A class of new fixed point theorems for 1-set-contractive operators and some variational iteration method

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Abstract

We give some new fixed point theorems for semi-closed 1-set-contractive operators and apply He’s variational iteration method to solve some integral equations (see [3]). We extend some conclusion and these methods are important meanings which are different from the recent works.

1. Introduction

It is well known, a lot of mathematical models of physical phenomena are arising in physics, mechanic, test theory, network and environmental monitoring, biology and engineering. The fixed point theory and variational iterative methods are important tools to study them in widely fields.

We study semi-closed 1-sets-contractive operators $A$ and investigate the boundary conditions under which the topological degrees of 1-set contractive fields, $\text{deg} \left(I - A, \Omega, p\right)$ are equal to 1 that along the discussion of [1]. Various kinds of analytical methods and numerical methods were used to solve integral equations. He’s variational iteration method is a powerful device for solving various nonlinear different equations. We apply He’s variational iteration method to solve some integral equations (see [3]).

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The topological degree theory and fixed point index theory play an important role in the study of fixed points for various classes of nonlinear operators in Banach spaces (see [1]-[10]).

Let $E$ be a real Banach space, $\Omega$ a bounded open subset of $E$ and $\theta$ the zero element of $E$. If $A : \overline{\Omega} \to E$ is a continuous continuous operator, we have some well known theorems as follows (see [1],[2]).

For convenience, we first recall theorem 1.1.

Theorem 1.1 (see Theorem 1. 1 in [1]) Suppose that $A$ has no fixed point on $\partial \Omega$, and one of the following conditions is satisfied:

(i) (Leray-Schauder) $\theta \in \Omega$, $Ax \neq \lambda x$, all $x \in \partial \Omega$ and $\lambda > 1$;

(ii) (Rothe) $||Ax|| \leq ||x||$,all $x \in \partial \Omega$;

(iii) (Petryshyn) $\theta \in \Omega$, $||Ax|| \leq ||Ax-x||$, all $x \in \partial \Omega$;

(iv) (Altman)$||Ax-x||^2 \geq ||Ax||^2-||x||^2$, all $x \in \partial \Omega$;

then $\deg (I - A, \Omega, \theta) = 1$, and hence $A$ has at least one fixed point in $\overline{\Omega}$.

Lemma 1.2 (see Corollary 2.1 [1]) Let $E$ be a real Banach space, $\Omega$ is a bounded open subset of $E$ and $\theta \in \Omega$. If $A : \overline{\Omega} \to E$ is a semi-closed 1-set -contractive operator such that satisfies the Leray-Schauder boundary condition,

$Ax \neq tx$ for all $x \in \partial \Omega$ and $t \geq 1$, then $\deg (I - A, \Omega, \theta) = 1$,

and so $A$ has a fixed point in $\Omega$.

Remark 1.3 This lemma 1.2 generalizes the famous Leray-Schauder theorem.

2.Main results

In the present paper, we extend theorem 2.5 in [1] at first. Theorem 2.1, theorem 2.3 and theorem 2.6 etc in [1] are also similar method to discuss in same important meanings.

Theorem 2.1 Let $E, \Omega, A$ be the same as in lemma 1.2. Moreover, if there exists $\alpha > 1$ and $\beta, \gamma \geq 0$, $n$ is a positive inthen there exists $x_0 \in \partial \Omega, \mu_0 \geq 1$, such that egal such that

$||Ax+\mu^{n(\alpha+\beta)+\gamma}|| \leq ||Ax-x||^{\alpha+\gamma} \cdot ||x||^\beta + ||x||^{n\alpha+\gamma} \cdot ||Ax||^\beta$ for all $x \in \partial \Omega$.

(1) $\deg (I - A, \Omega, \theta) = 1$, if $A$ has no fixed points on $\partial \Omega$, so $A$ has at least one fixed point in $\overline{\Omega}$.

Proof. If the operator $A$ has a fixed point on $\partial \Omega$, then $A$ has least one fixed point in $\overline{\Omega}$. Now suppose that $A$ has no fixed point on $\partial \Omega$. Next we shall prove that the Leray-Schauder condition is satisfied.

Suppose this is not true, then there exists $x_0 \in \partial \Omega, \mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$. So, we consider:

$f(t) = (t+1)^{n(\alpha+\beta)+\gamma} - (t-1)^{n\alpha+\gamma} - t^{n\beta}$, for any $t \geq 1$.

Sinc
\[ f'(x) = (n\alpha + \gamma)((t+1)^{n(\alpha + \beta) + \gamma - 1} - (t-1)^{n\alpha + \gamma - 1}) + n\beta \left((t+1)^{n(\alpha + \beta) + \gamma} - t^{n\beta - 1}\right) > 0, \]

\( f(t) \) is a strictly increasing function in \([1, \infty)\).

And so \( f(t) > f(1), t > 1 \). Thus, we have

\[ (t + 1)^{n(\alpha + \beta) + \gamma} > (t - 1)^{n\alpha + \gamma} + t^n\beta, t > 1. \]

And \( \|x_0\| \neq 0, \mu_0 > 1 \), we have

\[ \|Ax_0 + x_0\|^{(\alpha + \beta) + \gamma} = \|\mu x_0 + x_0\|^{(\alpha + \beta) + \gamma} = (\mu_0 + 1)^{n(\alpha + \beta) + \gamma} \|x_0\|^{(\alpha + \beta) + \gamma} = \|\mu_0 x_0 - x_0\|^{n\alpha + \gamma} + \|x_0\|^{\alpha + \gamma} \|\mu_0 x_0\|^{n\beta} = \|Ax_0 - x_0\|^{n\alpha + \gamma} \|x_0\|^{n\beta} + \|x_0\|^{\alpha + \gamma} \cdot \|Ax_0\|^{n\beta}. \]

It is contracted to (1), then by lemma 1.2, the conclusions of theorem 2.1 hold.

Corollary 2.2 Let \( n = 1, \gamma = 0 \), have theorem 2.5 in [1], by substituting (1) for (2) (easy get it):

\[ \|Ax\|^{\alpha} \cdot \|Ax + x\|^{\beta} \leq \|Ax - x\|^{\alpha} \cdot \|x\|^{\beta}, \text{ for all } x \in \partial \Omega(2) \]

3. Some notes for solution of integral equation (see [2])

We consider integral equation:

\[ x(t) = a(t) + (Tx)(t) \int_0^{\alpha(t)} u(t, s, x(s), x(\lambda s)) ds, 0 < \lambda < 1, (3) \]

where the function \( a(t), \sigma(t), u(t, s, x, y) \) and the operator \( T \) are given while \( x = x(t) \) is an unknown function. For convenience, by all condition (i) \( \text{--} (vi) \) theorem 2 in [2] as follows,

(i) \( a(t) \) is a non-decreasing and non-negative belongs to \( C(I) \);

(ii) \( \sigma(t) : I \rightarrow I \) is continuous;

(iii) The operator \( T : C(I) \rightarrow C(I) \) satisfies Darbo condition for the measure of non-compactness \( \mu \) is a positive operator, i.e., \( Tx \geq 0, \text{ if } x \geq 0 \)

\[ (\mu(X) = w_0(X) + i(X) \text{ in } [2]); \]

(iv) \( \|Tx\| \leq c + d\|x\| \) for each \( x \in C(I), c \geq 0; \)

(v) \( u : I \times I \times R \times R \rightarrow R \) continuous such that \( u : I \times I \times R_+ \times R_+ \rightarrow R_+ \) and for fixed \( s \in I \) and \( x, y \in R_+, t \rightarrow u(t, s, x, y) \) is non-decreasing on \( I \);
There exists a function \( f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) which is non-decreasing in each variable such that
\[
|u(t, s, x, y)| \leq f(|x|, |y|),
\]
for each \( t, s \in I \) and \( x, y \in \mathbb{R} \);

There exists \( r_0 > 0 \) with
\[
\|a\| + (c + dr_0) \cdot f(r_0, r_0) \leq r_0
\]
and \( f(r_0, r_0) \cdot Q < 1 \).

Theorem 3.1 (see theorem 2 in [2]) Under assumption \((i) - (vii)\), equation (3) has at least one solution \( x = x(t) \) which belongs to the space \( C(I) \) and is non-decreasing on the interval \( I \).

Example 1. \( x(t) = \left( \frac{t^2}{8} \right) + \left( 1 + x(t) \right) \cdot \int_0^t s \cdot x(s) \cdot x(\lambda s) \, ds \).

Let \( a(t) = \left( \frac{t^2}{8} \right) \), it is clear that it satisfies assumption \((i)\) with \( \|a\| = \frac{1}{8} \). We take the operator \( T \) defined by \((Tx)(t) = 1 + x(t)\), this operator verifies hypotheses \((iii)\) and \((iv)\) with \( c = 1, d = 1,\) and \( Q = 1 \).

We consider \( \sigma(t) = \frac{t}{1 + t^2} \), which satisfies assumption \((ii)\) and choose the function \( u(t, s, x, y) \) defined by the expression \( u(t, s, x, y) = t \cdot s \cdot x \cdot y \). The function is continuous and satisfies \((v)\) and \((vi)\) with \( f(x, y) = xy \).

In fact,
\[
|u(t, s, x, y)| = |t \cdot s \cdot x \cdot y| \leq |x| \cdot |y|.
\]

If we consider the previous functions the first inequality of assumption \((vii)\) takes the following form:
\[
\left( \frac{1}{8} \right) + (1 + r) \cdot r^2 \leq r.\]
Then it can be proved \( r_0 = \frac{3}{8} \) is a positive solution of this inequality, and
\[
f(r_0, r_0) \cdot Q = r_0^2 \cdot Q = \left( \frac{3}{8} \right)^2 < 1,
\]

is satisfied. By applying theorem 3.1, we get equation has at least one solution \( x^*(t) \in B_{3/8}^* \).

Example 2. \( x(t) = \left( \frac{t^2}{243} \right) + \left( 1 + x(t) \right) \cdot \int_0^t s \cdot x(s) \cdot x(\lambda s) \, ds \).

4. He’s iterative method for integral equation (see [3] and [5])

To illustrate the basic idea of the method, we consider:
\[
L[u(t)] + N[u(t)] = g(t) \ (see[3]),
\]
where $L$ is a linear operator, $N$ is a nonlinear operator and $g(t)$ is a continuous function. The basic character of the method is to construct functional for the system, which reads
\[ u_{n+1}(x) = u_n(x) + \int_0^1 \lambda(s)Lu[Lu_n(s) + Nu_n - g(s)]ds. \]

Where $\lambda$ is a Lagrange multiplier which can be identified optimally via variational theory, $u_n$ is the $n$th approximate solution, and $\overline{u}_n$ denotes a restricted variation, i.e.,
\[ \delta \overline{u}_n = 0. \]

There is an iterative formula
\[ u_{n+1}(x) = f(x) + \lambda \int_a^b k(x,t)u_n(t)dt \]

Of
\[ u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt. \]  \hspace{1cm} (4)

Theorem 4.1 (see theorem 2.1 in [3]) Consider the iteration scheme $u_0(x) = f(x)$, and the
\[ u_{n+1}(x) = f(x) + \lambda \int_a^b k(x,t)u_n(t)dt. \]  \hspace{1cm} (5)

For $n = 0, 1, 2, \ldots$ to construct a sequence of successive iterations $\{u_n(x)\}$ to the solution of Eq.(2.1). In addition, let
\[ \int_a^b \int_a^b k^2(x,t)dxdt = B^2 < \infty. \]

And assume that $f(x) \in L^2_{(a,b)}$. Then, if $|\lambda| < 1/B$, the above iteration converges in the norm of $L^2_{(a,b)}$ to the solution of Eq.(4).

Example 3  (similar example 2.2 in [3]) Consider the integral equation $(0 < \alpha < 1)$
\[ u(x) = x^\alpha + \lambda \int_0^b x \cdot t \cdot u(t)dt, \]  \hspace{1cm} (6)

Its iteration formula reads $u_{n+1}(x) = x^\alpha + \lambda \int_0^b x \cdot t \cdot u_n(t)dt$, and
\[ u_0(x) = x^\alpha. \]  \hspace{1cm} (7)

Substituting Eq.(7) in to Eq.(6), we have
\[ u_1(x) = x^\alpha + \lambda \int_0^b x \cdot t \cdot u_1(t)dt = x^\alpha + \left( \frac{\lambda}{\alpha + 2} \right)x, \]
and
\[ u_2(x) = x^\alpha + \left( \frac{\lambda}{\alpha + 2} + \frac{1}{3} \left( \frac{\lambda^2}{\alpha + 2} \right) \right)x, \ldots \]

inductively, we have that
\[ u_n(x) = x^\alpha + \left( \frac{\lambda}{\alpha + 2} + \left( \frac{1}{3} \right)^n \left( \frac{\lambda^2}{\alpha + 2} \right)^n + \cdots + \left( \frac{1}{3} \right)^{n-1} \left( \frac{\lambda^n}{\alpha + 2} \right) \right) x \quad (n = 1, 2, \ldots). \]

Then by theorem 4.1, that
\[
\int_a^b \int_a^b k^2(x, t) \, dx \, dt = \int_0^1 \int_0^1 (x \cdot t)^2 \, dx \, dt = \frac{1}{9} = B^2.
\]

Then if \( |\lambda| < 3 \), the Eq. (7) is convergent (as \( \alpha = \frac{1}{2} \), a exact solution

\[
u(x) = \sqrt{x} + \left( \frac{6 \lambda}{5(3 - \lambda)} \right) x
\]

(see [3]).

Example 4 (see example 3.2 in [3]). Consider the integral equation

\[
u^n(x) = -1 + \lambda \int_0^x (x - t) \nu(t) \, dt, \quad (8)
\]

We know \( \nu(x) = \cos x \) is exact solution of equation (8)

\[
\text{or equivalent } \nu(x) = 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x - t)^3 \nu(t) \, dt, \quad \text{as } \lambda = 1.
\]

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References


