HOMOTOPY CLASSIFICATION OF \((n, nz)\)-COMPLEXES

Michael N. DYER

University of Oregon, Dept. of Mathematics, Eugene, Or. 97403, USA

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0. Introduction

A \((n, m)\)-complex is a finite, connected \(m\)-dimensional CW complex \(X\) with fundamental group isomorphic to \(n\) whose universal cover \(\tilde{X}\) is \((m - 1)\)-connected. Let \(HT(n, m)\) denote the directed tree whose vertices are the set of homotopy types of \((n, m)\)-complexes. If \(X, Y\) are \((n, m)\)-complexes, the vertex \([X]\) is connected by an edge to vertex \([Y]\) if \(Y\) has the homotopy type of the sum \(X \vee S^m\) of \(X\) with the \(m\)-sphere \(S^m\). \(HT(n, m)\) is connected by \([3]\), Theorem 14, and clearly contains no circuits. A \((n, m)\)-complex \(X\) is said to be a root if the vertex \([X]\) has no predecessors. The stalk \([X]\) generated by a \((n, m)\)-complex \(X\) is the subtree with vertices

\([X], [X \vee S^m], \ldots, [X \vee nS^m], \ldots\),

where \(nS^m\) is the sum of \(n\) copies of the \(m\)-sphere \(S^m\). The level of a vertex \([X]\) is given by the directed Euler characteristic \((-1)^m \chi(X)\). \(x_{\min}\) denotes the minimum level occurring in the tree. The basic problem is to describe the homotopy tree \(HT(n, m)\). These trees have also been studied in \([3, 4, 5, 6, 7, 8, 9, 15, 28, 34]\).

This paper begins an attempt to carry out a program suggested by Whitehead in \([30, 7]\) and by MacLane and Whitehead in \([13]\). They point out the homotopy theory of \((n, m)\)-complexes can be carried out algebraically via the theory of algebraic \(m\)-types. The algebraic \(m\)-type of a connected complex \(X\) such that \(\tilde{X}\) is \((m - 1)\)-connected consists of the triple

\[T(X) = (\pi_1(X), \pi_m(X), k(X)),\]

where \(k(X) \in H^{m+1}(\pi_1(X); \pi_m(X))\) is the obstruction invariant of \([13]\). In this paper we apply the Swan–Wall invariant to the study of \(m\)-types (see section 1 for a definition).

As a first step, in section 1 we study the topological \(m\)-type of a complex. Here we ask: When does a CW complex \(X\) have the same topological \(m\)-type as a finite complex of dimension \(m\)? This is answered, in many cases, by the fact that the Swan–Wall class \(SW_m[X]\) in dimension \(m\) of the space \(X\) is an invariant of the
$m$-type of $X$. It was previously known from [28] that $\text{SW}_m [X]$ is a homotopy in-
variant. We then see that under certain conditions $X$ has the topological $m$-type of a
finite $m$-complex if and only if $\text{SW}_m [X] = 0$ (Proposition 1.4).

In section 2 we specialize to a finite fundamental group and algebraic $m$-types.
An algebraic $m$-type consists of a triple $T = (\pi, \pi_m, k)$, where $\pi$ is a group, $\pi_m$ is a
$\pi$-module, and $k$ is a cohomology class in $H^{m+1} (\pi, \pi_m)$. A homomorphism of algebraic
$m$-types $\varphi = (\theta, \beta) : T \to T' = (\pi', \pi'_m, k')$ consists of a group homomorphism
$\theta : \pi \to \pi'$ and a $\theta$-homomorphism $\beta : \pi_m \to \pi'_m (\beta(x \cdot a) = \theta(x) \cdot \beta(a), x \in \pi, a \in \pi_m)$
such that $\theta^* (k') = \beta_n (k)$ in the diagram:

$H^{m+1} (\pi, \pi_m) \overset{\varphi^*}{\longrightarrow} H^{m+1} (\pi', \pi'_m) \overset{\theta^*}{\longrightarrow} H^{m+1} (\pi', k')$.

Here $(\pi'_m)_{\theta}$ is $\pi'_m$ with a $\pi$-module structure induced via $\theta$. The homomorphism
$(\theta, \beta) : T \to T'$ is an isomorphism provided $\theta, \beta$ are bijective. Two $(\pi, m)$ complexes
$X, Y$ have the same homotopy type if and only if $T(X) \simeq T(Y)$, [13].

A natural question is: When is an algebraic $m$-type $(\pi, \pi_m, k) (m \geq 2)$ isomorphic
to the algebraic $m$-type of a $(\pi, m)$-complex? This was previously studied in [5] for
$m = 2$.

Call a $\pi$-module $\pi_m$ topologically realizable if $\pi_m = \pi_m (X)$ for some $(\pi, m)$-com-
plex $X$. For $\pi$ finite of order $n$ and $\pi_m$ topologically realizable we have $H^{m+1} (\pi, \pi_m)$
isomorphic to the finite cyclic group $\mathbb{Z}_n$ of order $n$. The only $k$-invariants which can
arise from $(\pi, m)$-complexes are generators of $\mathbb{Z}_n$ [10, Theorem 6.1]. Given an $m$-
type $T = (\pi, \pi_m, k)$, where $\pi_m$ is realizable and $k$ is a generator of $\mathbb{Z}_n$, let $T_p$ denote
the $m$-type $(\pi, \pi_m, p \cdot k)$ for any $p \in \mathbb{Z}_n^*$, the group of units of $\mathbb{Z}_n$. We compute the
Swan–Wall class $\text{SW}_m [T_p] \in \overline{K}_0 \mathbb{Z}_n$, the reduced projective class group of the integral
group ring $\mathbb{Z}_n$ of $\pi$, and show that the correspondence $p \mapsto \text{SW}_m [T_p]$ is a homomor-
phism. In fact, this homomorphism is precisely $-\delta$, where $\delta : \mathbb{Z}_n^* \to \overline{K}_0 \mathbb{Z}_n$ is the
boundary homomorphism in the Milnor Mayer–Vietoris sequence associated with the square of ring homomorphisms:

\[
\begin{array}{c}
\mathbb{Z}_n \\
\downarrow \epsilon \\
\mathbb{Z}/(N) \\
\end{array}
\]

Here $\epsilon$ is the augmentation homomorphism and $(N)$ is the ideal generated by $N = \sum_{x \in \pi} x$. Thus $T_p$ is the algebraic $m$-type of a $(\pi, m)$-complex if and only if $p$ is a
member of the kernel of $\epsilon$, provided $m \geq 3$ (Theorem 3.5).

This whole theory was suggested by Swan's fundamental paper [23]. In fact, the
above computation was essentially carried out [23] in the special case of $\pi$ a finite
periodic group of minimal free period $k$ and $m = ik - 1 (i > 0)$.

The ring $\mathbb{Z}_n \cong H^{m+1} (\pi; \pi_m)$ is called the classifying ring for the tree $\text{HT} (\pi, m)$
and $\partial : \mathbb{Z}_n^* \to \mathbb{Z}$, the classifying homomorphism. For a
study of the existence of classifying rings and homomorphisms for infinite groups, see [7, 8].

In order to ease the way for applications to homotopy trees, section 4 gives necessary and sufficient conditions for \( p \in \mathbb{Z}_n^* \) to be in the kernel of \( \partial : Z_n^* \to \tilde{K}_0 \mathbb{Z} \); section 5 studies the cancellation of realizable \( \pi \)-modules. Section 6 collects the pertinent results showing how much of the homotopy theory of \((\pi, m)\)-complexes can be described via algebraic \( m \)-types, while section 7 describes the roots of many of the trees with finite periodic fundamental group.

Section 8 computes isomorphism classes of algebraic \( m \)-types. Applications of the above to homotopy trees \( HT(\pi, m) \) are given in sections 9 and 10. For example, we prove the following theorem (10.1).

**Theorem.** Let \( \pi \) be a finite periodic group of order \( n \) and minimal free period \( k \). Furthermore, suppose that \( \pi \) is either abelian or that 8 does not divide the order of \( \pi \). Let \( R_m \) be a \((\pi, m)\)-complex of minimal directed Euler characteristic.

(a) Let \( m = ik \) or \( m = ik - 2 \neq 2 \) \((i > 0)\) and \( X \) be any \((\pi, m)\)-complex. Then \( X \) has the homotopy of the sum \( R_m \lor \alpha S^m \) of \( R_m \) with \( \alpha = (\chi(X) - 1) \) copies of the \( m \)-sphere \( S^m \).

(b) Let \( m = ik - 1 \) \((i > 0)\) and \( Y \) be any \((\pi, ik - 1)\)-complex whose absolute Euler characteristic \( |\chi(Y)| > 0 \). Then \( Y \) has the homotopy type of \( R_{ik - 1} \lor \beta S^{ik - 1} \) with \( \beta = |\chi(Y)| \).

(c) Let \( Aut_k \pi = \{ p \in \mathbb{Z}_n^* \mid \exists \alpha \in Aut \pi \exists \alpha_k^*(1) = p \text{ where } \alpha_k^*: H^k(\pi; \mathbb{Z}) \to H^k(\pi; \mathbb{Z}) \} \). The set of homotopy classes \( HT(\pi, ik - 1)_0 \) of \((\pi, ik - 1)\)-complexes with Euler characteristic zero is isomorphic to the group

\[
HT(\pi, ik - 1)_0 \cong \ker \{ \partial : Z_n^* \to \tilde{K}_0 \mathbb{Z}_n^* \} / (Aut_k \pi)^i.
\]

Thus the trees of homotopy types have the appearance of Fig. 1 below.

For example, if \( n \) is odd, the dihedral groups \( D_{2n} \) of order \( 2n \) satisfy the hypotheses of the theorem. For \( \pi = \mathbb{Z}_n^* \), the theorem gives a complete classification of the homotopy trees \( HT(\mathbb{Z}_n^*, t) \) \((t > 2)\). The roots of \( HT(\mathbb{Z}_n^*, \text{odd}) \) are the homotopy...
classes of the standard lens spaces and (c) suitably translated gives the classical homotopy classification of lens spaces [20,22,29]. \( HT(\mathbb{Z}_n, 2) \) was previously known (see [3] for \( n \) prime, [9 I] for arbitrary \( n \)).

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1. The Swan–Wall class and topological m-type

All CW-complexes in this section are connected and of finite type; i.e., the \( m \)-skeleton \( X^{(m)} \) is a finite complex for all \( m \geq 0 \).

**Definition.** Two CW-complexes \( X, Y \) have the same topological \( m \)-type \( (X \simeq_m Y) \) if one of the following (equivalent) statements is true:

(a) there are maps

\[ f : X^{(m+1)} \to Y^{(m+1)} : g \]

such that \( g \circ f \mid X^{(m)} \simeq (X^{(m)} \hookrightarrow X^{(m+1)}) \) and

\[ f \circ g \mid Y^{(m)} \simeq (Y^{(m)} \hookrightarrow Y^{(m+1)}) \; ; \]

(b) there exists a map \( f : X^{(m+1)} \to Y^{(m+1)} \) such that \( \pi_i f : \pi_i(X) \to \pi_i(Y) \) is an isomorphism for \( i \leq m [30 I] \). The maps \( f \) and \( g \) are called \( m \)-equivalences.

We recall from §0 that to each complex \( X \) with \( (m-1) \)-connected there is an associated algebraic \( m \)-type \( T(X) = (\pi_1(X), \pi_m(X), k(X)) \). It is known from [13] that two such complexes \( X, Y \) whose universal covers are \( (m-1) \)-connected have the same \( m \)-type if and only if \( T(X) \cong T(Y) \). It is also known from [30 II] that every abstract \( m \)-type \( T = (\pi, \pi_m, k) \) can be realized by a connected \( (m+1) \)-dimensional complex \( Y \) (not necessarily of finite type) in the sense that \( T(Y) \cong T \).

Let \( C(\pi) \) denote the abelian monoid of stable equivalence classes of finitely generated left \( \pi \)-modules. Thus, two finitely generated \( \pi \)-modules \( A \) and \( B \) represent the same element in \( C(\pi) \) if and only if

\[ A \oplus (\mathbb{Z} \pi)^\alpha \cong B \oplus (\mathbb{Z} \pi)^\beta \]

for suitable integers \( \alpha, \beta \). If \( [A] \) denotes the equivalence class of \( A \) in \( C(\pi) \), then \( [A] + [B] = [A \oplus B] \). \( \tilde{K}_0 \mathbb{Z} \pi \), the reduced projective class group of the integral group ring \( \mathbb{Z} \pi \), is the maximal subgroup of \( C(\pi) \). It is known that there are non-zero elements \( [P] \in \tilde{K}_0 \mathbb{Z} \pi \) such that \( [Z] + [P] = [Z] \), where \( Z \) is the trivial \( \pi \)-module.

See section 5. Thus \( C(\pi) \) is not a cancellation monoid.

If \( f : \pi_1 \to \pi_2 \) is a group homomorphism, then

\[ C(f) : C(\pi_1) \to C(\pi_2) \]
is defined by
\[ C(f)([M]) = [\mathbb{Z}\pi_2 \otimes_{\mathbb{Z}\pi_1} M] \in C(\pi_2) \]
for any \([M] \in C(\pi)\). This makes \(C\) into a functor from the category \(\mathcal{G}\) of groups to the category \(\mathcal{M}\) of abelian monoids.

**Definition.** For each CW complex \(X\) of finite type, we define the \textit{Swan–Wall class of \(X\) in dimension \(m\)}, \(SW_m\) \([X]\), as the class of the \(\pi\)-module \(C_m(\tilde{X})/B_m(\tilde{X})\) in \(C(\pi)\). Here \(C_m(\tilde{X})\) is the \(m\)th cellular chain complex of the universal cover \(\tilde{X}\) of \(X\) and \(B_m(\tilde{X}) = im\{C_{m+1}(\tilde{X}) \to C_m(\tilde{X})\}\).

In this section it will be shown that \(SW_m\) \([X]\) depends only on the \(m\)-type of \(X\). To be precise, let \(m \geq 2\). We say that the Swan–Wall \(SW_m\) \([X]\) class of a complex \(X\) is an \textit{invariant} of the topological \(m\)-type of \(X\) if given any complex \(Y\) and \(m\)-equivalence
\[ f: X^{(m+1)} \to Y^{(m+1)} \]
the isomorphism
\[ C(f_{\#}): C(\pi_1(X)) \to C(\pi_1(Y)) \]
induced by \(f_{\pi}: \pi_1(X) \to \pi_1(Y)\) carries \(SW_m\) \([X]\) \(\to SW_m\) \([Y]\).

1.1. **Theorem.** Let \(m \geq 2\). Suppose that \(X\) is a CW complex of finite type. Then the Swan–Wall class of \(X\) is an invariant of the topological \(m\)-type of \(X\).

Note that 1.1 implies that if \(X \simeq_m Y\), then \(SW_m\) \([X]\) \(\in \tilde{\mathbb{K}}_0Z\pi_1(X)\) iff \(SW_m\) \([Y]\) \(\in \tilde{\mathbb{K}}_0\mathbb{Z}\pi(Y)\).

The above theorem is a corollary to the following theorem of W. Cockcroft and R. Swan, which also implies the uniqueness of \(SW_m\) \([T]\) for algebraic \(m\)-types \(T = (\pi, \pi_m, k)\).

1.2. **Theorem** [3, appendix, corollary]. Let \(\pi\) be any group; \(C\) and \(C'\) be two finitely generated projective chain complexes of \(\mathbb{Z}\pi\)-modules each having length \(\leq m + 1\). Let \(f: C \to C'\) be a chain map inducing isomorphisms \(f_\ast: H_i(C) \to H_i(C')\) for \(i \leq m\). Thus we have the following commutative diagram:

\[
\begin{array}{cccccc}
C: 0 & \to & C_{m+1} & \xrightarrow{\partial_{m+1}} & C_m & \xrightarrow{\partial_m} & C_{m-1} & \to & \cdots & \xrightarrow{\partial_1} & C_0 & \to & 0 \\
| & f_{m+1} & & f_m & & f_{m-1} & & & & f_0 & & & \\
C': 0 & \to & C'_{m+1} & \xrightarrow{\partial'_{m+1}} & C'_m & \xrightarrow{\partial'_m} & C'_{m-1} & \to & \cdots & \xrightarrow{\partial'_1} & C'_0 & \to & 0
\end{array}
\]

where each \(C_i (C'_i)\) is a finitely generated projective \(\mathbb{Z}\pi\)-module. Then there exists a \(\pi\)-isomorphism
where \( B_m = \text{im} \partial_{m+1} \).

**Proof of 1.1.** Let \( f : X^{(m+1)} \to Y^{(m+1)} \) be a cellular \( m \)-equivalence, where \( Y \) has finite type. Let
\[
\tilde{f}_* : C_*(\tilde{X}^{(m+1)}) \to C_*(\tilde{Y}^{(m+1)})
\]
be the induced chain map on the cellular chain complexes. \( f \) is an \( m \)-equivalence implies that
\[
\tilde{f}_* : C_*(\tilde{X}^{(m+1)}) \to C_*(\tilde{Y}^{(m+1)})
\]
satisfies the hypotheses of the Cockcroft–Swan theorem. Here
\[
\theta = f_{1#} : \pi_1(X) \to \pi_1(Y).
\]
Thus in \( C(\pi_1(X)) \),
\[
[C_m/B_m] = [C_m(\tilde{X}^{(m+1)})/B_m(\tilde{X}^{(m+1)})]
\]
\[
= \left[ \{C_m(\tilde{Y}^{(m+1)})/B_m(\tilde{Y}^{(m+1)})\}_\theta \right] \equiv \left[ \{C'_m/B'_m\}_\theta \right].
\]
It is easy exercise in the definition of
\[
C(\theta) : C(\pi_1(X)) \to C(\pi_1(Y))
\]
to show that
\[
C(\theta) ([C'_m/B'_m]_\theta) = [C'_m/B'_m] = \text{SW}_m[Y].
\]

Note the similarity between 1.2 and Schanuel's theorem of [23, §1].

In order to clarify somewhat the use of the Swan–Wall invariant for \( m \)-types and the obstruction to finiteness of [28], we offer the next proposition.

1.3. **Proposition.** Let \( X \) be a CW-complex such that the Hurewicz homomorphism \( h : \pi_m(\tilde{X}^{(m)}) \to H_m(\tilde{X}^{(m)}) \) is a monomorphism. The complex \( X \) has \( \text{SW}_m[X] \in \tilde{K}_0\mathbb{Z}\pi_1(X) \) iff \( X \) has the \( m \)-type of one dominated by a finite complex of dimension \( m \).

Note that the hypothesis is satisfied if \( \tilde{X} \) is \( (m - 1) \)-connected.

**Sketch of a Proof.** By hypothesis \( Q = C_m(\tilde{X})/B_m(\tilde{X}) \) is a finitely generated projective module. We conclude that \( P = B_m(\tilde{X}) \) and \( R = H_{m+1}(\tilde{X}^{(m+1)}) \) are also finitely generated projective modules. Note that \( P \oplus Q \cong C_m(\tilde{X}) \) and \( P \oplus R \cong C_{m+1}(\tilde{X}) \). We build a CW complex \( \tilde{X} \) of finite type starting with the \((m + 1)\)-skeleton \( \tilde{X}^{(m+1)} = X^{(m+1)} \) so that \( \tilde{X} \) has the following chain complex:
$C_*(\tilde{X}): \cdots \xrightarrow{q_R} C_{m+1}(\tilde{X}) \xrightarrow{q_P} C_{m+1}(\tilde{X}) \xrightarrow{q_R} C_{m+1}(\tilde{X})$

$\delta_{m+1} \xrightarrow{} C_m(\tilde{X}) \xrightarrow{\delta_m} \cdots \xrightarrow{\delta_2} C_0(\tilde{X})$,

where the alternating homomorphisms $q_R, q_P : C_{m+1}(\tilde{X}) \to C_{m+1}(\tilde{X})$ denote the natural retractions onto $R, P \subset C_{m+1}(\tilde{X})$. The hypothesis that $h$ is monic allows such an $\tilde{X}$ to be constructed. Clearly $\tilde{X} \simeq mX$. But $\tilde{X}$ also satisfies property $D_m$ of [28]. This together with the fact that $SW_m[\tilde{X}] \in \tilde{K}_0Z\pi_1(\tilde{X})$ implies that $\tilde{X}$ is dominated by a finite complex of dimension $m$ [28, Theorem F].

For example, any space $X$ having $SW_2[X] \in \tilde{K}_0Z\pi_1(X)$ has the 2-type of a complex dominated by a finite 2-complex.

1.4. Proposition. Let $m \geq 3$. Suppose that $X$ has the property that the universal cover $\tilde{X}$ is $(m-1)$-connected. Then $X$ has the $m$-type of a finite $m$-complex iff its Swan--Wall class $SW_m[X] = 0$.

In this case, we say that the topological $m$-type of $X$ is $m$-realizable.

Proof. If $X \simeq mY$, where $Y$ is a finite $m$-complex, then by 1.1, $SW_m[Y] = SW_m[X] = 0$. Suppose that $SW_m[X] = 0$. Let $\pi = \pi_1(X)$ and choose an integer $s$ such that $C_m(\tilde{X})/B_m(\tilde{X}) \oplus \pi^s \cong (\pi^s)^t$.

Then the chain complex

$C : C_m(\tilde{X})/B_m(\tilde{X}) \oplus (\pi^s)^s \xrightarrow{\partial_m \oplus \text{id}} C_{m-1}(\tilde{X}) \oplus (\pi^s)^s$

$\delta_m \downarrow \quad \downarrow \quad \downarrow$

$C_{m-2}(\tilde{X}) \quad \delta_m^{m-2} \quad \cdots$

is free. Because $m \geq 3$, the $(m-1)$-skeleton $C^{(m-1)}$ is realizable by the complex $X^{(m-1)} \cup sS^{m-1}$. Build an $m$-complex

$Y = (X^{(m-1)} \cup sS^{m-1}) \cup \bigcup_{i=1}^t e_i^m$

by attaching $m$-cells to mimic the map $\delta_m : (\pi^s)^s \rightarrow C_{m-1}(\tilde{X}) \oplus (\pi^s)^s$. This can be done because each element of $\ker \delta_m$ is spherical by hypothesis. We claim that $Y \simeq mX$. This follows because $X$ and $Y$ have the same algebraic $m$-type. □
We may now define the Swan–Wall class $\text{SW}_m\{[T]\}$ of an isomorphism class $\{T\}$ of an algebraic $m$-type $T = (\pi, \pi_m, k)$ provided $T$ is realizable by an $(m + 1)$-dimensional finite complex $Y (T(Y) \cong T)$. Let $\text{Aut} \pi$ act in the natural way as a group of automorphisms on $C(\pi)$ and $\overline{C}(\pi) = C(\pi)/\text{Aut} \pi$ be the monoid of orbits. Note that the maximal subgroup of $\overline{C}(\pi)$ is $K_0 Z/\text{Aut} \pi$. If $(\theta, \theta') : T(Y) \to T$ is an isomorphism of $m$-types, then $\text{SW}_m\{[T]\} = \text{class of } C(\theta)(\text{SW}_m\{Y\})$ in $\overline{C}(\pi)$. This is well-defined by 1.1. In the cases under consideration for most of this paper (a finite) that orbit is a single element in $C(\pi)$ (see the proof of 2.6).

2. The Swan–Wall class of an algebraic $m$-type

Let $\pi$ be a finite group of order $n$. Choose a free resolution of the trivial $\pi$-module $Z$:

$$\mathcal{R}(\pi) : \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} Z \to 0,$$

where each $C_i$ is a finitely generated free module. $\mathcal{R}(\pi)$ is called the reference resolution. In this section we will compute the Euler characteristic of certain partial projective resolutions and show that it depends only on the $m$-type of the resolution.

For the resolution $\mathcal{R}(\pi)$, let $\pi_m = \ker \partial_m$. Any such $\pi$-module $\pi_m$ is called realizable. For $m \geq 3$, $\pi_m$ is realizable implies that $\pi_m \cong \pi_m(X)$ for some $(\pi, m)$-complex $X$. This follows from [23, Proposition 3.1]. For $m = 2$, it is unknown whether $\pi_2$ is realizable implies $\pi_2 \cong \pi_2(X)$ for some 2-complex $X$.

Let

$$\mathcal{P}(m) : 0 \to \pi_m' \to P_m \xrightarrow{p_m} P_{m-1} \xrightarrow{p_{m-1}} \cdots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} Z \to 0,$$

be a partial resolution with each $P_i$ finitely generated and projective. Since $\pi$ is finite, we have $H^{m+1}(\pi; \pi_m') \cong H^0(\pi; Z) \cong Z$. To see this, one uses the fact that $H^i(\pi; P) = 0$ for all $i > 0$ and any projective $\pi$-module $P$ [2, p. 199]. Using the partial resolution $\mathcal{P}(m)$ and the short exact sequences

$$0 \to \ker p_i \to P_i \to \text{im } p_i \to 0,$$

we obtain the isomorphism $\alpha_i : \tilde{H}^i(\pi, \text{im } p_i) \to \tilde{H}^{i+1}(\pi, \ker p_i)$. Then $\alpha(\mathcal{P}) = \alpha_0 \cdot \alpha_1 \cdot \cdots \cdot \alpha_m : H^0(\pi; Z) \to H^{m+1}(\pi, \pi_m')$ is the desired isomorphism.

The $m$-type $T(\mathcal{P}) = (\pi, \pi_m', k)$ is the triple consisting of $\pi$, the $\pi$-module $\pi_m'$, and the cohomology class $k \in H^{m+1}(\pi; \pi_m')$ determined as follows: Choose any chain map

$$f : \mathcal{P}(m) \to \mathcal{P}(m)$$

$(\mathcal{P}(m))$ is the $m$-skeleton of $\mathcal{R}(\pi))$

inducing the identity on $Z$. Thus $f$ is a sequence of maps:
\[ \cdots \to C_{m+2} \xrightarrow{\partial_{m+2}} C_{m+1} \xrightarrow{\partial_{m+1}} C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} Z \to 0. \]

\( f_m \cdot \partial_{m+1} \) has image in \( \pi'_m = \ker p_m \cdot (f_m \cdot \partial_{m+1}) \cdot \partial_{m+2} = 0 \) implies that \( f_m \cdot \partial_{m+1} \) represents an element \( k = \{f_m \cdot \partial_{m+1}\} \in H^{m+1}(\pi; \pi'_m) \). \( k \) is independent of the choice of \( f \) because any other \( f' \) is homotopic to \( f \) \([10]\). It follows from Theorem 6.3 of \([10]\) that

\[ \alpha(\mathcal{P}(m)) : H^0(\pi; Z) \xrightarrow{\cong} H^{m+1}(\pi; \pi'_m) \]

carries \( 1 \to k \). Thus each such \( k \) is a generator of \( H^{m+1}(\pi; \pi'_m) \).

If \( \pi'_m = \pi_m = \ker \partial_m \), we may identify \( k \) with an integer modulo \( n \). Consider

\( g = f_m|_{\pi_m} : \pi_m \to \pi_m \) induced by the chain map \( f : \mathcal{R}(m) \to \mathcal{D}(m) \). This induces an isomorphism

\[ g_* : H^{m+1}(\pi, \pi_m) \to H^{m+1}(\pi, \pi_m), \]

thus there exists a unit \( p \in \mathbb{Z}'_n \) such that \( g_*(i) = p \cdot i \) for any \( i \in \mathbb{Z}_n \). We identify \( k \) with \( p = g_*(1) \). This follows because the following diagram commutes:

Thus \( k = \alpha(\mathcal{P}(m))(1) = g_* \alpha(\mathcal{R}(m)) (1) = g_*(1) \). Here we identify \( k(\mathcal{R}(m)) \) with \( 1 = \alpha(\mathcal{R}(0m)) (1) \) in \( H^{m+1}(\pi; \pi_m) \).

We formalize the notion of degree as follows: Suppose \( A_m \) is any \( \pi \)-module such that \( H^{m+1}(\pi; A_m) = \mathbb{Z}_n \). We say that a \( \pi \)-homomorphism \( f : A_m \to A_m \) has degree \( p \in \mathbb{Z}_n \) if the induced homomorphism \( f_* : H^{m+1}(\pi; A_m) \to H^{m+1}(\pi; A_m) \) has degree \( p \).

**Definition.** A projective partial resolution

\[ \mathcal{P}_p^{(m)} : 0 \to \pi_m \to P_m \to \cdots \to P_1 \to P_0 \to Z \to 0 \]

realizes the \( m \)-type \((\pi, \pi_m, p)\) if, with respect to the reference resolution \( \mathcal{R}(\pi) \),

\[ k(\mathcal{P}_p) = p \in \mathbb{Z}_n'. \]
Let $p$ be an integer prime to $n$ and $N = \Sigma_{x \in \pi} x$ be the norm element in $\mathbb{Z}_n$. Let $(p, N)$ denote the projective left ideal in $\mathbb{Z}_n$ generated by $p$ and $N$. $(p, N)$ is projective by [23, Lemma 6.1]. If $p'$ is congruent to $p$ modulo $n$, then $(p', N) \cong (p, N)$ [23, 6.1 (b)]. Thus for any $p \in \mathbb{Z}_n^*$, we will abuse the notation and write $(p, N)$.

Since $(p, N) \oplus (p', N) \cong (pp', N) \oplus \mathbb{Z}_n$, the correspondence $p \rightarrow [(p, N)]$ defines a homomorphism

$$\nu : \mathbb{Z}_n^* \rightarrow \tilde{K}_0\mathbb{Z}_n.$$ 

Let

$$\begin{array}{ccc}
\mathbb{Z}_n & \xrightarrow{pr} & \mathbb{Z}_n/(N) \\
\downarrow{\epsilon} & & \downarrow{\epsilon'} \\
\mathbb{Z} & \xrightarrow{pr'} & \mathbb{Z}_n
\end{array}$$

be the square of ring homomorphisms with $pr$ and $pr'$ the natural projections. Consider the reduced Mayer–Vietoris sequence in algebraic $K$-theory due to Milnor [16]:

$$K_1\mathbb{Z}_n \rightarrow K_1\mathbb{Z} \oplus K_1\mathbb{Z}_n/(N) \rightarrow K_1\mathbb{Z}_n \rightarrow \tilde{K}_0\mathbb{Z}_n \oplus \tilde{K}_0\mathbb{Z}_n/(N) \rightarrow \tilde{K}_0\mathbb{Z}_n.$$ 

It follows without difficulty from the definition of $\partial$ in [16] that $\partial = \nu$.

The following lemma of Swan has proved very useful for the present work (see 5.3, 3.2 as well as [23, §2]). Furthermore, an alternate proof of 1.1 in the case that $\pi$ is finite and $SW_m[X]$ is a member of $\tilde{K}_0\mathbb{Z}_n$ can be based on this lemma.

2.1. Lemma (Swan [23]). Let $\pi$ be a finite group and

$$0 \rightarrow M' \otimes P \rightarrow M \rightarrow M'' \rightarrow 0$$

be an exact sequence of $\pi$-modules with $P$ projective and $M''$ free as an abelian group. Then $P$ is a direct summand of $M$.

We are now ready for the existence theorem for $m$-types $(\pi, \pi_m, p)$.

2.2. Theorem. Let $\pi_m$ be a realizable $\pi$-module. For each $p \in \mathbb{Z}_n^*$, $(\pi, \pi_m, p)$ can be realized by a partial projective resolution whose Euler characteristic is $-[(p, N)]$ $\in \tilde{K}_0\mathbb{Z}_n$.

Proof. Let $A(\pi)$ be the augmentation ideal. We will show (2.4) that for each $p \in \mathbb{Z}_n^*$ there is an isomorphism

$$\alpha = \alpha(p) : A(\pi) \otimes (p, N) \rightarrow A(\pi) \otimes \mathbb{Z}_n$$

of degree $p^{-1}$. Assuming this, consider the exact sequence

$$0 \rightarrow A(\pi) \xrightarrow{i} \mathbb{Z}_n \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.$$
Adding on a copy of $\mathbb{Z}_n$ gives

$$0 \to A(\pi) \oplus \mathbb{Z}_n \xrightarrow{i \oplus \text{id}} (\mathbb{Z}_n)^2 \xrightarrow{e + 0} Z \to 0.$$  

Then use $\alpha$ to obtain

$$0 \to A(\pi) \oplus (p, N) \xrightarrow{i \cdot \alpha} (\mathbb{Z}_n)^2 \xrightarrow{e + 0} Z \to 0.$$  

By the lemma of Swan (2.1) $\bar{\alpha}((p, N))$ is a direct summand of $(\mathbb{Z}_n)^2$. Let $\rho : (\mathbb{Z}_n)^2 \to (p, N)$ be the retraction $(\rho \circ \bar{\alpha}(p, N) = \text{id})$ and $P$ the complementary summand. Since

$$(p, N) \oplus (p^{-1}, N) \cong (\mathbb{Z}_n)^2 \cong (p, N) \oplus P$$

we observe that $[P] = [(p^{-1}, N)] \in K_0 \mathbb{Z}_n$. Thus the following is exact

$$0 \to A(\pi) \xrightarrow{\beta \cdot (\bar{\alpha}(A(\pi)))} P \xrightarrow{e'} Z \to 0$$

where $\beta : (\mathbb{Z}_n)^2 \to P$ is the natural projection and $e' = (e + 0) \frac{1}{P}$.

**2.3. Lemma.** $\mathcal{P}$ realizes the 0-type $(\pi, A(\pi), p)$, $p \in \mathbb{Z}_n^*$.  

**Proof.** Let $0 \to A(\pi) \to \mathbb{Z}_n \to Z \to 0$ be the reference resolution and consider the following composite chain map:

$$0 \to A(\pi) \xrightarrow{i} Z \xrightarrow{e} Z \to 0$$

$$0 \to A(\pi) \oplus \mathbb{Z}_n \xrightarrow{i} (\mathbb{Z}_n)^2 \xrightarrow{e + 0} Z \to 0$$

$$0 \to A(\pi) \oplus (p, N) \xrightarrow{\bar{\alpha}} (\mathbb{Z}_n)^2 \xrightarrow{e + 0} Z \to 0$$

Clearly $\bar{\beta}(\text{proj.}) = \beta \cdot \bar{\alpha}$. Thus the chain map induces a map $A(\pi) \to A(\pi)$ which factors as
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\[
A(\pi) \oplus Z\pi \xrightarrow{\alpha} A(\pi) \oplus (p, N) \xrightarrow{\text{proj.}} A(\pi).
\]

This clearly has degree \(p\).

Now let

\[
\mathcal{R}(m)(\pi) : 0 \to \pi_m \to C_m \to C_{m-1} \to \cdots \to C_1 \to C_0 \xrightarrow{\partial_0} Z \to 0
\]

be the reference resolution. If the rank of \(C_0 = 3\), then \(A(\pi) = \ker \partial_0\) and the resolution \(\mathcal{R}_p\) is given by

\[
\mathcal{R}_p : 0 \to \pi_m \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} p \xrightarrow{e'} \to Z \to 0
\]

where \(\partial'_1(c) = \tilde{\partial} \cdot \partial_1(c)\) (\(c \in C_1\)).

If \(\text{rank } C_0 \geq 2\), then by using Schanuel's lemma [23, section 1] and (*) there is an isomorphism \(\alpha' : \ker \partial_0 \oplus Z\pi \oplus (p, N) \to \ker \partial_0 \oplus Z\pi \oplus Z\pi\) of degree \(p^{-1}\). Then use \(\alpha'\) and apply the above construction to

\[
0 \to \ker \partial_0 \to C_0 \xrightarrow{\partial_0} Z \to 0
\]

to obtain a resolution \(\mathcal{P}\) realizing \((\pi, \ker \partial_0, p)\)

\[
\mathcal{P} : 0 \to \ker \partial_0 \xrightarrow{\tilde{\partial}} P \xrightarrow{e'} Z \to 0
\]

where \([P] = [(p^{-1}, N)]\) in \(\tilde{K}_0 Z\pi\). Then \(\mathcal{R}_p\) is given as above.

By alternately adding on \((p, N)\) to \(C_{\text{even}}\) and \((p^{-1}, N)\) to \(C_{\text{odd}}\) this partial resolution can be transformed into

\[
\mathcal{R}^{(m)}_p : 0 \to \pi_m \to C_m \oplus Q \to C'_m \to \cdots \to C'_1 \to C'_0 \to Z \to 0
\]

where \([Q] = (-1)^{m+1} [(p, N)]\) and each \(C'_i\) is a free module. This completes the proof of 2.2 modulo (*).

\[\square\]

2.4. Lemma. For each \(p \in \mathbb{Z}_n^*\), there is an isomorphism

\[
\alpha_p : A(\pi) \oplus (p^{-1}, N) \to A(\pi) \oplus Z\pi
\]

of degree \(p\).

\[\text{Proof}\]. Consider the exact sequence

\[
0 \to A(\pi) \xrightarrow{\beta} (p', N) \xrightarrow{e'} Z \to 0
\]

where \(p' \in p^{-1}\), \(e' = e|_{(p', N)}\), and \(\beta : A(\pi) \to p'A(\pi)\) is multiplication by \(p'\). Choose
integers $r, s$ so that $rp' + sn = 1$. Consider the chain map

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A(n) & \longrightarrow & Z\pi & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow g & & \downarrow f & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(n) & \longrightarrow & (p', N) & \longrightarrow & Z & \longrightarrow & 0
\end{array}
\]

where $f(1) = rp' + sN$ and $g(\alpha) = r\alpha (\alpha \in A(n))$. Then for any $\alpha \in A(n), f(\alpha) = \alpha rp' + c(\alpha)sN = \alpha rp' = \beta g(\alpha)$. Therefore the degree of $g$ is $[p] = p \in\mathbb{Z}_n^*$. Now apply Schanuel’s Lemma [23, § 1] to the two exact sequences to obtain an isomorphism $\alpha_p : A(n) \oplus (p^{-1}, N) \rightarrow A(n) \oplus Z\pi$

such that $g = q \circ \alpha_p \circ i$, where $i : A(n) \rightarrow A(n) \oplus (p^{-1}, N)$ is the natural inclusion and $q : A(n) \oplus Z\pi \rightarrow A(n)$ is the projection.

Now we state the uniqueness theorem for the Swan–Wall class of $(\pi, \pi_m, p)$.

**2.5. Theorem.** If $\mathcal{R}_p^{(m)}$ realizes $(\pi, \pi_m, p)$, then the Euler characteristic

$$\chi(\mathcal{R}_p^{(m)}) = -[(p, N)].$$

Thus the correspondence $p \rightarrow \chi(\mathcal{R}_p^{(m)})$ defines a homomorphism $\nu(\pi, \pi_m) : \mathbb{K}_n^* \rightarrow \tilde{K}_0\mathbb{Z}\pi$. In fact, this correspondence depends only upon the isomorphism class of $(\pi, \pi_m, p)$.

**2.6. Proposition.** If $(\pi, \pi_m, p) \cong (\pi, \pi'_m, p')$, then

$$\nu(\pi, \pi_m)(p) = \nu(\pi, \pi'_m)(p').$$

**2.7. Corollary.** The correspondence $p \rightarrow -\nu(\pi, \pi_m)(p)$ $(p \in\mathbb{Z}_n^*)$ is the boundary operator $\partial : K_1\mathbb{Z}\pi \rightarrow \tilde{K}_0\mathbb{Z}\pi$ in the Milnor Mayer–Vietoris sequence.

**Proof of 2.5.** Let $^{i}\mathcal{R}_p^{(m)} : 0 \rightarrow \pi_m \rightarrow ^{i}P_m \rightarrow ^{i}C_{m-1} \rightarrow \cdots \rightarrow ^{i}C_0 \rightarrow Z \rightarrow 0$ denote any two resolutions realizing $(\pi, \pi_m, p)$ $(i = 1, 2)$ such that each $^{i}C_j$ is free and $^{i}P_m$ projective. By standard obstruction arguments, there is a chain map $f : ^{1}\mathcal{R}_p^{(m)} \rightarrow ^{2}\mathcal{R}_p^{(m)}$

covering the identity on $Z$ and inducing the identity on $\pi_m$. Thus, it follows from 1.2 [3, appendix, Corollary 4] that $\left[^{1}P_m\right] = \left[^{2}P_m\right] \in \tilde{K}_0\mathbb{Z}\pi$.

\qed
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**Prop 2.6.** Let \((\theta, \theta') : (\pi, \pi_m, p) \to (\pi, \pi'_m, p')\) be the isomorphism. Realize each \(m\)-type by partial projective resolutions \(\mathcal{P}\) and \(\mathcal{P}'\). Standard arguments on the \(k\)-invariant [10, Theorem 4] show that there is a \(\theta\)-chain map \(f : \mathcal{P} \to \mathcal{P}'\) such that \(f_m|\pi_m = \theta'\). The proof of 1.1 shows that

\[
C(\theta)(\chi(\mathcal{P})) = \chi(\mathcal{P}') \in \tilde{K}_0\pi.
\]

By 2.2, \(\chi(\mathcal{P}) = -[(p, N)]\), \(\chi(\mathcal{P}') = -[(p', N)]\). Thus

\[
C(\theta)\left([(p, N)]\right) = [(p', N)].
\]

We will be done once we show: \(C(\theta)\left([(p, N)]\right) = [(p, N)]\) for all \(\theta \in \text{Aut} \pi\), all \(p \in \mathbb{Z}_n^\ast\). This follows because

\[
C(\theta)\left([(p, N)]\right) = [(Z\pi)_\theta \oplus_{\pi} (p, N)] = [(p, N)_{\theta^{-1}}].
\]

But it is clear that \((p, N) \cong (p, N)_{\theta^{-1}}\) as \(\pi\)-modules. \(\square\)

Notice that the argument that \([(p, N)]\) is fixed under the action of \(\text{Aut} \pi\) on \(C(\pi)\) generalizes to the following situation: Let \(M\) be any finitely generated \(\pi\)-module (\(\pi\) not necessarily finite) such that, for each \(\theta \in \text{Aut} \pi\), there exists an \(\theta\)-automorphism \(M \to M\). Then \([M] \in C(\pi)^{\text{Aut} \pi}\).

**3. \(p\)-stability**

Let \(\pi\) be a finite group of order \(n\). A \(\pi\)-module \(A_m\) such that \(H^{m+1}(\pi; A_m) = \mathbb{Z}_n\) is called \(p\)-stable (of height \(\leq i\)) if there exists an isomorphism

\[
\alpha(p) : A_m \oplus (p^{-1}, N) \oplus (Z\pi)^i \to A_m \oplus (Z\pi)^{i+1} \quad (i \geq 0)
\]

of degree \(p \in \mathbb{Z}_n^\ast\). \(A_m\) is called stable if \(A_m\) is \(p\)-stable for all \(p \in \mathbb{Z}_n^\ast\). For example, the trivial \(\pi\)-module \(Z\) [23, Lemma 6.1] and the augmentation ideal \(A(\pi)\) (?4) are stable (with height 0).

For finite groups, this definition is quite general. It follows from 1.2 and 2.1 that if \(M\) is any \(\pi\)-module such that \(\alpha : \pi_m \otimes M \to \pi_m \otimes (Z\pi)^i\)

then the degree of \(\alpha = p \in \mathbb{Z}_n^\ast\) and \(M\) is projective. Furthermore, an easy consequence of 3.1 and 2.5 is that

\[
[M] = \pm [(p, N)] \text{ in } \tilde{K}_0\pi.
\]

Note that \(\pi_m\) is \(p\)-stable iff \(\pi_m\) is \(p^{-1}\)-stable iff \(\pi_m\) is \((-p)\)-stable [23, Lemma 6.1].

**3.1. Proposition.** If \(\pi_m\) is a realizable \(\pi\)-module, then \(\pi_m\) is stable. If \(\pi_m\) is \(p\)-stable and realizable, then the \(m\)-type \((\pi, \pi_m, p)\) is realizable by a projective resolution.
Proof. The first statement follows from Schanuel's Lemma and 2.5. The second is proved in [23, section 2]. \end{proof}

3.2. Conjecture. Let $\pi_m$ be a realizable $\pi$-module and $p \in \mathbb{Z}^*_n$. There is an isomorphism

$$\alpha : \pi_m \oplus (p(-1)^{m+1}N) \to \pi_m \oplus \mathbb{Z}\pi$$

of degree $p$. In other words, each realizable $\pi_m$ is stable with height zero.

As we have seen, this conjecture is true for $Z$ and $A(\pi)$. We show that the ring $\mathbb{Z}\pi/(N)$ also is stable with height zero. If $p \in \mathbb{Z}^*_n$, we say that $p$ is $s$-free if $[(p, N)] = 0$ in $\tilde{K}_0\mathbb{Z}\pi$.

3.3. Corollary. Let $\pi_m$ be realizable and $p \in \mathbb{Z}^*_n$. $p$ is $s$-free iff there exists an integer $i \geq 0$ and an automorphism

$$\pi_m \oplus (\mathbb{Z}\pi)^i \to \pi_m \oplus (\mathbb{Z}\pi)^i$$

of degree $p$.

Proof. Let $p$ be $s$-free. 3.1 implies there exists an

$$\alpha : \pi_m \oplus (p(-1)^{m+1}N) \oplus (\mathbb{Z}\pi)^i \to \pi_m \oplus (\mathbb{Z}\pi)^{i+1} (i \geq 0).$$

If $[(p, N)] = 0$, then $\beta : (p^{i+1}N) \oplus \mathbb{Z}\pi \mathbin{\xrightarrow{\cong}} (\mathbb{Z}\pi)^2$. The composite

$$\pi_m \oplus (\mathbb{Z}\pi)^i \oplus (\mathbb{Z}\pi)^2 \xrightarrow{id \oplus id \oplus \beta^{-1}} \pi_m \oplus (\mathbb{Z}\pi)^i \oplus (p(-1)^{m+1}N) \oplus \mathbb{Z}\pi$$

then has degree $p$. If such an automorphism $\pi_m \oplus (\mathbb{Z}\pi)^i \to \pi_m \oplus (\mathbb{Z}\pi)^i$ exists of degree $p$, then, using the Swan construction of [23, \S 2], we may realize $(\pi, \pi_m, p)$ by a free resolution. It follows from the uniqueness Theorem 2.5 that $p$ is $s$-free. \end{proof}

For an example which is useful for periodic groups, we show that 3.2 is true for $\mathbb{Z}\pi/(N)$.

3.4. Proposition. For each $p \in \mathbb{Z}^*_n$, there is an isomorphism

$$\alpha_p : \mathbb{Z}\pi/(N) \oplus (p^{-1}N) \to \mathbb{Z}\pi/(N) \oplus \mathbb{Z}\pi$$

of degree $p$.

Proof. Choose integers $s \in p$ and $q \in p^{-1}$ and consider the isomorphism

$$\tau : (s, N) \oplus (q, N) \cong (\mathbb{Z}\pi)^2$$
of [23, Lemma 6.1]. Represent $(s, N)$ abstractly as the left $\pi$-module generated by $u_1, v_1$ with the single relation $Nu_1 = sv_1$. Similarly $(q, N)$ has generators $u_2, v_2$. Then \( \tau(v_1) = (v_1^1, v_2^1, N), \tau(v_2) = (v_1^2, v_2^2, N) \) where \( v_q \in \mathbb{Z} \) and the determinant \( u_1^1v_2^2 - v_1^2v_1^2 = \pm 1 \). By diagonalizing this integer matrix using only row operations we obtain a new basis for $(Z\pi)^2$ with respect to which

\[
\tau(v_1) = (N, 0), \quad \tau(v_2) = (0, \pm N).
\]

Thus \( \tau \) induces an isomorphism

\[
\tilde{\tau} : (s, N)/(N) \oplus (q, N) \to Z\pi/(N) \oplus Z\pi.
\]

Now \( \tau(u_1) = (\alpha_1, \beta_1) \) with \( \epsilon_1, \beta_1 \in \mathbb{Z} \). \( \tau(sv_1, 0) = \tau(Nu_1, 0) = N(\alpha_1, \beta_1) = (\epsilon(\alpha_1) N, \epsilon(\beta_1) N) = s(N, 0) \). Thus \( \epsilon(\alpha_1) = s \). But

\[
g : Z\pi/(N) \to (s, N)/(N)
\]

where \( g(1 + (N)) = u_1 + (N') \) is an isomorphism. Thus \( \alpha = \tilde{\tau} (g \oplus \text{id}) \) gives an isomorphism of degree \( p \). \( \Box \)

Finally, using the notion of stability we may characterize those algebraic $m$-types which are realizable by $(\pi, m)$-complexes.

3.5. Theorem. Let \( \pi \) be a finite group of order \( n \) and \( \pi_m \) be a realizable \( \pi \)-module. If \( m \geq 0 \) and \( p \in \mathbb{Z}^* \), then \( (\pi, \pi_m, p) \) is realizable by a free complex iff \( [(p, N)] = 0 \). If \( m \geq 3 \), \( (\pi, \pi_m, p) \) is the $m$-type of a $(\pi, m)$-complex iff \( [(p, N)] = 0 \).

Proof. The first statement follows from 2.2. To prove the second statement, we must be more careful. By a remark at the beginning of section 2, we may assume that the reference resolution

\[
\mathcal{R}^{(m)}(\pi) : 0 \to \pi_m \to C_m \xrightarrow{\delta m} \cdots \to C_0 \to Z \to 0
\]

is the cellular chain complex of the universal cover \( \tilde{X} \) of a $(\pi, m)$-complex \( X \). By 3.1, \( \pi_m \) is $p$-stable. Again by means of the construction of Swan [23, § 2] we may use the isomorphism \( \pi_m \oplus (Z\pi)^i \to \pi_m \oplus (Z\pi)^i \) of degree $p^{-1}$ to construct a new resolution

\[
\mathcal{R}_p^{(m)} : 0 \to \pi_m \to C'_m \xrightarrow{\delta m} C_{m-1} \oplus (Z\pi)^i \to C_{m-2} \to \cdots.
\]

Since \( m \geq 3 \), \( C_{m-1} \in (Z\pi)^i \to C_{m-2} \to \cdots \to C_0 \to Z \to 0 \) is realizable by \( C_*(X^{(m-1)} \vee iS^{m-1}) \). Then one defines the $(\pi, m)$-complex

\[
X_p = (X^{(m-1)} \vee iS^{m-1}) \cup \bigcup_{j=1}^{k} e_j^m
\]
obtained by attaching $k = \text{rank } C'_m$ cells of dimension $m$ mimicking the homomorphism $\partial'_m : C'_m \to C'_{m-1} \cong (Z\pi)^{\ell}$. If $X$ is a $(\pi, m)$-complex such that $T(X) \cong (\pi, p, m, p)$, then $[(p, N)] = 0$ by 2.6.

4. Stably free modules $(p, N)$

This section is devoted to the study of necessary and sufficient conditions that the Swan projectives $(p, N)$ be (stably) free. Recall the following conventions: $\pi$ is a finite group of order $n$. An integer $p \in \mathbb{Z}_n^*$ is called free if the projective left ideal $(p, N)$ is free. $p$ is called s-free if $(p, N)$ is stably free. Let $\epsilon' : Z\pi/(N) \to Z_n$ be given by $\epsilon'(\alpha + (N)) = \epsilon(\alpha) + n\mathbb{Z}$ for each $\alpha \in \mathbb{Z}\pi$. By $A \cong pA$ we mean there is an automorphism $A \to A$ of degree $p$.

4.1. Proposition. The following are equivalent for $p \in \mathbb{Z}_n^*$.

(a) $p$ is free,
(b) $Z\pi/(N) \cong p Z\pi/(N)$,
(c) $Z \oplus Z\pi \cong p Z \oplus Z\pi$,
(d) $A(\pi) \cong p A(\pi)$.

Proof. (a) $\Rightarrow$ (b) follows from Lemma 6.3 of [23].
(a) $\Rightarrow$ (c) is a consequence of Lemmas 6.3 and 6.1 (b) of [23].
(c) $\Rightarrow$ (a): Let $h : Z \oplus Z\pi \to Z \oplus Z\pi$ be an automorphism of degree $p$. $(Z \oplus Z\pi)^\pi = Z \oplus (N)$ implies $h(Z \oplus (N)) \subset Z \oplus (N)$. In fact, $h' = h|Z \oplus (N)$ is bijective because $h$ is an automorphism. Thus we have the commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
Z \oplus (N) & \overset{h'}{\longrightarrow} & Z \oplus (N) \\
\downarrow & & \downarrow \\
Z \oplus Z\pi & \overset{h}{\longrightarrow} & Z \oplus Z\pi \\
\downarrow & & \downarrow \\
Z\pi/(N) & \overset{\tilde{h}}{\longrightarrow} & Z\pi/(N) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
\]

The induced map $\tilde{h} : Z\pi/(N) \to Z\pi/(N)$ is an isomorphism. Hence, if $[\alpha] = \alpha + (N)$ for any $\alpha \in Z\pi$, then $\tilde{h}[1] = \kappa$ is a unit in $Z\pi/(N)$. We will show that $pe'(k) = \pm 1$. Because $\epsilon'(k)$ is free we have $p$ is free. Under $h : (1, 0) \to (p', rN), (0, 1) \to (s, k')$, where $p' \in p, r, s \in \mathbb{Z},$ and $k' \in Z\pi$ is any preimage of $k$ under the projection $Z\pi \to Z\pi/(N)$. Let $\epsilon(k') = q$. Notice that $h(0, N) = (qs, \epsilon(k')N)$. The integer matrix of $h'$ with respect to the basis \{(1, 0), (0, N)\} is
Hence $p'q - rsn = \pm 1$. This completes (c) $\Rightarrow$ (a).

(b) $\iff$ (d) is left to the reader. 

4.2. Proposition. Let $p \in \mathbb{Z}_n^*$. The following statements are equivalent:

(a) $p$ is $s$-free,
(b) $(p, N) \oplus \mathbb{Z}_n \cong (\mathbb{Z}_n)^2$,
(c) $\mathbb{Z}_n/(N) \oplus \mathbb{Z}_n \cong_p \mathbb{Z}_n/(N) \oplus (\mathbb{Z}_n)^s \cong_p \mathbb{Z}_n/(N) \oplus (\mathbb{Z}_n)^r$ ($s \geq 1$),
(d) $Z \oplus (\mathbb{Z}_n)^2 \cong_p Z \oplus (\mathbb{Z}_n)^2$, 
(e) $Z \oplus (\mathbb{Z}_n)^t \cong_p Z \oplus (\mathbb{Z}_n)^t$ ($t \geq 2$),
(f) $A(\pi) \oplus \mathbb{Z}_n \cong_p A(\pi) \oplus \mathbb{Z}_n$, 
(g) $A(\pi) \oplus (\mathbb{Z}_n)^u \cong_p A(\pi) \oplus (\mathbb{Z}_n)^u$ ($u \geq 1$).

A string of equivalence: like this deserves a conjecture.

4.3. Conjecture. Let

$$\delta(m) = \begin{cases} 
0 & \text{if } m \text{ even}, p \in \mathbb{Z}_n^*; \\
1 & \text{if } m \text{ odd}.
\end{cases}$$

and $\pi_m$ be realizable of minimal $\mathbb{Z}$-rank. Define $(\mathbb{Z}_n)^0 = 0$.

(a) $p$ is free iff $\pi_m \oplus (\mathbb{Z}_n)^{6(m)} \cong_p \pi_m \oplus (\mathbb{Z}_n)^{6(m)}$,
(b) $p$ is $s$-free iff $\pi_m \oplus (\mathbb{Z}_n)^{6(m)+1} \cong_p \pi_m \oplus (\mathbb{Z}_n)^{6(m)+1}$.

Answering this question in some generality seems the next important step in the determination of the homotopy trees for $(\pi, m)$-complexes. For example, if $\pi$ has free period 4 and

$$0 \rightarrow Z \rightarrow \mathbb{Z}_n \rightarrow C_2 \rightarrow C_1 \rightarrow \mathbb{Z}_n \rightarrow 0$$

is a free periodic resolution, is 4.3 true for ker $\partial_1$? This is the only remaining case for such groups.

Also, as we shall see in section 9, the following question has significance in the determination of the homotopy trees: Does there exist a finite (periodic) group $\pi$ for which some $(p, N)$ is stably free but not free?

5. The cancellation property

Throughout this section $\pi$ denotes a finite group of order $n$. A finitely generated, $\mathbb{Z}$-torsion free $\pi$-module $M$ has the cancellation property (CP) iff for any $\pi$-module $M'$ such that $M' \oplus (\mathbb{Z}_n)^{\alpha} \cong M \oplus (\mathbb{Z}_n)^{\beta}$ with $\alpha \leq \beta$, we have $M' \cong M \oplus (\mathbb{Z}_n)^{\beta - \alpha}$. We say that $\pi$ has CP$_0$ iff the 0-module has CP.
In this section we show that if \( \pi_m = \pi_m(X) \) with \( X \) a \((\pi, m)\)-complex and \( m \) even, then \( \pi_m \oplus \mathbb{Z}\pi \) has CP. Furthermore, we show that if \( \pi \) has \( \text{CP}_0 \), then the trivial module \( \mathbb{Z} \), the augmentation ideal \( A(\pi) \), and the ring \( \mathbb{Z}\pi/(N) \) all have CP.

The chief technical condition is the \textit{Eichler condition}: A finitely generated \( \pi \)-module \( M \) satisfies the Eichler condition iff the semisimple \( \mathbb{Q} \)-algebra (\( \mathbb{Q} \) is the rational numbers) \( \text{End}_{\mathbb{Q}}(QM) \) has no simple component which is a totally definite quaternion algebra over its center (see [25, p. 176] for a definition). \( \pi \) satisfies the \textit{Eichler condition} provided the \( \pi \)-module \( \mathbb{Z}\pi \) does. It follows from [25, Theorem 19.8] that if \( \pi \) satisfies the Eichler condition, then \( \pi \) has \( \text{CP}_0 \). Such groups as finite abelian groups, finite simple groups and groups of odd order satisfy the Eichler condition [25, p. 178].

The following corollary to the theorem of Jacobinski [11, 25] was pointed out to me by Swan [see 9 II for a proof].

**5.1. Proposition.** Let \( X \) be a \((\pi, m)\)-complex such that \( (-1)^m \chi(X) \geq 1 \). Then the \( \pi \)-module \( \pi_m(X) \oplus \mathbb{Z}\pi \) has the cancellation property. \( \square \)

As an example, let \( G \) be a finite abelian group. A simple calculation using Theorem 1.1 of [24] shows that any \((G, m)\)-complex \( X \) has \( (-1)^m \chi(X) \geq 1 \). Thus any such \( \pi_m(X) \oplus \mathbb{Z}G \) has the cancellation property. The next corollary generalizes this example. Let \( s_m(\pi) \) be the minimal number of generators of the group \( H^m(\pi; \mathbb{Z}) \).

**5.2. Corollary.** Let \( X \) be any \((\pi, m)\)-complex. If \( m \) is even or if \( s_m(\pi) \geq 2 \), then \( \pi_m(X) \oplus \mathbb{Z}\pi \) has the cancellation property; if \( m \) is odd, then \( \pi_m(X) \oplus (\mathbb{Z}\pi)^2 \) has the cancellation property. \( \square \)

For a moment, let us study the \( \pi \)-modules \( A(\pi) \) and \( \mathbb{Z}\pi/(N) \). Using a combination of Swan’s Lemma (2.1) and the argument of [9 II], the following proposition is shown.

**5.3. Proposition.** \( A(\pi) \oplus \mathbb{Z}\pi \) has the cancellation property; if \( \pi \) has \( \text{CP}_0 \), then \( A(\pi) \) has CP.

Thus, for example, if \( \pi \) is a finite abelian group, \( A(\pi) \) has the cancellation property.

**Proof.** Let \( A = A(\pi) \) and

\[
0 \to A \xrightarrow{i} \mathbb{Z}\pi \xrightarrow{\epsilon} \mathbb{Z} \to 0
\]

be the exact sequence of \( \pi \)-modules, where \( \epsilon \) is the augmentation. Suppose \( \alpha : A \oplus (\mathbb{Z}\pi)^{\alpha_1} \xrightarrow{\alpha} M \oplus (\mathbb{Z}\pi)^{\alpha_2} \) (\( \alpha_1 \geq \alpha_2 \)) is an isomorphism. Consider the exact sequence
Using \( \alpha \), obtain the exact sequence

\[
0 \to M \oplus (\mathbb{Z}\pi)^{\alpha_2} \xrightarrow{1 + \alpha} (\mathbb{Z}\pi)^{1 + \alpha_1} \xrightarrow{(e,0,\ldots,0)} \mathbb{Z} \to 0.
\]

\( \mathbb{Z}\pi \) is a projective \( \pi \)-module \( \Rightarrow \mathbb{Z}\pi \) is weakly injective ([2], chapter XII, prop. 1.1) \( \Rightarrow (\mathbb{Z}\pi)^{\alpha_2} \) is a direct summand of \( (\mathbb{Z}\pi)^{1 + \alpha_1} \) (2.1). If \( \pi \) has \( \text{CP}_0 \) or if \( \alpha_1 > \alpha_2 \), then the complementary summand to the above splitting is also free. The second case follows from the cancellation theorem of Bass [1]. Thus there is a new basis for \( (\mathbb{Z}\pi)^{1 + \alpha_1} \) such that \( l \cdot \alpha \) equals:

\[
(*) \quad 0 \to M \oplus (\mathbb{Z}\pi)^{\alpha_2} \xrightarrow{\bar{\alpha} \oplus \text{id}} (\mathbb{Z}\pi)^{1 + \alpha_1 - \alpha_2} \oplus (\mathbb{Z}\pi)^{\alpha_2} \xrightarrow{(\bar{e},0)} \mathbb{Z} \to 0.
\]

So

\[
0 \to M \xrightarrow{\bar{\alpha}} (\mathbb{Z}\pi)^{1 + \alpha_1 - \alpha_2} \xrightarrow{\bar{e}} \mathbb{Z} \to 0
\]

is exact. The idea is to change the basis in \( C \) so that the "matrix" of \( \bar{e} \) with respect to the new basis in \( C \) is

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

This will show that \( M \cong A \oplus (\mathbb{Z}\pi)^{\alpha_1 - \alpha_2} \). Let \( x_1, \ldots, x_m \) \( (m = 1 + \alpha_1 - \alpha_2) \) be the given basis for \( C \). Then \( \bar{e}(x_i) = \beta_i \in \mathbb{Z} \) \( (i = 1, \ldots, m) \). \( \bar{e} \) is surjective \( \Rightarrow \exists \gamma_i \in \mathbb{Z} \) \( (i = 1, \ldots, m) \) \( \sum \gamma_i \beta_i = 1 \). It follows that there is a unimodular matrix over \( \mathbb{Z} \) which transforms the basis \( \{x_1, \ldots, x_m\} \) to a basis \( \{\bar{x}_1, \ldots, \bar{x}_m\} \) having the above matrix for \( \bar{e} \). If \( \alpha_1 = \alpha_2 \), then a similar argument works by cancelling \( (\mathbb{Z}\pi)^{\alpha_1 - 1} \) from sequence (*).

By combining the argument of [3] with 5.3, we have a new proof of the fact that \( \text{HT}(\mathbb{Z}\pi, 2) = \langle S^1 \cup_n e^2 \rangle \), [9 1].

By using an argument dual to 5.3, one may easily show the following proposition.

**5.4. Proposition.** \( \mathbb{Z}\pi/(N) \oplus \mathbb{Z}\pi \) has the cancellation property; if \( \pi \) has \( \text{CP}_0 \), then \( \mathbb{Z}\pi/(N) \) has the cancellation property.

For the rest of this section we will study the cancellation of the trivial \( \pi \)-module \( \mathbb{Z} \). The non-cancellation set for \( \mathbb{Z} \oplus \mathbb{Z}\pi \) will be completely determined in [34].
5.5. Proposition. Let $Z$ be the trivial $\pi$-module. The isomorphism

$$Z \oplus (Z\pi)^i \cong M \oplus (Z\pi)^j \quad (i \geq j)$$

implies that

(a) $M \cong Z \oplus (Z\pi)^i$ provided $i \neq j - 1$

(b) $Z \oplus (Z\pi) \cong M \oplus Z\pi$ if $i = j - 1$.

In case (b), $M \cong Z \oplus P$, where $P$ is a projective $\pi$-module of rank 1. Furthermore, there is an integer $p$ prime to the order of $\pi$ such that

$$[P] = [(p, N)] \in \tilde{K}_0Z\pi :$$

i.e., $[P]$ is in the image of $\partial : Z^n_* \to \tilde{K}_0Z\pi$.

5.6. Corollary. If $\pi$ satisfies the Eichler condition, then $Z$ has the cancellation property.

The corollary follows because $Z \oplus Z\pi$ has CP by [11]. The proposition follows easily from Lemmas 4.7 and 4.2 of [23].

It might be tempting to conjecture that $P$ must be stably free, but this is false, as the isomorphism

$$Z \oplus (p, N) \cong Z \oplus Z\pi$$


We note finally that recent work of Williams [32] has shed some light on the cancellation properties of the relation module $\pi_1$.

6. The homotopy theory of $(\pi, m)$-complexes

Recall that a $(\pi, m)$-complex is a finite, connected $m$-dimensional CW-complex $X$ such that

(a) $\pi_1(X) = \pi$

(b) $\pi_i(X) = 0, \quad 1 < i < m$.

To each connected CW-complex $X$ having $\tilde{X}(m - 1)$-connected and $\pi_1(X) = \pi$ we associate its algebraic $m$-type $T(X) = (\pi, \pi_m(X), k(X))$, where $k(X) \in H^{m+1}(\pi, \pi_m(X))$ is the obstruction invariant of [13]. Each map $f : X \to Y$ of $(\pi, m)$-complexes induces a homomorphism $f_* : T(X) \to T(Y)$ of their algebraic $m$-types. Basic references for $m$-types are [13], [29], and [30].

Let $X, Y$ be $(\pi, m)$-complexes. The following Theorems 6.1-6.4 are known [13] and show that most of the homotopy theory of $(\pi, m)$-complexes can be carried out algebraically.

6.1. Theorem [13, Theorem 3]. Any homomorphism $\varphi : T(X) \to T(Y)$ is realized by a map $f : X \to Y$ such that $f_* = \varphi$. 


6.2. **Theorem** [13, Theorem 1]. \( X \cong Y \) if and only if \( T(X) \cong T(Y) \).

6.3. **Theorem.** Let \( f, g : X \rightarrow Y \) be maps.

(a) If \( f \cong g : X \rightarrow Y \), then \( f_* = g_* : T(X) \rightarrow T(Y) \).

(b) If \( f_* = g_* : T(X) \rightarrow T(Y) \), then there exists \( \sigma(f, g) \in H^m_{\pi_1(X), (\pi_m(Y))_{f_*}} \)

with \( \sigma(f, g) = 0 \iff f \cong g \).

**Definition.** Let \( [X, Y]_\theta \) denote the set of homotopy classes of maps \( X \rightarrow Y \) inducing \( \theta : \pi_1(X) \rightarrow \pi_1(Y) \).

6.4. **Theorem.** The following sequence of sets is exact

\[ 0 \rightarrow H^m(\pi, \pi_m(Y)_\theta) \rightarrow [X, Y]_\theta \rightarrow \text{Hom}_\theta(T(X), T(Y)) \rightarrow 0 \]

in the sense that there is an action of \( H^m(\pi, \pi_m(Y)_\theta) \) on \([X, Y]_\theta\) whose orbits consist of classes \([f] \in [X, Y]_\theta\) that induce a given homomorphism \( \varphi : T(X) \rightarrow T(Y) \).

6.3 and 6.4 are contained in [19, §27].

Let \( \xi[X, Y] \) be the subset of \([X, Y]\) consisting of homotopy equivalences.

6.5. **Corollary.** The following is an exact sequence of sets:

\[ 0 \rightarrow H^m(\pi, \pi_m(Y)_\theta) \rightarrow \xi(X, Y)_\theta \rightarrow \text{Iso}_\theta(T(X), T(Y)) \rightarrow 0. \]

6.6. **Theorem (Sieradski).** Let \( \pi \) be a finite group and \( X, Y \) be \((\pi, m)\)-complexes. Then

(a) \( \xi(X, Y) \cong \text{Iso}(T(X), T(Y)) \) (bijection)

(b) \( \xi(X) \cong \text{Aut}(T(X)) \) (group isomorphism).

Theorem 6.6. follows from 6.5 and the following lemma of A.J. Sieradski which, in turn, follows easily from the fact that \( H^i(\pi; F) = 0 \) for any finite \( \pi \), free \( \pi \)-module \( F \), and \( i > 0 \).

6.7. **Lemma.** Let \( \pi \) be any finite group. Let \( Y \) be a \((\pi, m)\)-complex and \( \theta : \pi \rightarrow \pi_1(Y) \) be an isomorphism. Then \( H^m(\pi, (\pi_m(Y))_\theta) = 0 \).

7. **Generalized lens spaces**

This section is devoted to an exposition of pertinent facts about the minimal roots for trees having periodic fundamental groups. We collect it here for use in later sections.
Definition. Let $X$ be a simply connected, $m$-dimensional, finite CW complex having the homotopy type of the $m$-sphere $S^m$ ($m \geq 3$). Suppose that a finite group $\pi$ acts freely and cell-wise on $X$ in the sense that if $g \in \pi$ and $e^a$ is a cell of $X$, then $g(e^a)$ is also a cell of $X$. We call the orbit complex $L = X/\pi$ a generalized lens space (GLS).

Each such $\pi$ is necessarily periodic [2, 17] of period $m + 1$. Any such $L$ is a $(\pi, m)$-complex and is a minimal root of $HT(\pi, m)$. The usual lens spaces

$$L(p : q_1, \ldots, q_t) = S^{2l-1}/Z_p \quad (p \geq 2, q_i \geq 1, (p, q_i) = 1)$$

are examples of generalized lens spaces. See [23, appendix] and [2, p. 253] for other examples. According to [23], there is a generalized lens space for any finite periodic group. For a complete classification of such groups, see [33, p. 179], [26].

Let $\pi$ be a finite group having periodic cohomology of minimal period $k$. A periodic projective resolution of period $h$ over the trivial $\pi$-module $Z$ is an exact sequence

$$E_h : 0 \to Z \to P_{h-1} \to P_{h-2} \to \cdots \to P_1 \to P_0 \to Z \to 0$$

where each $P_i$ is a finitely generated projective $\pi$-module. $E$ is free if each $P_i$ is free. If $E$ is free and $h \geq 4$, then there exists a generalized lens space $L(\pi, h - 1)$ as a root of the homotopy tree [23, Proposition 3.1]. Also it is known that there always exists a periodic projective resolution $E_k$ of period $k$, the minimal period of $\pi$ [23, Theorem 4.1].

However, it is not known whether a free periodic resolution of period $k$ exists. Let $mE_k$ denote the process of stringing together $m$ copies of $E_k$ [23] to obtain a periodic resolution of period $mk$. Let $d$ be the greatest common divisor of $m$, the order of $\pi$, and $\phi(n)$, where $\phi$ is the Euler $\phi$-function. Then there exists a periodic free resolution of period $d \cdot k$ [23, Theorem B].

Definition. We say that the minimal free period of $\pi$ is $p \cdot k$, the smallest multiple of the minimal period $k$ admitting a periodic free resolution of period $p \cdot k$. For example, $\Sigma_3$ and $\SigmaQ(4t)$ have minimal free period 4.

The following lemma is useful in studying the homotopy trees $HT(\pi, ik - 2)$.

7.1. Lemma. Let $\pi$ be a finite periodic group of (not necessarily minimal) free period $k$. There exists a $(\pi, k - 2)$-complex $X$ such that $\pi_{k-2}(X) \cong Z\pi/(N)$ as a $\pi$-module, with the possible exception of $k = 4$ and $\pi$ non-abelian. $X$ will be the $(k - 2)$-skeleton of an appropriate GLS and is necessarily a minimal root of $HT(\pi, k - 2)$.

The proof of 7.1 is an easy exercise in the use of the Lemma of Swan (2.1) that projective $\pi$-modules behave as injectives in certain settings.

If $\pi$ is non-abelian and $k = 4$, the existence of the 2-complex in 7.1 turns on
whether there are any finite groups of period 4 with negative deficiency (see [24], section 2 and Corollary 5.1 for a related discussion).

8. Isomorphisms between algebraic $m$-types

This section is devoted to the computation of the set

$$Q_m(\pi_m) = \{ p \in \mathbb{Z}_n^* | (\pi, \pi_m, 1) \equiv (\pi, \pi_m, p) \}$$

for certain $\pi$-modules $\pi_m$. Here, as usual, $n$ is the order of $\pi$.

We will assume that the following set is “known”. Its computation in many cases is not difficult. Let $\pi$ be a periodic of minimal period $k$ and let $\text{Aut}_{ik} \pi = \{ p \in \mathbb{Z}_n^* | \exists \theta \in \text{Aut} \pi \text{ such that } \theta(p) = 1, \text{ where } \theta_{ik} : H_{ik}(\pi; Z) \rightarrow H_{ik}(\pi; Z) \}$. It follows from [2, chapter XII, section 11] that $\text{Aut}_{ik} \pi = (\text{Aut}_k \pi)'$.

Examples

1. If $\pi = \mathbb{Z}_n$, the finite cyclic group of order $n$, then $\text{Aut}_{2i} \mathbb{Z}_n = (\mathbb{Z}_n^*)^i$.

2. If $\pi$ is the quaternion group $Q(8)$ of order 8, then

$$\text{Aut}_{4i} Q(8) = \{ 1 \}.$$  

3. If $\pi$ is the generalized quaternion group $GQ(4t)$ of order $4t \geq 16$, then

$$\text{Aut}_{4i} GQ(4t) = (\mathbb{Z}_4^*)^{2i}.$$  

4. If $\pi$ is the dihedral group $D_{2n}$ of order $2n$ (odd), then $\text{Aut}_{4i} D_{2n} = (\mathbb{Z}_2^*)^{2i}$ (see section 10).

Definition

(a) $F(\pi) = \{ p \in \mathbb{Z}_n^* | (p, N) \text{ is free} \}$.

(b) $SF(\pi) = \ker \{ \delta : \mathbb{Z}_n^* \rightarrow \mathbb{Q}_0 \mathbb{Z}_n \}$.

If follows from [23] that $p \in F(\pi)$ iff there is a unit $u \in \mathbb{Z}_n/(N)$ of augmentation $p$. Recall that $p \in \mathbb{Z}_n^*$ is free iff $p \in F(\pi)$.

8.1. Proposition. Let $\pi$ have minimal period $k$ and order $n$.

(a) If $m = sk - 1$ ($s > 0$) and $\pi_m = \mathbb{Z}$, then

$$Q_{sk-1}(Z) = _{\pi} \text{Aut}_{sk} \pi .$$

(b) $Q_{sk-1}(Z) \oplus (Z \pi) = \text{Aut}_{sk} \pi \cdot F(\pi)$.

(c) $Q_{sk-1}(Z) \oplus (Z \pi)^t = SF(\pi)$ ($t \geq 2$).

Proof. (a) If $\theta_{sk}^*(p) = 1$, then $zp \in Q_{sk-1}(Z)$ because $(\theta, \pm \text{id}) : (\pi, Z, 1) \rightarrow (\pi, Z, sp)$ gives the isomorphism. If $(\theta, \theta') : (\pi, Z, 1) \rightarrow (\pi, Z, p)$ is an isomorphism, then $\theta' = \pm \text{id}$ and thus $(\theta_{sk}^*)^{-1} \theta^*(p) = 1$. Thus $\theta^*(p) = \pm 1$, which implies that either $p$ or $-p$ is a member of $\text{Aut}_{sk} \pi$.

(b) and (c) will follow from the next lemma.
8.2. Lemma. Let \( \theta \in \text{Aut} \pi \) and \( t \) be a non-negative integer. Any \( \theta \)-automorphism \( \theta' : Z \oplus (Z\pi)^t \to Z \oplus (Z\pi)^t \) splits as the composite \( \varphi \circ Z(\theta) \) of a \( \pi \)-automorphism \( \varphi \) and a degree 1 \( \theta \)-automorphism \( Z(\theta) \).

Proof. Let \( \overline{Z(\theta)} = \text{id} \oplus Z(\theta) \oplus \cdots \oplus Z(\theta) : Z \oplus (Z\pi)^t \to Z \oplus (Z\pi)^t \). 

Let \( \varphi : Z \oplus (Z\pi)^t \to Z \oplus (Z\pi)^t \) be defined by \( \varphi(e_i) = \theta'(e_i) \), where \( e_i \) is the element of \( Z \oplus (Z\pi)^t \) having a 1 in the \( i \)th coordinate and zeros elsewhere. It follows that \( \varphi \) is an automorphism and that \( \theta' = \varphi \circ Z(\theta) \). To see the latter, let

\[
\sum_{i=1}^{t+1} \alpha_i e_i \in Z \oplus (Z\pi)^t \quad (\alpha_1 \in Z, \alpha_i \in Z\pi \text{ for } i \geq 2);
\]

\[
\varphi(\underbrace{Z(\theta)(\sum \alpha_i e_i)}_{t \text{ times}}) = \alpha_1 \varphi(e_1) + \sum_{i=1}^{t+1} (\theta)(\alpha_i) \theta'(e_i) = \theta'(\sum \alpha_i e_i).
\]

We see that if \( \text{Aut}(Z \oplus (Z\pi)^t) \) denotes the set of all \( \theta \)-automorphisms, \( \theta \in \text{Aut} \pi \), then, as a corollary to 8.2, the sequence \( 1 \to \text{Aut}_n(Z \oplus (Z\pi)^t) \to \text{Aut}(Z \oplus (Z\pi)^t) \to \text{Aut}(Z \oplus Z\pi) \to 1 \) is split exact. The splitting \( s : \text{Aut} \pi \to \text{Aut}(Z \oplus (Z\pi)^t) \) is such that the degree of \( s(\theta) = 1 \) for all \( \theta \in \text{Aut} \pi \). We call such a module \( Z \oplus (Z\pi)^t \) finely split. Let us finish the proof of 8.1 and then develop this theme further.

Proof of 8.1 (b). Let \( (\theta, \theta') : (\pi, Z \oplus Z\pi, 1) \to (\pi, Z \oplus Z\pi, p) \) be an isomorphism. By 8.2, \( \theta' = \varphi \circ Z(\theta) \), where \( Z(\theta) \) is a \( \theta \)-automorphism of degree 1 and \( \varphi \) is a \( \pi \)-automorphism. Thus \( (\theta, \theta') = (\text{id}, \varphi) \circ (\theta, Z(\theta)) \). By the Proposition 4.1 (c), the degree of \( \varphi \) is a member of \( F(\pi) \). So, letting \( \bar{\varphi} : (Z \oplus Z\pi)_\theta \to (Z \oplus Z\pi)_0 \) be the \( \pi \)-homomorphism induced by \( \varphi \), we have \( 1 = (\theta^*_{\pi})^{-1} \theta^*(p) = Z(\theta)^{-1} \bar{\varphi}_{\pi}^*(\theta^*(p)) = \varphi_{\pi}^{-1}(\theta^*(p)) \).

Thus, if \( \varphi_{\pi}(1) = \theta^*(p) = r, p \) is free and \( p = r \cdot (p/r) \) where \( \theta^*(p/r) = 1 \). Hence \( p/r = \text{Aut}_{sk} \pi \). Suppose \( p = q \cdot r \) where \( q \in \text{Aut}_{sk} \pi, r \in F(\pi) \). Then again by Proposition 4.1 there is an automorphism \( \varphi : Z \oplus Z\pi \to Z \oplus Z\pi \) of degree \( r \). Choose \( \theta \in \text{Aut} \pi \) so that \( \theta^*_{sk}(q) = 1 \). Then \( (\theta, \theta') = (\theta, \varphi \cdot Z(\theta)) \) is a member of \( F(\pi) \) is an automorphism of \( (\pi, Z \oplus Z\pi, 1) \to (\pi, Z \oplus Z\pi, p) \) because \( (\varphi \cdot Z(\theta))_{\pi}(1) = r = \theta^*_{sk}(p) \).

(c) If \( p \in Q_{sk-1}(Z \oplus (Z\pi)^t) \) then by the preceding argument (using Proposition 4.2, instead of 4.1) \( p = q \cdot r \), where \( q \in \text{Aut}_{sk} \pi \) and \( r \in F(\pi) \). But clearly \( \text{Aut}_{sk} \pi \subset \text{SF}(\pi) \) by 2.6. Thus \( p \in S_{\pi}(\pi) \). Then by 4.2 there is a \( \pi \)-automorphism \( \varphi : Z \oplus (Z\pi)^t \to Z \oplus (Z\pi)^t \) of degree \( p \) and \( (\text{id}, \varphi) \) gives the necessary isomorphism.

We return now to finely split modules. Let \( \pi_m \) be a realizable \( \pi \)-module. For each \( \theta \in \text{Aut} \pi \), there is a natural isomorphism

\[
\alpha(\theta) : H^{m+1}(\pi ; \pi_m) \to H^{m+1}(\pi ; (\pi_m)_0)
\]

included by the isomorphism
End_{\pi}(\pi_m) \approx End_{\pi}((\pi_m)_0)
given by carrying a \pi-homomorphism \varphi : \pi_m \to \pi_m to a \pi-homomorphism \tilde{\varphi} : (\pi_m)_0 \to (\pi_m)_0 such that \tilde{\varphi} = \varphi.

Letting Aut_{\pi} denote the group of all \theta-automorphisms of \pi_m \to \pi_m and Aut_{\pi}m
the group of \pi-automorphisms of \pi_m \to \pi_m, we have the following exact sequence:

$$1 \to \text{Aut}_{\pi} \pi_m \to \text{Aut}_{\pi} \pi_m \xrightarrow{h} \text{G}(\pi) \to 1 \quad (\text{G}(\pi) \subset \text{Aut}_{\pi})$$

where \( h (\theta-automorphism) \to \theta \). We say that \( \pi_m \) is finely split if \( \text{G}(\pi) = \text{Aut}_{\pi} \) and there is a splitting \( s : \text{Aut}_{\pi} \to \text{Aut}_{\pi} \pi_m \) (\( hs = \text{id} \)) such that

$$\alpha(\theta)^{-1} \cdot s(\theta)(1) = 1$$

where

$$s(\theta)^*: H^{m+1}_\pi(\pi, \pi_m) \to H^{m+1}_\pi(\pi, (\pi_m)_0)$$

is induced by the \( \pi \)-isomorphism \( s(\theta) : \pi_m \to (\pi_m)_0 \). Thus \( s(\theta) \) has “degree 1”.

In order to state the generalization of 8.1 that we seek, let

$$\text{Aut}_{\pi}^{m+1} \pi_m = \left\{ q \in Z_\pi^* \mid \exists \beta \in \text{Aut}_{\pi} \pi_m \text{ such that } \beta^*(1) = q \right\}$$

where \( \beta^* : H^{m+1}_\pi(\pi, \pi_m) \to H^{m+1}_\pi(\pi, \pi_m) \)

and

$$\text{Aut}_{\pi}^{m+1} \pi_m = \left\{ q \in Z_\pi^* \mid \exists \theta \in \text{Aut}_{\pi} \text{ such that } \alpha(\theta)^{-1} \cdot \theta^*(1) = q \right\},$$

where \( \theta^* : H^{m+1}_\pi(\pi, \pi_m) \to H^{m+1}_\pi(\pi, (\pi_m)_0) \)

is induced by \( \theta \).

\( \text{Aut}_{\pi}^{m+1} \pi_m \) is independent of the resolution chosen (hence independent of \( \pi_m \)).
\( \text{Aut}_{\pi}^{m+1} \pi_m \) is independent of the resolution, provided both end with \( \pi_m \).

8.3. Theorem. Let \( \pi_m \) be a finely split, realizable \( \pi \)-module. Then \( Q_m(\pi_m) = (\text{Aut}_{\pi}^{m+1} \pi_m) \cdot (\text{Aut}_{\pi}^{m+1} \pi_m) \).

Before giving the proof of 8.3, let us give several examples. If \( \pi \) is periodic of minimal free period \( k \), then \( \pi_{sk} \cong A(\pi) \) and \( \pi_{sk-2} \cong \mathbb{Z}/(N) \) \( (s > 0) \) are finely split. Furthermore, if \( \pi_m \) is finely split, then \( \pi_m \oplus (\mathbb{Z}/(N)) \) is finely split, for any \( i > 0 \).

Proof of 8.3. If \( q \in Q_m(\pi_m) \), then there is an isomorphism

$$\theta(\theta') : (\pi, \pi_m, 1) \to (\pi, \pi_m, q).$$

Write \( \theta' = \varphi \cdot s(\theta) \) and let \( r = \varphi_s(1) \in \text{Aut}_{\pi}^{m+1} \pi_m \). Then \( \theta_{m+1}^*(q) = \theta'_*(1) \) implies that
\[ a = \alpha(\theta)^{-1} \cdot \theta_{m+1}^* (q) = \alpha(\theta)^{-1} \cdot \theta_{*}^* (1) \]
\[ = \alpha(\theta)^{-1} \cdot (\varphi \cdot s(\theta))_{*} (1) \]
\[ = \alpha(\theta)^{-1} \cdot \tilde{\varphi}_{*} \cdot s(\theta)_{*} (1) \]

where \( \tilde{\varphi} : (\pi_m)_\theta \rightarrow (\pi_m)_\theta \) is the \( \pi \)-homomorphism defined by \( \varphi \).

\[ \therefore a = \alpha(\theta)^{-1} \cdot \tilde{\varphi}_{*} \cdot s(\theta)_{*} (1) \]
\[ = \varphi_{*} \cdot \alpha(\theta)^{-1} \cdot s(\theta)_{*} (1) \]
\[ = \varphi_{*} (1) = r . \]

Hence \( q = r \cdot q/r \), where \( r \in \text{Aut}_{m+1} \pi_m \) and \( q/r \in \text{Aut}_{m+1} \pi \). The converse is proved similarly.

8.4. Corollary. Let \( \pi \) have minimal free period \( k \) and let \( \pi_{sk} = A(\pi), \pi_{sk-2} = Z\pi/(n) \).

Then

- (a) \( Q_{sk}(A(\pi)) = \text{Aut}_{sk} \pi \cdot F(\pi) \)
- (b) \( Q_{sk}(A(\pi) \oplus (Z\pi)^t) = SF(\pi) \) \( (t \geq 1) \)
- (c) \( Q_{sk-2}(Z\pi/(n)) = \text{Aut}_{sk-1} \pi \cdot F(\pi) \)
- (d) \( Q_{sk-2}(Z\pi/(n) \oplus (Z\pi)^t) = SF(\pi) \) \( (t \geq 1) \).

We leave the proof to the reader.

One might think that in (c), \( \text{Aut}_{sk-1} \pi = \text{Aut}_{sk} \pi \), but this is false. Consider \( \pi = Z_n \).

By direct computation one may show that \( \text{Aut}_{2s-1} Z_n = \text{Aut}_{2s-2} Z_n = (Z_n^*)^{s-1} \) and \( \text{Aut}_{2s} Z_n = (Z_n^*)^s \). It would be interesting to know \( \text{Aut}_{sk-1} \pi \) (for periodic \( \pi \)) in terms of (say) \( \text{Aut}_{sk-2} \pi \).

9. Applications: Homotopy trees for periodic groups

In this section we will describe certain homotopy trees for periodic groups. Let \( \ast \) denote the class of \( \pi \)-modules \( \theta \)-isomorphic to \( Z \oplus Z\pi \) for some \( \theta \in \text{Aut} \pi \). Define the set \( NC_{\pi} Z \oplus Z\pi (\text{Aut} \pi) \) to be the set of \( (\text{Aut} \pi) \)-isomorphism classes of non-cancellation examples of the trivial \( \pi \)-module \( Z \oplus Z\pi \) together with the trivial element \( \ast \). Thus, a nontrivial element of \( NC_{\pi} Z \oplus Z\pi (\text{Aut} \pi) \) is an \( (\text{Aut} \pi) \)-isomorphism class of \( \pi \)-modules \( M \) such that

- (a) \( M \oplus Z\pi \cong Z \oplus (Z\pi)^2 \)
- (b) \( M \) is not \( \theta \)-isomorphic to \( Z \oplus Z\pi \) for any \( \theta \in \text{Aut} \pi \).
9.1. **Theorem.** Let \( \pi \) be a finite group with minimal free period \( k \) and \( L_i \) be a GLS which is a \((\pi, ik - 1)\)-complex \((i > 0)\). The tree \( HT(\pi, ik - 1) \) can be described as follows:

(a) If \( X \) is a \((\pi, ik - 1)\)-complex with absolute Euler characteristic \( |\chi(X)| \geq 2 \), then \( X \) has the homotopy type of the sum \( L_i \vee \alpha S^{ik-1} \) of the GLS \( L_i \) and \( \alpha = |\chi(X)| \) copies of the \((ik - 1)\)-sphere \( S^{ik-1} \).

(b) If \( |\chi(X)| = 1 \), then \( X \vee S^{ik-1} \simeq L_i \vee 2S^{ik-1} \).

(c) The set \( HT(\pi, ik - 1) \) of homotopy types of \((\pi, ik - 1)\)-complexes with Euler characteristic zero is in one-to-one correspondence with the group

\[ HT(\pi, ik - 1) \cong SF(\pi)/\pm \text{Aut}_k \pi. \]

(d) Let \( L_i(p) \) be the GLS corresponding to each \( p \in SF(\pi) \). Let \( HT(\pi, ik - 1)^N \) be the set of homotopy classes of non-roots with absolute Euler characteristic \( 1 \) (i.e., \( |\chi(X)| = 1 \) and \( X \simeq L_i(p) \vee S^{ik-1} \) for some \( p \in SF(\pi) \)). Then,

\[ HT(\pi, ik - 1)^N \cong SF(\pi)/\text{Aut}_k \pi \cdot F(\pi). \]

(e) Let \( HT(\pi, ik - 1)^R \) be the set of roots of absolute Euler characteristic \( 1 \). Then \( HT(\pi, ik - 1)^R \) is isomorphic to disjoint union of groups \( \bigcup_{M \in \pi} G(M) \), where each \( G(M) = SF(\pi)/Q_{ik-1}(Z \oplus P) \) for \( Z \oplus P \in M \).

**Note.**

(a) Let \( p, q \in SF(\pi) \). Then \( L_i(p) \simeq L_i(q) \) iff there is a \( \theta \in \text{Aut} \pi \) such that

\[ p = \pm(\theta_k^*(1))^{\dagger} \cdot q. \]

Similarly, \( L_i(p) \vee S^{ik-1} \simeq L_i(q) \vee S^{ik-1} \) iff there is a \( \theta \in \text{Aut} \pi \) and a \( p' \in F(\pi) \) such that

\[ q = (\theta_k^*(1))^{\dagger} \cdot p' \cdot p. \]

(b) Theorem 10.1 (c) states in part that the boundary homomorphism \( \partial : Z_n^* \rightarrow K_0Z\pi \) factors as follows:

\[ Z_n^* \xrightarrow{\partial} K_0Z\pi \]

\[ Z_n^*/\pm \text{Aut}_k \pi. \]

(c) At the present writing, we know no example of a finite group \( \pi \) for which

\[ \mathcal{C}(\pi) = SF(\pi)/F(\pi) \neq 0. \]

However, for \( \pi = GQ(32) \), the generalized quaternion group of order 32, it is known [34] that

\[ \mathcal{C}_{Z_{gZ}\pi}(\text{Aut} \pi) \neq * \]

modulo this unknown, the tree \( HT(\pi, ik - 1) \) looks like Fig. 2; where \( S = S^{ik-1} \) and the dashed lines represent the unsolved question.
Proof of 9.1.

(a) Let $X$ be any $(\pi, ik - 1)$-complex with absolute Euler characteristic greater than one. By Theorem 14 of [31], $\pi_{ik-1}(X) \oplus (Z\pi)^{\ell} \cong Z \oplus (Z\pi)^{\ell}$ for certain integers $r$, $t$. 5.5 (a) implies that $\pi_{ik-1}(X) \cong (Z\pi)^{\alpha} \oplus Z$. Thus $T(X) \cong (\pi, Z \oplus (Z\pi)^{\alpha}, p) = T_p$ for some $p \in Z_n^*$. But $p$ must be $s$-free by 3.5. $\gamma \geq 2$ and 4.2 (d) yield an automorphism $\gamma : Z \oplus (Z\pi)^{\alpha} \to Z \oplus (Z\pi)^{\alpha}$ of degree $p$. Thus $T_p \cong T_{ik}$ and hence $X \cong L_{ik}$. 

(b) Follows from 3.5, 5.5 and 4.2 as in (a).

(c) Suppose $\chi(X) = 0$. $\pi_{ik-1}(X) \oplus (Z\pi)^{\ell} \cong Z \oplus (Z\pi)^{\ell}$ implies (by 5.5 (b)) that $\pi_{ik-1}(X) \cong Z$. Thus $T(X) \cong (\pi, Z, p)$, where $p$ is $s$-free. Then

$$\text{HT}(\pi, ik - 1)_0 \cong SF(\pi)/\mathbb{Z} \text{Aut}_{ik},$$

follows from 8.1 (a), 3.5, and 6.2.

(d) Follows as in (c), except that one uses 8.1 (b).

(e) For each non-trivial non-cancellation example $M \in \mathcal{K}(\pi, Z \oplus Z\pi)$, we have $M \cong Z \oplus P$ where $P$ is a rank 1 projective ideal and $[P] = [(q, N)] \in K_0Z\pi$ for some $q \in Z_q^* [5.5]$. Each $(ik - 1)$-type $(\pi, Z \oplus P, p)$ is realizable as a $(\pi, ik - 1)$-complex iff $p \in q^{-1}SF(\pi)$. Thus

$$\text{HT}(\pi, ik - 1)_1^R \cong \bigcup_{M \in \mathcal{K}(\pi, Z \oplus Z\pi)} SF(\pi)/Q_{ik-1}(M).$$

Recall that, in a homotopy tree $\text{HT}(\pi, m)$, the stalk $\langle X \rangle$ generated by a $(\pi, m)$-complex $X$ is the subtree with vertices $\{[X], [X \vee S^m], \ldots, [X \vee nS^m], \ldots\}$.

9.2. Theorem. Let $\pi$ be a finite group of minimal free period $k$ and let $X_m$ be a minimal root for $\text{HT}(\pi, m)$. For $m - ik$ or $ik - 2$ $(i > 0, m \neq 2)$ the subtree $\text{HT}(\pi, m)^1$ of homotopy types of $(\pi, m)$-complexes with Euler characteristic greater
than \( \chi_{\text{min}} (= \text{minimum } \{ |X| \mid X \text{ is a } (\pi, m) \text{-complex} \} ) \) is a single stalk \( \langle X_m \vee S^m \rangle \) generated by \( X_m \vee S^m \). If \( m = 2 \) \((k = 4, i = 1)\) we must add the hypothesis that the 2-type \((\pi, \mathbb{Z}_n/(N), 1)\) is realizable by a 2-complex.

**Proof.** 9.2 is a consequence of 5.3, 5.4, and 4.2 (c), (e).

The following theorem is an example of the kind of application one can make. It is a generalization of a theorem of [12].

**9.3. Theorem.** Let \( \pi \) be a finite periodic group of minimal free period \( k \). For each \( p \in \text{SF}(\pi) \), let \( L_i(p) \) denote \( \pi \in (\pi, ik - 1) \)-lens space corresponding to \( p \). Then the set of homotopy equivalences \( \xi(L_i(p), L_i(q)) \) from \( L_i(p) \to L_i(q) \) is in one to one correspondence with \( \{ \alpha \in \text{Aut}(\pi) = \pm(\alpha_k^1(1)) \cdot p \} \).

**Proof.** Apply 6.6 and then study \( \text{Iso}_{ik-1}(\pi, Z, p, \pi, Z, q) \).

10. Finite periodic groups satisfying Eichler’s condition

In this section we prove the theorem of the introduction. Let \( GQ(4n) \) be the group of order \( 4n \) (of minimal free period four) having presentation \( \{ a, b : a^n = b^2, ba = a^{-1}b \} \) (see [2, chapter XIII] and [18]). If \( n \) is a power of two, then \( GQ(4n) \) is a generalized quaternion group. Let \( D_{2n} \) be the dihedral group of order \( 2n \) having presentation \( \{ x, y : y^2, xy = x^{n-1}y \} \). It is well known that \( D_{2n} \) is periodic iff \( n \) is odd.

Let \( \mathcal{C} = \{ GQ(4 \cdot 2^i) \mid i = 1, 2, \ldots \} \cup \{ P_{24}, P_{48}, P_{120} \} \) where \( P_{24} \) is the binary tetrahedral group, \( P_{48} \) the binary octahedral group, \( P_{120} \) the binary icosahedral group. According to [25, p. 178], a finite group \( \pi \) satisfies the Eichler condition (section 5) if \( \pi \) does not admit any surjection onto any group in the class \( \mathcal{C} \). For example, if \( \pi \) is finite abelian or \( 8 \) does not divide the order of \( \pi \), then \( \pi \) satisfies the Eichler condition. [26] contains a convenient listing of the finite periodic groups.

If \( \pi \) satisfies the Eichler condition and is periodic of minimal free period \( k \), then we can give very simple descriptions of the homotopy trees \( HT(\pi, m) \) for \( m = ik \), \( ik - 1, ik - 2 \) \((i > 0)\). Let \( L_i \) denote any GLS which is a \((\pi, ik - 1)\)-complex such that the \((ik - 2)\)-skeleton \( K_i \) of \( L_i \) has \( \pi_{ik-2}(K_i) \cong \mathbb{Z}_n/(N) \). This is always possible (7.1) if \( \pi \) is abelian or, when \( \pi \) is non-abelian, \( ik - 2 \neq 2 \). Let \( M_i = L_i \cup \phi^{ik} \), where \( \phi : S^{ik-1} \to L_i \) has degree 1.

**10.1. Theorem.** Let \( \pi \) have minimal free period \( k \), order \( n \), and suppose that \( \pi \) satisfies the Eichler condition.

(a) The homotopy tree \( HT(\pi, ik - 1) \) \((i > 0)\) may be described as follows:

(i) If \( X \) is a \((\pi, ik - 1)\)-complex such that the absolute Euler characteristic \( |\chi(X)| > 0 \), then \( X \cong L_i \vee |\chi(X)|S^{ik - 1} \).

(ii) The set of roots \( HT(\pi, ik - 1)_0 \cong F(\pi)/\pm(\text{Aut}_k(\pi))^j \).
(b) The homotopy tree $\text{HT}(\pi, ik) \ (i > 0)$ is a single stalk $\langle M_i \rangle$ generated by the root $M_i$; i.e., if $Y$ is any $(\pi, ik)$-complex, $Y \simeq M_i \vee \omega S^i k$ for $\alpha = \chi(Y) = 1$.

(c) If $\pi$ is abelian or, for $\pi$ non-abelian, $ik - 2 \neq 2$, $\text{HT}(\pi, ik - 2)$ is a single stalk $\langle K_i \rangle$ generated by $K_i$ If $\pi$ is non-abelian and $ik - 2 = 2$, the same is true, provided we assume the existence of $K_i$.

Proof. If $X$ is a $(\pi, ik - 1)$-complex, then $T(X) \cong \langle A_\pi \oplus (Z\pi)^\alpha, \rho \rangle$ where $\rho \in SF(\pi) = F(\pi)$ and $\alpha = |\chi(X)|$. If $\alpha \geq 1$, then 9.1 together with $\text{SF}(\pi) = F(\pi)$ assures that $T(X) \cong T(L_i \vee \omega S^{ik-1})$. If $\alpha = 0$, the 8.1 (a) yields $\text{HT}(\pi, ik - 1) \cong \langle \text{S}F(\pi)/\pm \text{Aut}_{ik}\pi \rangle$.

(b) and (c) follow similarly from 8.4 and the fact that $A(\pi)$ and $Z\pi(N)$ have CP. □

Theorem 10.1 generalizes the classical homotopy classification of the lens spaces [12], [20], [29]. To see this we will completely describe $\text{HT}(Z_m, m) \ (m \geq 2)$. If $\pi = Z_m$, then the boundary homomorphism $\partial : Z^*_m \to K_0 Z(Z_m)$ is zero [23, § 6]. Let $q$ be an integer prime to $m$ and such that $0 < q < m$. Let $\chi$ be a generator of $Z_m$. For $m$ an odd integer greater than two, build the GLS $P^m(n; q)$ so that the cellular chain complex of the universal cover $\tilde{P}^m(n; q)$ is:

$$C^m \xrightarrow{(x^q - 1)} C_{m-1} \xrightarrow{(N)} C_{m-2} \xrightarrow{(x - 1)} \cdots \xrightarrow{(x^q - 1)} C_0$$

where each $C_i = Z(Z_m)$. Each $P^m(n; q)$ has the homotopy type of a standard lens space. For $m$ an even integer greater than or equal to two, let $P_m(n)$ be the $m$-skeleton of $P^{m+1}(n; 1)$.

10.2. Corollary. Let $s \geq 1$

(a) For $m = 2s$, $\text{HT}(Z_m, m) = \langle P^m(n) \rangle$.

(b) Let $m = 2s - 1 \ (s \geq 2)$.

(i) Any $(Z_m, m)$-complex $X$ with absolute Euler characteristic $|\chi(X)| > 0$ has the homotopy type of $P^m(n; 1) \vee |\chi(X)| \text{S}^m$.

(ii) The set of roots $\text{HT}(Z_m, m)_0 \cong Z_*^m(\text{S}^m)^\hat{m}$.

See Fig. 1 (section 0) for a picture of the trees.

For the next application, let $\pi = \Sigma_3$, the symmetric group on three letters. $\Sigma_3$ has minimal free period four, order 6, and satisfies the Eichler condition [23, appendix]. Furthermore, the realization $K_1$ of the efficient presentation $\mathcal{P} : \{x, y : y^2, x, y = x^2\}$ of $\Sigma_3$ has $\pi_2(K_1) \cong Z\pi(N)$ as a $\pi$-module. Simple computations then yield the following corollary.

10.3. Corollary. Let $i > 0$. The homotopy tree $\hat{\text{HT}}(\Sigma_3, m)$ is a single stalk $\langle R_i \rangle$ generated by the root $R_i$ where

$$R_i = \begin{cases} K_i & \text{if } m = 4i - 2 \\ L_i & \text{if } m = 4i - 1 \\ M_i & \text{if } m = 4i \end{cases}$$
Note. $H^1(\Sigma_3, 2) = \langle K_1 \rangle$ completes a partial result of [9, II].

The third application involves the quaternion group $Q(8)$ of order 8 and minimal free period 4. The computations of [14] and [21] show that $\tilde{K}_0ZQ(8) \cong Z_2$, generated by $[(3, N)]$. Thus $\partial : Z^*_8/\{\pm 1\} \rightarrow \tilde{K}_0ZQ(8)$ is an isomorphism.

10.4. Corollary. The homotopy tree $HT(Q(8), 4i - 1)$ is a single stalk $\langle L_i \rangle$ generated by the GLS $L_i$.

Note. Recall that $d(n) = SF(n) / f(n)$. If $d(n) = 0$, then $Aut_k \pi \subset SF(n) = F(n) = \{p \in \mathbb{Z} | \text{p is free}\}$. By 4.1, we may possibly say something positive about units of certain augmentation in the ring $Z\pi/(N)$ simply by computing $Aut_k \pi$.

For the remainder of this section we will work out a fourth application. Let $\pi = D_{2n}$, for $n$ odd. An efficient presentation of $D_{2n}$ is given by $\mathcal{P} : \{x, y : y^2, yxyx^{-n+1}\}$ (see [18]). $D_{2n}$ has minimal free period four. Since this does not seem to be known, we will present a periodic free resolution of period four. Each element of $D_{2n}$ can be written in the form $x^iy^j, 0 < i < n, j = 0, 1$. For $i > 0$, let $\langle x, d \rangle = 1 + x + \cdots + x^{i-1} \in ZD_{2n}$. The following resolution generalizes the corresponding one for $D_6 = \Sigma_3$ given in [23, appendix]. Let $D = D_{2n}$,

$$
0 \rightarrow Z \xrightarrow{d_4} ZD \xrightarrow{d_3} (ZD)^2 \xrightarrow{d_2} (ZD)^2 \xrightarrow{d_1} ZD \xrightarrow{e} Z \rightarrow 0.
$$

Let $\{e_i\}$ be a basis for $C_i$ ($i = 0, 1, 2, 3$). The boundary operators are given thusly:

$\partial_1(e_1^i) = (x - 1)e^0$

$\partial_1(e_2^i) = (y - 1)e^0$

$\partial_2(e_1^2) = -(x, n - 1) - y)e_1^1 + (1 + yx)e_2^1$

$\partial_2(e_2^2) = (1 + y)e_2^1$

$\partial_3(e_3) = \left(\frac{x}{2}, \frac{n + 1}{2}\right) - 1\left( (y + 1)(x - 1)e_1^2 + (y - 1)e_2^2 \right)$

$\partial_4(e_4) = N e^3$.

By using the spectral sequence of the exact sequence

$$
1 \rightarrow Z_n \rightarrow D_{2n} \rightarrow Z_2 \rightarrow 1
$$

one may easily see that $Aut_4D_{2n} = (Z^*_n)^2$. If we specialize to the case $n = p^n$ ($p$ odd
prime, \( s > 0 \), then it is a well known result from number theory that \( \mathbb{Z}_2^* / \mathbb{Z}_2^* (\mathbb{Z}_2^* )^2 = 1 \). Hence in this case the boundary homomorphism \( \partial : \mathbb{Z}_2^* \to \widehat{K}_0 \mathbb{Z} \mathbb{D}_2^* \) is zero. As a corollary we have: if \( n \) is the power of an odd prime, then for each \( p \in \mathbb{Z}_2^* \), there is a unit of augmentation \( p \) in the ring \( \mathbb{Z} \mathbb{D}/(N) \), (see [35]).

10.5. Corollary. If \( n \) is any odd integer, then \( \text{HT}(D_{2n}, 4s - 1)_0 \cong \mathbb{Z}_2^* / (\mathbb{Z}_2^*)^{2s} \). In particular, if \( n \) is the power of an odd prime, then \( \text{HT}(D_{2n}, 3) = \langle L_1 \rangle \).

Recent computations by Ullom [27] of \( \text{im} \{ 0 : \mathbb{Z}_n^* \to \widehat{K}_0 \mathbb{Z}_n \} \) for many finite groups should make a large number of examples accessible.

References


S.V. Ullom, Nontrivial lower bounds for class groups of integral group rings (preprint).


