case there would be a deck transformation of $\tilde{X}$ (corresponding to an element of G) taking $\tilde{C}_j$ to itself and $\bar{F}_1$ to $\bar{F}_2$ homeomorphically, which is not possible. Therefore, $N_{i,j}$ is a T-bundle over C_i of type (c). Moreover, $p(\bar{F}_1)$ is an annulus with boundary $S_1 \cup C_i$, and $p(\bar{F}_2)$ is a Möbius band with boundary S_2. Let n be the degree of the restriction map $p|\tilde{C}_j : \tilde{C}_j \to C_i$. We attach a 2-cell to X along S_1 with an attaching map of degree n, and a second 2-cell along S_2 via an attaching map of degree m, with $2m = l.c.m.(2,n)$. We denote by e_1 and e_2 the closed 2-cells we have just attached along S_1 and S_2, respectively. Observe that the integers n and m are precisely the orders of the elements of $\pi_1(X) \cong G$ determined by S_1 and S_2, respectively. Thus, the resulting complex $X \cup e_1 \cup e_2$ also has G as fundamental group. We denote this new complex by K, with universal cover $\tilde{K}$ and covering map $p' : \tilde{K} \to K$. Observe that the circles $\tilde{S}_k \subset \tilde{K}$, $k = 1,2$, consist now of edges of order ≥ 3. Note that the inclusions $e_k \subset K$, $k = 1,2$, induce monomorphisms $\pi_1(e_k) \to \pi_1(K)$, and hence each component of $(p')^{-1}(e_k) \subset \tilde{K}$ is a copy of the universal cover of e_k, which consists of a collection of disks attached along their boundaries via a map of degree 1. We choose disks $D_k \subset \tilde{K}$, $k = 1,2$, with $\partial D_k = \tilde{S}_k$ and $p'(D_k) = e_k$. Next, we take circles B_1 and B_2, parallel to $\tilde{C}_j$, in the components F_1 and F_2 of $\tilde{N}_{i,j} - \tilde{C}_j$, respectively. By Lemma 2.7, if we cut $\tilde{K}$ along B_1 (resp. B_2) we get two connected components, since the closure of the 2-component of the 1-acyclic fake surface $\tilde{X} \cup D_1 \cup D_2$ containing B_1 (resp. B_2) is a product $S^1 \times [0,1]$. See Fig. 2.

We will denote by $\tilde{K}_1$ (resp. $\tilde{K}_2$) the closure of the component of $\tilde{K} - B_1$ (resp. $\tilde{K} - B_2$) not containing $\tilde{C}_j$ (see Fig. 2). Observe that $\tilde{K}$ has the same proper homotopy type as the adjunction complex $\tilde{K}_1 \cup_h \tilde{K}_2$, where $h : B_2 \to B_1$ is a map of degree ± 2. Furthermore, this complex $\tilde{K}_1 \cup_h \tilde{K}_2$ is proper homotopy equivalent to the adjunction complex $W = \tilde{K}_1 \cup_g \tilde{K}_2$, where $g : B_2 \to B_1$ is now a map of degree 1, since B_1 bounds a disk $\Delta \supset D_1$ in $\tilde{K}_1$ and hence any two maps $B_2 \to B_1$ are homotopic in $\tilde{K}_1$. Notice that as a consequence of this alteration of $\tilde{K}$, the complex W does not contain a circle $\tilde{C}_j$ of triple edges. Observe that if v is a vertex of type II in $\tilde{X} \subset \tilde{K}$ whose link has already been embedded in $\mathbf{R}^2$ in the process of the proof, then $v \in W$ and $lk(v,W) = lk(v,\tilde{X})$, and so we can keep that embedding. Finally, we are able to choose embeddings ϕ_v

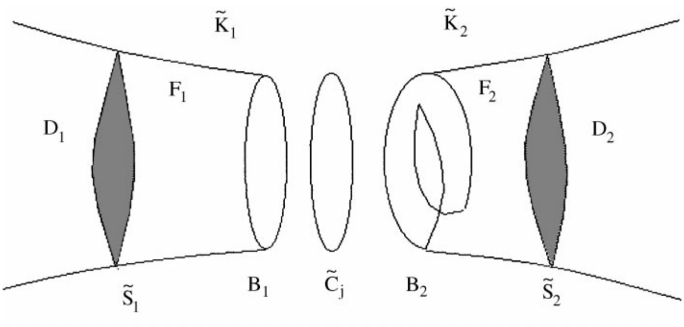


Fig. 2.

for the link of each vertex v on the circles $\tilde{S}_k \subset W$, $k=1,2$ (which consist of edges of order ≥ 3) proceeding in a similar way as in Case 1, and so that the corresponding cochain ω_Φ vanishes on them. To see this, note that if $\tilde{N}_k$ is a regular neighborhood of $\tilde{S}_k$ in W, then the closure of each component of $\tilde{N}_k - \tilde{S}_k$ is an annulus.

This way, after these alterations of $\tilde{K}$ within its proper homotopy type, we get a complex W and a family Φ consisting of an embedding for the link of each vertex of W in $\mathbf{R}^2$, so that the associated cochain ω_Φ is trivial. Thus, by Theorem 3.2, W thickens to a 3-manifold and this concludes the proof, since this thickening is proper. □

Proof of Corollary 1.3. Let X be a simply connected fake surface with no vertices of type III. If X is the universal cover of a finite fake surface Y, then the group $G=\pi_1(Y)$ is in the class $\mathscr{C}$ and hence there is a finite 2-complex K with $\pi_1(K)\cong G$ and whose universal cover $\tilde{K}$ is proper homotopy equivalent to a 3-manifold M, by Theorem 1.2. Let L be a $K(G,1)$ complex having finite 2-skeleton $L^2=K$. By [4, 4.1] we have an isomorphism

$$H^2(G;\mathbf{Z}G)\oplus(\text{free abelian})\cong H^2_c(\tilde{L}^2;\mathbf{Z})=H^2_c(\tilde{K};\mathbf{Z}).$$

By Lefschetz duality, we have that $H^2_c(\tilde{K};\mathbf{Z})\cong H_1(M,\partial M;\mathbf{Z})$ is free abelian, whence $H^2(G;\mathbf{Z}G)$ is also free abelian. Next, we take a $K(G,1)$ complex Z with 2-skeleton $Z^2=Y$. Again, by [4, 4.1], we get that $H^2_c(\tilde{Z}^2;\mathbf{Z})=H^2_c(X;\mathbf{Z})$ is free abelian, since we have just shown that $H^2(G;\mathbf{Z}G)$ is free abelian, and this concludes the proof. □

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