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The Spectrum of a Modified Linear Pencil

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Abstract—Suppose the spectrum of a symmetric definite linear pencil is known. This paper addresses the question of what can be said about the spectrum when scalar multiples of a rank-one update are added to each matrix in the pencil.

The secular equation for this problem is derived, and from it, a certain separation property is found which gives insight into the connection between the eigenvalues before and after modification.

In the context of structural dynamics, the result characterises the behaviour of a finite-dimensional vibrating system undergoing mass and stiffness modifications.

The result also leads to applications such as a divide and conquer algorithm for the eigenvalues of the modified system (so-called matrix tearing) and spectral shifting. An illustrative example is also given.

Keywords—Generalized eigenvalue problem, Secular equation, Interlacing eigenvalues, Modified vibrating system, Divide and conquer, Matrix tearing.

1. INTRODUCTION

The pencil $P(\lambda) = \mathbf{A} - \lambda\mathbf{B}, \mathbf{A}, \mathbf{B} \in \mathcal{R}^{n\times n}$, real n-square matrices, is said to be symmetric definite if $\mathbf{A}$ is symmetric and $\mathbf{B}$ is symmetric positive definite. The real scalar λ and the associated vector $\mathbf{y}$ which satisfy

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{y} = \mathbf{0}, \qquad \mathbf{y}^T\mathbf{B}\mathbf{y} = 1,$$

are called an eigenvalue and normalized eigenvector of $P(\lambda)$. The set of eigenvalues of P is called its spectrum and is denoted

$$\sigma(\mathbf{A}, \mathbf{B}) = \{\lambda_j\}_{j=1}^{n}. \tag{1.1}$$

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Eigenproblems such as this arise in the modelling of conservative vibrating systems. There $\sqrt{\lambda}$ is called the natural frequency and the vector $\mathbf{y}$ its associated mode-shape. Our interest centers on systems in which the matrices $\mathbf{A}$ and $\mathbf{B}$ are modified by the addition to each of a symmetric rank-one correction. More precisely, for a given vector $\mathbf{u} \in \mathcal{R}^n$ and the two real scalars α and β, we define the eigenvector $\mathbf{x}$ and its corresponding eigenvalue μ as those which satisfy

$$\left(\mathbf{A} + \alpha \mathbf{u}\mathbf{u}^T\right) \mathbf{x} = \mu \left(\mathbf{B} + \beta \mathbf{u}\mathbf{u}^T\right) \mathbf{x}, \qquad \mathbf{x}^T \left(\mathbf{B} + \beta \mathbf{u}\mathbf{u}^T\right) \mathbf{x} = 1. \tag{1.2}$$

In the context of vibrations, modifications such as these represent structural changes in the masses and stiffness of the system. We will consider only those modifications $\beta \mathbf{u}\mathbf{u}^T$ which are such that the matrix $\mathbf{B} + \beta \mathbf{u}\mathbf{u}^T$ is positive definite.

Denote the spectrum of this modified system by

$$\sigma\left(\mathbf{A} + \alpha \mathbf{u}\mathbf{u}^T, \mathbf{B} + \beta \mathbf{u}\mathbf{u}^T\right) = \{\mu_j\}_{j=1}^n.$$

In Section 2, we derive the secular equation which charactertises the eigenvalues $\{\mu_j\}_{j=1}^n$ of the modified system in terms of the $\{\lambda_j\}_{j=1}^n$, the eigenvalues of the original system, and the modifying parameters α, β, and $\mathbf{u}$. From this characterisation, a certain separation property of the eigenvalues follows.

These results may be used to shift the eigenvalues of symmetric pencils (Section 3.2), or more importantly, they lead, in Section 3 to a divide and conquer algorithm for the solution of a large generalized eigenvalue problem with a pair of symmetric tridiagonal matrices. These often arise in the finite element models of Sturm-Liouville systems, such as wave propagation in horns or the vibrating rod and string. For the standard eigenvalue problem, where only the eigenvalues of a large tridiagonal symmetric matrix are required, the current method of choice is divide and conquer, sometimes called *matrix tearing*. However, the QZ method for the generalized eigenvalue problem does not respect tridiagonal structure or symmetry and so is inefficient for these problems. Thus, there is a need for methods that are efficient on the generalized eigenvalue problem for large symmetric tridiagonals.

The divide and conquer algorithm we present is a generalization of a technique which appears to have been first proposed in [1] and later used as the basis for a parallel algorithm in [2]. The algorithm is well suited to parallel computation although we do not pursue that here.

Results which correspond to those in this paper, but which apply to the the standard eigenvalue problem, are to be found in [3]. Changes in the spectrum due to mass and stiffness modification have also been addressed in [4,5].

In Section 3, we describe the divide and conquer method which exploits the secular equation of Section 2 to efficiently compute the eigenvalues of a symmetric tridiagonal pair. In Section 4, we derive a second-order method for computing the zeros of the secular equation which was presented in Section 2. In Section 5.1, we illustrate the method on a small matrix tearing problem and in Section 5.2 we report on some numerical tests. Finally, in Section 6, we draw some conclusions regarding this work.

2. THE EIGENVALUES OF THE MODIFIED SYSTEM

Let $\mathbf{Y}$, regular, be the matrix which simultaneously diagonalizes $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{Y}^T \mathbf{A} \mathbf{Y} = \mathbf{\Lambda} = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \qquad \mathbf{Y}^T \mathbf{B} \mathbf{Y} = \mathbf{I}. \tag{2.1}$$

Multiplying (1.2) on the left by $\mathbf{Y}^T$ gives

$$\mathbf{Y}^T \left(\mathbf{A} + \alpha \mathbf{u}\mathbf{u}^T\right) \mathbf{Y}\mathbf{Y}^{-1}\mathbf{x} = \mu \mathbf{Y}^T \left(\mathbf{B} + \beta \mathbf{u}\mathbf{u}^T\right) \mathbf{Y}\mathbf{Y}^{-1}\mathbf{x},$$

from which

$$\left(\mathbf{\Lambda} + \alpha \mathbf{Y}^T \mathbf{u}\mathbf{u}^T \mathbf{Y}\right) \left(\mathbf{Y}^{-1}\mathbf{x}\right) = \mu \left(\mathbf{I} + \beta \mathbf{Y}^T \mathbf{u}\mathbf{u}^T \mathbf{Y}\right) \left(\mathbf{Y}^{-1}\mathbf{x}\right)$$

easily follows. Setting $\hat{\mathbf{u}} = \mathbf{Y}^T \mathbf{u}$ and $\mathbf{Y}\hat{\mathbf{x}} = \mathbf{x}$ gives

$$\left(\mathbf{\Lambda} + \alpha \hat{\mathbf{u}}\hat{\mathbf{u}}^T\right) \hat{\mathbf{x}} = \mu \left(\mathbf{I} + \beta \hat{\mathbf{u}}\hat{\mathbf{u}}^T\right) \hat{\mathbf{x}}. \tag{2.2}$$

For later use, we denote by $\mathbf{e}_j$ the j^{th} column of an identity matrix of appropriate dimension.

LEMMA 2.1. *Suppose the λ_j of (1.1) are distinct and $\mathbf{Y}$ is such that*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n. \tag{2.3}$$

Assume further, that $\mathbf{e}_j^T \hat{\mathbf{u}} \neq 0$ and $\lambda_j \neq \alpha/\beta$ for all j. Then,

(a) $(\mathbf{\Lambda} - \mu\mathbf{I})$ *is invertible,*
(b) $\mu \neq \alpha/\beta$,
(c) $\hat{\mathbf{x}}^T \hat{\mathbf{u}} \neq 0$,
(d) *the eigenvector $\hat{\mathbf{x}}$ of the rank-modified diagonal system (2.2) is a scalar multiple of $(\mathbf{\Lambda} - \mu\mathbf{I})^{-1}\hat{\mathbf{u}}$.*

PROOF. Rearranging (2.2) gives

$$(\mathbf{\Lambda} - \mu\mathbf{I})\hat{\mathbf{x}} + (\alpha - \mu\beta)\hat{\mathbf{u}}\hat{\mathbf{u}}^T\hat{\mathbf{x}} = \mathbf{0}.$$

Suppose that $\mu = \lambda_j$ for some j. Then,

$$\begin{aligned} 0 &= \mathbf{e}_j^T \left((\mathbf{\Lambda} - \lambda_j\mathbf{I})\hat{\mathbf{x}} + (\alpha - \lambda_j\beta)\hat{\mathbf{u}}\hat{\mathbf{u}}^T\hat{\mathbf{x}} \right) \\ &= (\alpha - \lambda_j\beta)\mathbf{e}_j^T\hat{\mathbf{u}} \left(\hat{\mathbf{u}}^T\hat{\mathbf{x}} \right). \end{aligned}$$

This implies that $\hat{\mathbf{u}}^T\hat{\mathbf{x}} = 0$ and so

$$(\mathbf{\Lambda} - \lambda_j\mathbf{I})\hat{\mathbf{x}} = \mathbf{0}.$$

But this together with (2.3), implies that the eigenvector $\hat{\mathbf{x}}$ is a multiple of $\mathbf{e}_j$, $\hat{\mathbf{x}} \in \text{span}(\mathbf{e}_j)$, which in turn implies that $\hat{\mathbf{x}}^T\hat{\mathbf{u}} = \mathbf{e}_j^T\hat{\mathbf{u}} = 0$, a contradiction. Hence, $(\mathbf{\Lambda} - \mu\mathbf{I})$ is invertible and we can write

$$\hat{\mathbf{x}} = (\beta\mu - \alpha)(\mathbf{\Lambda} - \mu\mathbf{I})^{-1}\hat{\mathbf{u}}\hat{\mathbf{u}}^T\hat{\mathbf{x}}. \tag{2.4}$$

Since $\mathbf{x} \neq 0$, we have $\mu \neq \alpha/\beta$ and $\hat{\mathbf{u}}^T\hat{\mathbf{x}} \neq 0$, and the form of $\hat{\mathbf{x}}$ as required by (d) in Lemma 2.1. ∎

THEOREM 2.2. *Let $\mathbf{u}$ be such that $\mathbf{u}^T\mathbf{Y}\mathbf{e}_j \neq 0$ for $j = 1, 2, \ldots, n$ and let $\mathbf{Y}$ be such that (2.3) holds. Suppose, without loss of generality, that the μ_j are labeled in nondecreasing order*

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n. \tag{2.5}$$

Then, the μ_j are the zeros of the secular equation

$$1 - (\beta\mu - \alpha)\hat{\mathbf{u}}^T(\mathbf{\Lambda} - \mu\mathbf{I})^{-1}\hat{\mathbf{u}} = 0, \tag{2.6}$$

and consequently,

$$\begin{aligned} &\frac{\alpha}{\beta} \leq \mu_1 \leq \lambda_1, && \text{if } \frac{\alpha}{\beta} \leq \lambda_1, \\ &\lambda_j \leq \mu_j \leq \lambda_{j+1}, && \text{if } \lambda_j \leq \frac{\alpha}{\beta}, \\ &\lambda_j \leq \mu_j \leq \frac{\alpha}{\beta} \leq \mu_{j+1} \leq \lambda_{j+1}, && \text{if } \lambda_j \leq \frac{\alpha}{\beta} \leq \lambda_{j+1}, \\ &\lambda_j \leq \mu_{j+1} \leq \lambda_{j+1}, && \text{if } \frac{\alpha}{\beta} \leq \lambda_j, \\ &\lambda_n \leq \mu_n \leq \frac{\alpha}{\beta}, && \text{if } \lambda_n \leq \frac{\alpha}{\beta}. \end{aligned} \tag{2.7}$$

Geometrically, this result can be expressed as follows: the eigenvalues of the new system interlace the elements of the set containing the old eigenvalues and the number α/β.

PROOF. Suppose first, that the λ_j of (1.1) are distinct from each other and the ratio α/β. We first prove Theorem 2.2 for the case where the weak inequalities in (2.7) and (2.5) are replaced by the strict inequalities.

Multiplying (2.4) on the left by $\hat{\mathbf{u}}^T$ and canceling the term $\hat{\mathbf{u}}^T\hat{\mathbf{x}} = \mathbf{u}^T\mathbf{x} \neq 0$ gives the secular equation (2.6) and any μ which satisfies (2.6) is an eigenvalue of the modified system.

We can consider the left-hand side of (2.6) as a function of the variable μ and write it in component form as

$$g(\mu) \stackrel{\text{def}}{=} 1 - (\beta\mu - \alpha)\sum_{j=1}^{n} \frac{\hat{u}_j^2}{\lambda_j - \mu} = 0. \tag{2.8}$$

Note that $g(\mu)$ has, since the $\hat{u}_j^2$ do not vanish, exactly n zeros. We now show that these zeros of g satisfy the interlacing property (2.7). No component of $\hat{\mathbf{u}}$ vanishes and so $g(\mu)$ is a rational function with denominator and numerator, each of degree n. It therefore has exactly n zeros and n poles.

Consider first, the case where $\lambda_k < \alpha/\beta < \lambda_{k+1}$. Then, for all $j \leq k$ we see, by taking account of the signs of $(\beta\mu - \alpha)$ and $(\lambda_j - \mu)$, that

$$\lim_{\mu \to \lambda_j -} g(\mu) = +\infty$$

and

$$\lim_{\mu \to \lambda_j +} g(\mu) = -\infty.$$

Therefore, by the continuity of g between its poles, $g(\mu)$ has at least one zero in each of the intervals $(\lambda_j, \lambda_{j+1})$, $j = 1, 2, \ldots, k-1$.

Similarly, for the range $j > k+1$, where $\mu > \alpha/\beta$, we easily see that

$$\lim_{\mu \to \lambda_j -} g(\mu) = -\infty$$

and

$$\lim_{\mu \to \lambda_j +} g(\mu) = +\infty,$$

and we can say that $g(\mu)$ has at least one zero in each of the intervals $(\lambda_j, \lambda_{j+1})$, $j = k+1$, $k+2, \ldots, n-1$. This accounts for at least $n-2$ of its zeros.

At the point $\mu = \alpha/\beta$ the function g takes the value 1. Putting this together with

$$\lim_{\mu \to \lambda_k +} g(\mu) = \lim_{\mu \to \lambda_{k+1} -} g(\mu) = -\infty,$$

we see that g has two zeros in $(\lambda_k, \lambda_{k+1})$: one between λ_k and α/β and the other between α/β and λ_{k+1}. This accounts for all n zeros in the case where $\lambda_k < \alpha/\beta < \lambda_{k+1}$, and therefore, establishes the result for that case.

Now consider the case where $\alpha/\beta < \lambda_1$. It is easy to see from its form, that at its poles, λ_j, $j = 1, 2, \ldots, n$, g satisfies

$$\lim_{\mu \to \lambda_j -} g(\mu) = -\infty$$

and

$$\lim_{\mu \to \lambda_j +} g(\mu) = +\infty.$$

Thus, g has at least one zero between each pair of consecutive λ_j. This accounts for at least $n-1$ zeros. At the point α/β, g takes the value 1. This, together with

$$\lim_{\mu \to \lambda_1 -} g(\mu) = -\infty,$$

means that g has at least one zero in the interval between α/β and λ_1. Since g has exactly n zeros this establishes the result for the case where $\alpha/\beta < \lambda_1$. By a similar argument it follows that when $\alpha/\beta > \lambda_n$, the interlacing property is satisfied for the case where (2.7), with the weak inequality replaced by the strict inequality, holds.

By a simple continuity argument, it follows that if the assumption of distinctness is dropped, then the result (2.7) follows and the theorem is proved. ∎

The function g describes the relation between the eigenvalues of the two systems. Its n zeros $\{\mu_j\}_{j=1}^n$ are the eigenvalues of the modified system, expressed in terms of the eigenvalues of the original system.

In Figure 1, we display an example of the function $g(\mu)$

$$g(\mu) = 1 - (3\mu - 7)\left(\frac{1/4}{1-\mu} + \frac{1/9}{2-\mu} + \frac{1/16}{3-\mu}\right). \tag{2.9}$$

The poles, at $1, 2, 3$, are the eigenvalues of the original system and the zeros, approximately $1.4196, 2.0913, 2.9233$ are the eigenvalues of the system after rank-one corrections are added to each of the matrices in the pencil.

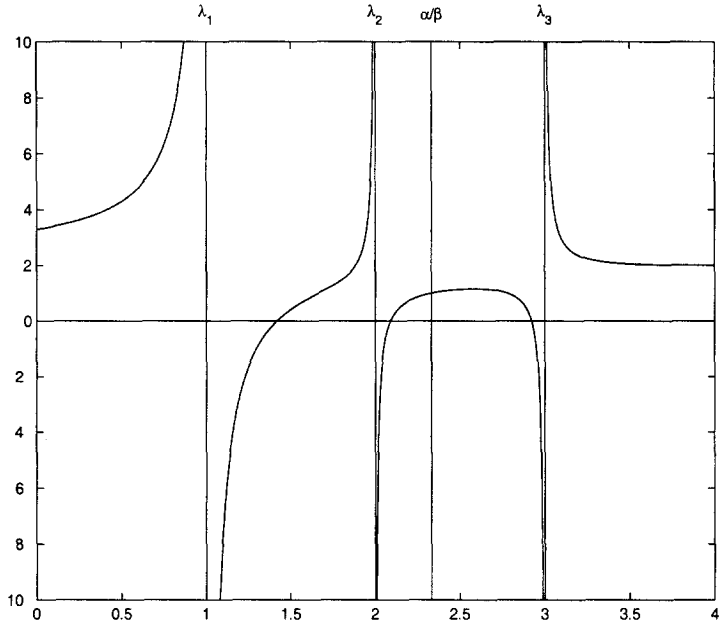

Figure 1. The function $g(\mu)$ of (2.9).

3. APPLICATIONS

3.1. A Divide and Conquer Method for the Eigenvalues of a Tridiagonal Symmetric Positive Definite Pair

In this section, we describe one stage of a divide and conquer method which finds the eigenvalues of a symmetric positive definite pair and which, for simplicity we assume have dimension $2n$.

The starting data at a typical stage are

(a) the eigenvalues $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2$ of two pairs of tridiagonal, symmetric matrices $\mathbf{A}_1, \mathbf{B}_1$ and $\mathbf{A}_2, \mathbf{B}_2$, the $\mathbf{B}_i$ symmetric positive definite, $\mathbf{A}_i$ symmetric positive semidefinite, all in $\mathcal{R}^{n\times n}$,

(b) for $j = 1, 2$, the first and last rows $\mathbf{e}_1^T\mathbf{Y}_j$ and $\mathbf{e}_n^T\mathbf{Y}_j$ of the normalized ($\mathbf{Y}_j^T\mathbf{B}\mathbf{Y}_j = \mathbf{I}_n$) eigenvector matrices for the pairs $\mathbf{A}_j, \mathbf{B}_j$,

(c) two real scalars α, β.

Define the vector $\mathbf{u} \in \mathcal{R}^{2n}$,

$$\mathbf{u} = \begin{pmatrix} \mathbf{e}_n \\ \mathbf{e}_1 \end{pmatrix},$$

where $\mathbf{e}_1$ is the first column on n-dimensional identity and $\mathbf{e}_n$, the last. Denote the direct sum of $\mathbf{A}_1$ and $\mathbf{A}_2$ by $\mathbf{A}$ and similarly for $\mathbf{B}$,

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \\ & \mathbf{A}_2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & \\ & \mathbf{B}_2 \end{pmatrix}. \tag{3.1}$$

We can use the results of the previous section to determine the eigenvalues of the $2n \times 2n$ symmetric tridiagonal matrix pencil

$$P(\lambda) = \mathbf{A} + \alpha \mathbf{u}\mathbf{u}^T - \lambda \left(\mathbf{B} + \beta \mathbf{u}\mathbf{u}^T \right), \tag{3.2}$$

$$= \hat{\mathbf{A}} - \lambda \hat{\mathbf{B}}, \tag{3.3}$$

with the obvious definitions.

The diagonal matrix $\mathbf{\Lambda}$ is the direct sum of $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, and $\hat{\mathbf{u}}$ is $\mathbf{Y}^T\mathbf{u}$ where $\mathbf{Y}$ is the direct sum of $\mathbf{Y}_1$ and $\mathbf{Y}_2$. Thus,

$$\hat{\mathbf{u}} = \mathbf{Y}^T\mathbf{u} = \begin{pmatrix} \mathbf{Y}_1^T\mathbf{e}_n \\ \mathbf{Y}_2^T\mathbf{e}_1 \end{pmatrix}. \tag{3.4}$$

With these starting data, we can write the secular equation which relates the eigenvalues $\{\lambda_j\}_{j=1}^{2n}$ of the (unmodified) $2n \times 2n$ pencil $P(\lambda) = \mathbf{A} - \lambda\mathbf{B}$ with $\mathbf{A}$ and $\mathbf{B}$ as in (3.1) to the eigenvalues of the modified system of (3.2) as

$$g(\mu) = 1 - (\beta\mu - \alpha) \sum_{j=1}^{2n} \frac{\hat{u}_j^2}{\lambda_j - \mu}, \tag{3.5}$$

where, as before

$$\mathbf{\Lambda} = \operatorname{diag}\{\mathbf{\Lambda}_1, \mathbf{\Lambda}_2\} = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\},$$

and the $\hat{u}_j$ are the components defined by (3.4).

We discuss a method to find the zeros of g, which are all simple, in Section 4.

Once all the eigenvalues $\{\mu_j\}_{j=1}^{2n}$ of the modified system have been found, we can use the fact established in (d) of Lemma 2.1 to find the eigenvectors of the modified system as follows. It suffices to describe the determination of the eigenvector $\hat{\mathbf{x}}$ corresponding to the eigenvalue μ.

We compute $\mathbf{w} = (\mathbf{\Lambda} - \mu\mathbf{I})^{-1}\hat{\mathbf{u}}$, a scalar multiple of the eigenvector $\hat{\mathbf{x}}$. This calculation requires just n flops because $\hat{\mathbf{u}} = \mathbf{Y}^T\mathbf{u}$ is only a selection operation and $(\mathbf{\Lambda} - \mu\mathbf{I})$ is diagonal. From the lemma, we know that $\mathbf{w}$ is a scalar multiple of $\hat{\mathbf{x}}$, say $\mathbf{w} = \theta\hat{\mathbf{x}}$. We can find the normalizing constant θ by noting that, substituting $\mathbf{Y}\hat{\mathbf{x}}$ for $\mathbf{x}$ in the normalizing relation of (1.2), we get

$$\hat{\mathbf{x}}^T\mathbf{Y}^T \left(\mathbf{B} + \beta\mathbf{u}\mathbf{u}^T \right) \mathbf{Y}\hat{\mathbf{x}} = 1$$

whence,

$$\hat{\mathbf{x}}^T \left(\mathbf{I} + \beta\mathbf{Y}^T\mathbf{u}\mathbf{u}^T\mathbf{Y} \right) \hat{\mathbf{x}} = 1. \tag{3.6}$$

We require the first and last elements of $\mathbf{x}$, so we actually compute $\mathbf{e}_1^T\mathbf{Y}\hat{\mathbf{x}}$ and $\mathbf{e}_{2n}^T\mathbf{Y}\hat{\mathbf{x}}$, and these are rows of $\mathbf{Y}$ which were in the data set. This takes us to the point where we have all $2n$ eigenvalues and the first and last eigenvectors of the full tridiagonal matrix pair $\hat{\mathbf{A}}, \hat{\mathbf{B}}$.

This describes a process that gets us to the next stage of a divide and conquer scheme for all the eigenvalues and the first and last rows of the eigenvector matrix of a large tridiagonal symmetric positive definite pencil from similar information about two subpencils.

Computing all the eigenvector data needed for the next stage of this process requires approximately $3(2n)^2 + 4(2n) = 12n^2 + 8n$ flops.

A process similar to that of this section can be used to compute the Schur decomposition of a symmetric positive definite pencil from the Schur decompositions of two subpencils by exploiting the fact that there are very economical and effective ways of computing the Schur decomposition of a diagonal matrix pair

$$\mathbf{\Lambda} + \alpha\mathbf{u}\mathbf{u}^T, \qquad \mathbf{I} + \beta\mathbf{u}\mathbf{u}^T.$$

3.2. Spectral Shift

Another use of our result is in the stabilisation of systems modelled by a symmetric positive definite system with pencil $P(\lambda) = \mathbf{A} - \lambda\mathbf{B}$. Suppose an estimate ξ exists for the smallest eigenvalue λ_1. Then any rank one modification of the form (1.2) where $\alpha/\beta < \xi$ will shift the whole spectrum towards the region of stability. It is clear that after such a shift, the largest eigenvalue of the shifted system cannot be smaller than the second largest eigenvalue of the original system.

4. THE ZEROS OF THE SECULAR EQUATION

We can simplify the problem in some situations. Assume for a moment, without loss of generality, that $\|\hat{\mathbf{u}}\|_2 = 1$.

If $\hat{u}_j = 0$, we can deflate the problem into two smaller subproblems because then the eigenpair $\lambda_j, \mathbf{e}_j$ of the matrix pair $(\mathbf{\Lambda}, \mathbf{I})$ is also an eigenpair of the modified system $(\mathbf{\Lambda} + \alpha\hat{\mathbf{u}}\hat{\mathbf{u}}^T, \mathbf{I} + \beta\hat{\mathbf{u}}\hat{\mathbf{u}}^T)$. This follows from the fact that $\mathbf{e}_j^T\mathbf{u}_j = 0$ implies

$$\begin{aligned}\left(\mathbf{\Lambda} + \alpha\hat{\mathbf{u}}\hat{\mathbf{u}}^T\right)\mathbf{e}_j - \lambda_j\left(\mathbf{I} + \beta\hat{\mathbf{u}}\hat{\mathbf{u}}^T\right)\mathbf{e}_j &= \mathbf{\Lambda}\mathbf{e}_j + \alpha\hat{\mathbf{u}}\hat{\mathbf{u}}^T\mathbf{e}_j - \lambda_j\mathbf{e}_j - \beta\lambda_j\hat{\mathbf{u}}\hat{\mathbf{u}}^T\mathbf{e}_j \\ &= \mathbf{\Lambda}\mathbf{e}_j - \lambda_j\mathbf{e}_j = 0.\end{aligned}$$

Thus, we can restrict our consideration to the case where $\hat{u}_j \neq 0$ for any j.

Another case which simplifies occurs when $\hat{\mathbf{u}} = \mathbf{e}_j$, i.e., $\hat{u}_j = 1$, and $\hat{u}_i = 0$, $i \neq j$. Then $\mu_j = (\lambda_j + \alpha)/(1 + \beta)$ is the only solution of the secular equation, all the other eigenvalues are in the spectrum of the pair $(\mathbf{\Lambda}, \mathbf{I})$ and all the eigenvectors remain unchanged. In particular, $\mathbf{e}_j$ is the eigenvector corresponding to μ_j because

$$\begin{aligned}\left(\mathbf{\Lambda} + \alpha\mathbf{e}_j\mathbf{e}_j^T\right)\mathbf{e}_j - \mu_j\left(\mathbf{I} + \beta\mathbf{e}_j\mathbf{e}_j^T\right)\mathbf{e}_j &= \mathbf{\Lambda}\mathbf{e}_j + \alpha\mathbf{e}_j\mathbf{e}_j^T\mathbf{e}_j - \mu_j\mathbf{e}_j - \beta\mu_j\mathbf{e}_j\mathbf{e}_j^T\mathbf{e}_j \\ &= \lambda_j\mathbf{e}_j + \alpha\mathbf{e}_j - \mu_j\mathbf{e}_j - \beta\mu_j\mathbf{e}_j \\ &= (\lambda_j + \alpha - \mu_j(1+\beta))\mathbf{e}_j \\ &= 0.\end{aligned}$$

A fast solver, such as Newton's method, protected by bisection, would seem to be a natural choice for computing the zeros of the secular equation because we know upper and lower bounds for each zero and for the derivative of g we have the convenient form

$$g'(\mu) = -\beta\sum_{j=1}^{2n}\hat{u}_j^2\frac{(\lambda_j - \alpha/\beta)}{(\lambda_j - \mu)^2}.$$

Instead, however, we follow the line taken in [6,7] and rather than use a locally linear approximation as in Newton's method, we devise a method based on a simple rational, local approximation. Thus, at the approximation point $\mu_j^{(k)}$, we approximate the function $g(\mu)$ of (3.5) by

$$f(\mu) = 1 - \frac{c(\beta\mu - \alpha)}{(d - \mu)}, \tag{4.1}$$

where the constants c and d are chosen so that

$$\begin{aligned}f\left(\mu_j^{(k)}\right) &= g\left(\mu_j^{(k)}\right), \\ f'\left(\mu_j^{(k)}\right) &= g'\left(\mu_j^{(k)}\right).\end{aligned}$$

The required c and d are

$$c = \frac{g\left(\mu_j^{(k)}\right)^2 - 2g\left(\mu_j^{(k)}\right) + 1}{\beta\left(g\left(\mu_j^{(k)}\right) - 1\right) - g'\left(\mu_j^{(k)}\right)\left(\beta\mu_j^{(k)} - \alpha\right)},$$
$$d = \frac{\alpha\left(g\left(\mu_j^{(k)}\right) - 1\right) - \mu_j^{(k)} g'\left(\mu_j^{(k)}\right)\left(\beta\mu_j^{(k)} - \alpha\right)}{\beta\left(g\left(\mu_j^{(k)}\right) - 1\right) - g'\left(\mu_j^{(k)}\right)\left(\beta\mu_j^{(k)} - \alpha\right)}.$$

Our new approximation $\mu_j^{(k+1)} = (d + c\alpha)/(c\beta + 1)$, the zero of f, thus becomes

$$\mu_j^{(k+1)} = \frac{\alpha g\left(\mu_j^{(k)}\right)\left(g\left(\mu_j^{(k)}\right) - 1\right) - \mu_j^{(k)} g'\left(\mu_j^{(k)}\right)\left(\beta\mu_j^{(k)} - \alpha\right)}{\beta g\left(\mu_j^{(k)}\right)\left(g\left(\mu_j^{(k)}\right) - 1\right) - g'\left(\mu_j^{(k)}\right)\left(\beta\mu_j^{(k)} - \alpha\right)} \tag{4.2}$$
$$\stackrel{\text{def}}{=} h\left(\mu_j^{(k)}\right).$$

Suppose $\mu \neq \beta/\alpha$. Clearly, any point η for which $g(\eta) = 0$ is a fixed point of the equation $\mu = h(\mu)$. Furthermore, if the fixed-point iteration defined by (4.2) converges, then it does so with a rate of convergence which is at least quadratic. This follows easily from the fact that the numerator of $h'(\mu)$ is

$$-(\beta\mu - \alpha)g(\mu)\left[g''(\mu)(g(\mu) - 1)(\beta\mu - \alpha) + 2\,g'(\mu)\left(\beta(g(\mu) - 1) - g'(\mu)(\beta\mu - \alpha)\right)\right],$$

which vanishes when $g(\mu)$ does, and its denominator is

$$\left(\beta g(\mu)(1 - g(\mu)) + g'(\mu)(\beta\mu - \alpha)\right)^2$$

which does not vanish where $g(\mu)$ vanishes. Of course, g and g' cannot vanish together by Theorem 2.2.

Any point ξ where $g(\xi) = 1$ is also a fixed point of $\mu = h(\mu)$ but a simple calculation with h' shows that $h'(\xi) = 2$ and so such a ξ is not a point of attraction for this iteration.

We indicate below how the eigenvaules of the modified system can thus be found reliably with a hybrid method that ideally has quadratic convergence.

5. EXAMPLES

5.1. An Illustration of Matrix Tearing

In this section, we illustrate the use of the method with a small example.

Consider an axially vibrating rod fixed at one end and free to oscillate at the other, with length L, modulus of elasticity E, cross-sectional area A, and mass density ρ. The finite element model of this rod leads to a generalized eigenvalue problem with matrices $\hat{\mathbf{K}}, \hat{\mathbf{M}}$ which are symmetric, tridiagonal, and positive definite. The eigenvalues, which represent the squares of the resonant frequencies, together with the squares of the last components of the $\hat{\mathbf{M}}$-normalized eigenvectors, determine the rod's free end point frequency response function due to a harmonic excitation.

We consider the uniform case, $EA = \rho A = L = 1$, where the rod is modelled by four equal elements. The *stiffness* $\hat{\mathbf{K}}$ and *mass* $\hat{\mathbf{M}}$ matrices, both in $\mathcal{R}^{2n\times 2n}$, for this case are

$$\hat{\mathbf{K}} = 6\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \tag{5.1}$$

$$\hat{\mathbf{M}} = \frac{1}{36}\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{5.2}$$

The first row of Table 1 shows the eigenvalues of this pair and the second and third rows show the first and last components of the corresponding eigenvectors.

Table 1. The eigenvalues and the first and last components of the eigenvectors of the pair $\hat{\mathbf{K}}$ and $\hat{\mathbf{M}}$.

j	1	2	3	4	5	6
μ_j	2.4815	23.3699	70.8756	156.1612	285.2015	410.6475
$\mathbf{e}_1^T\hat{\mathbf{Y}}\mathbf{e}_j$	0.3681	1.0527	1.5743	1.7931	1.5233	0.6234
$\mathbf{e}_{2n}^T\hat{\mathbf{Y}}\mathbf{e}_j$	1.4223	−1.4888	1.6298	−1.8563	2.1542	−2.4088

The starting data, shown in Tables 2 and 3, are the eigenvalues and first and last components of the eigenvectors of the $n \times n$ submatrices $\mathbf{K}_1, \mathbf{M}_1$, and $\mathbf{K}_2, \mathbf{M}_2$ where

$$\mathbf{K} = 6\begin{pmatrix} \mathbf{K}_1 & \\ & \mathbf{K}_2 \end{pmatrix} = 6\begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 3 & & & \\ & & & 3 & -1 & 0 \\ & & & -1 & 2 & -1 \\ & & & 0 & -1 & 1 \end{pmatrix},$$

$$\mathbf{M} = \frac{1}{36}\begin{pmatrix} \mathbf{M}_1 & \\ & \mathbf{M}_2 \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 4 & 1 & 0 & & & \\ 1 & 4 & 1 & & & \\ 0 & 1 & 3 & & & \\ & & & 3 & 1 & 0 \\ & & & 1 & 4 & 1 \\ & & & 0 & 1 & 2 \end{pmatrix}.$$

Thus, with $\alpha = -6$, $\beta = 1/36$, and $\mathbf{u} = (0,0,1,1,0,0)^T$, we have

$$\hat{\mathbf{K}} = \mathbf{K} + \alpha\mathbf{u}\mathbf{u}^T, \qquad \hat{\mathbf{M}} = \mathbf{M} + \beta\mathbf{u}\mathbf{u}^T,$$

Table 2. Spectral data for $\mathbf{K}_1, \mathbf{M}_1$.

j	1	2	3
λ_j	30.9992	148.5613	373.6102
$\mathbf{e}_1^T\mathbf{Y}_1\mathbf{e}_j$	−1.5481	−2.3452	1.3274
$\mathbf{e}_n^T\mathbf{Y}_1\mathbf{e}_j$	−0.8591	1.8807	2.9825

Table 3. Spectral data for $\mathbf{K}_2, \mathbf{M}_2$.

j	1	2	3
λ_j	14.6857	167.2091	432
$\mathbf{e}_1^T\mathbf{Y}_2\mathbf{e}_j$	0.6997	2.3609	2.6833
$\mathbf{e}_n^T\mathbf{Y}_2\mathbf{e}_j$	2.2641	−2.9182	2.6833

so $\hat{\mathbf{u}} = \mathbf{Y}^T\mathbf{u}$ is

$$\hat{\mathbf{u}} = (-0.8591, 1.8807, 2.9825, 0.6997, 2.3609, 2.6833)^T$$

and the eigenvalues of the pair $\mathbf{K}, \mathbf{M}$ are (in the corresponding order)

$$\begin{aligned}\mathbf{\Lambda} &= \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_6\} \\ &= \operatorname{diag}\{30.9992, 148.5613, 373.6102, 14.6857, 167.2091, 432.0000\}.\end{aligned}$$

The secular equation is

$$g(\mu) = 1 - (36\mu + 6)\sum_{j=1}^{6} \frac{\hat{u}_j^2}{\lambda_j - \mu}$$

and its zeros are the μ_j shown in Table 1.

We now find the first and last rows of the eigenvector corresponding to the eigenvalue $\lambda = 2.4815$. The eigenvector we seek is a scalar multiple θ of $\hat{\mathbf{x}}$

$$\mathbf{w} = \theta\hat{\mathbf{x}} = (\mathbf{\Lambda} - 2.4815\mathbf{I})^{-1}\hat{\mathbf{u}}.$$

This matrix multiplication requires only $2n = 6$ flops. We find θ by using the normalization condition (3.6). Thus,

$$\mathbf{w}^T\left(\mathbf{I} + \frac{1}{36}\hat{\mathbf{u}}^T\hat{\mathbf{u}}\right)\mathbf{w} = \theta^2\hat{\mathbf{x}}^T\hat{\mathbf{B}}\hat{\mathbf{x}} = \theta^2,$$

whence, $\theta = 0.0736$ and so

$$\hat{\mathbf{x}} = (-0.4091, 0.1748, 0.1091, 0.7785, 0.1946, 0.0848)^T.$$

Note that the last n components of the $2n$-dimensional row vector

$$\mathbf{e}_1^T\mathbf{Y} = \left(\mathbf{e}_1^T\mathbf{Y}_1, \mathbf{o}^T\right)$$

vanish, as do the first n of $\mathbf{e}_{2n}^T\mathbf{Y}$. Thus, we compute the required first component of the eigenvector $\mathbf{x}$ by multiplying $\mathbf{e}_1^T\mathbf{Y}_1$ with the first three components of $\hat{\mathbf{x}}$, and similarly, we get the last component of $\mathbf{x}$ by multiplying $\mathbf{e}_n^T\mathbf{Y}_2$ with the last three components of $\hat{\mathbf{x}}$.

The first phase is completed once we have the first and last components of the eigenvectors for all the eigenvalues. This process encapsulates the essentials of one complete stage in a divide and conquer algorithm which may be applied to find all the eigenvalues of a large tridiagonal symmetric positive definite pencil.

We emphasize that the whole calculation was done without ever using eigenvector components not in the first or last row. This fact is important in large problems.

5.2. Numerical Results

In this section, we report on the performance of our method for the problem of the previous section but with $n = 128$ (instead of $n = 6$ as in the illustrative example above). All calculations were performed in MATLAB, running on an IEEE floating point standard machine with a *machine epsilon* of 2.22×10^{-16}.

The matrices $\hat{\mathbf{K}}, \hat{\mathbf{M}}$ for this example have diagonal and subdiagonal elements like those of (5.1) and (5.2) except that the scale factors 6 and $1/36$ are here replaced by n and $1/(6n)$, with $n = 128$. The required eigenvalues

$$\mu_j = 6n^2 \frac{(1 - \cos t_j)}{(2 + \cos t_j)}, \qquad t_j = \pi \left(\frac{j - 1}{2} \right), \quad j = 1, 2, \ldots, n,$$

now range (approximately) from 2.46743 to 196585. The exact eigenvector matrix is $\mathbf{X} = [x_{ij}] = \sin(it_j)$.

A history table showing the number of iterations used for each eigenvalue and showing each bisection step by the letter "b" and each rational step by the letter "r" is displayed in the Appendix. The relative error for each of the computed eigenvalues is also displayed.

All the eigenvalues were found to a relative error smaller than 9.9×10^{-13}, i.e., within the preset tolerance of 10^{-12}. In fact, many eigenvalues were found to limiting accuracy, a result of the quadratic convergence of the rational iteration.

We also tested the method on a variety of other matrix problems, some with random matrices and some with structured form. The empirical evidence from the application of this method suggests that the number of iterations required for each eigenvalue is, on average, about eight. We therefore estimate the number of flops required to find all $2n$ eigenvalues from the secular equation to be $18(2n)(2n + 5)$

Together with the flop count for the eigenvector calculations of Section 3, the total flop count for this process is $21(2n)(2n + 5)$. Of course, for an $n \times n$ system the count would be $21n(n + 5)$.

5.3. The Practical Zero-Finding Algorithm

As noted before, the algorithm used is bisection protected rational approximation iteration.

ALGORITHM STEPS.

(a) Merge the ratio α/β with the λ_j and sort them.
(b) Set a tolerance ϵ.
(c) Start the search for the next μ. Go to the next interval.
  (i) If the interval width is smaller than 2ϵ, set $\mu_j = \lambda_j$.
  (ii) Compute where to set left and right boundaries so that the computation of g will not overflow.
  (iii) If g vanishes at either end we have the next μ. Go to the next interval.
  (iv) If g has the same sign at both ends of the interval then, compute g at the left-hand side of the interval. If $g > 0$ and the left end point is smaller than α/β or if $g < 0$ and the left end point is greater than α/β, then set μ to the left-hand side end of the interval. Otherwise, set μ to the right-hand side end of the interval.
  (v) If none of the above conditions are met, then the function has a sign change between the currently set left and right endpoints. We can now iterate for the zero. The iteration starts with the midpoint of the interval and computes a rational approximation according to (4.2). If the resulting $\mu_j^{(k+1)}$ is inside the current bracketing interval, choose two of the endpoints which preserve bracketing. Otherwise, choose the midpoint of the interval as the next approximation.
  (vi) Termination criteria. We stop if
    (1) after two rational steps, the difference between successive iterates is smaller than ϵ,
    (2) the length of the bracketing interval falls below 2ϵ, or
    (3) if the function value goes to floating zero.

6. CONCLUSIONS

We have characterised the eigenvalues of a symmetric definite linear pencil modified by having scalar multiples of a rank-one update added to each of the matrices in the pencil. This is done

via the secular equation for the system, which expresses the eigenvalues of the modified system in terms of the eigenvalues of the original system and the modification. An interlacing property, which generalizes the well-known interlacing property for the corresponding modified standard eigenvalue problem, has been presented. These results provide a means of developing algorithms for the eigenvalues of certain modified systems (so-called matrix tearing) and spectral shifting. The algorithm suggested in the text may have application in the determination of the eigenvalues of large tridiagonal systems on parallel architectures.

APPENDIX

NUMERICAL RESULTS

Here we display, for the problem treated in Section 5, the eigenvalues of the $\hat{\mathbf{K}}, \hat{\mathbf{M}}$ pair, the relative error of the computed eigenvalues (compared against the exact eigenvalues for the problem), and an iteration history. Each letter "b" represents a bisection step that occurred in the solution process for that eigenvalue and each "r" represents a rational iteration step.

E'value	Rel Error		Iteration history
2.5	-3E-013	1	bbbbbbbbbbbbbrrrrrr
22.2	4E-014	2	rrrrr
61.7	-4E-015	3	rrrr
121.0	5E-015	4	rrrr
200.1	-9E-015	5	rrrr
299.0	1E-014	6	rrr
417.9	7E-015	7	rrr
556.7	4E-015	8	rrrr
715.7	1E-015	9	rrrr
894.8	3E-015	10	rrrr
1094.2	-1E-015	11	rrrr
1313.9	3E-015	12	rrrr
1554.3	-2E-015	13	rrrr
1815.3	-2E-015	14	rrrr
2097.1	0E+000	15	rrrbrbbbbbbbbbbbbbbbbbbbbbbrb
2399.9	3E-015	16	rrrbrbbbbbbbbbbbbbbbbbbbbbbbbrb
2723.9	-8E-016	17	rrrbbbbrb
3069.3	-3E-013	18	rrr
3436.3	-2E-015	19	rrrr
3825.1	-4E-014	20	rrr
4235.9	2E-016	21	rrrr
4669.0	-8E-016	22	rrr
5124.7	0E+000	23	rrrr
5603.2	0E+000	24	rrr
6104.8	-3E-016	25	rrrbbbbbbrb
6629.8	-2E-014	26	rrr
7178.5	8E-016	27	rrrbbbbrr
7751.3	-1E-013	28	rrr
8348.5	-4E-016	29	rrrbrbbbbbbbbbbbbbbbbbbbbbrb
8970.4	-6E-013	30	rrr
9617.4	-1E-015	31	rrrbrbbbbbbbbbbbbbbbbbbbbbbbbrb
10290.0	-8E-013	32	rrrbb
10988.4	-1E-015	33	rrrbrbbbbbbbbbbbbbbbbbbbrb
11713.1	-1E-012	34	rrrbbr
12464.6	-6E-016	35	rrrbrbbbbbbbbbbbbbbbbbbbbrb

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 13243.2  1E-016  36 rrrr
 14049.4 -6E-016  37 rrrbrbbbbbbbbbbbbbbbbbrb
 14883.8 -1E-015  38 rrrbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbrr
 15746.7  1E-016  39 rrrr
 16638.6  2E-016  40 rrrbrb
 17560.2  6E-016  41 rrrr
 18511.8  5E-013  42 rrrbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbb
 19494.1 -1E-015  43 rrrr
 20507.5 -4E-016  44 rrrr
 21552.7 -1E-015  45 rrrr
 22630.2  5E-016  46 rrrr
 23740.5 -8E-016  47 rrrr
 24884.4 -4E-016  48 rrrr
 26062.3 -2E-015  49 rrrr
 27275.0  9E-016  50 rrrbbbbbbbbbbbbbbbbbbbbbbbbrb
 28522.9 -2E-015  51 rrrr
 29806.9 -2E-015  52 rrrr
 31127.4 -3E-015  53 rrrr
 32485.2 -4E-015  54 rrrr
 33880.8 -3E-015  55 rrrr
 35315.0 -2E-014  56 rrrr
 36788.4 -5E-015  57 rrrr
 38301.5 -2E-013  58 rrrr
 39855.2 -8E-015  59 rrrr
 41450.0 -9E-016  60 rrrrbbbbbbbbbbbbbbbbbbbbbbbbrb
 43086.5 -1E-014  61 rrrr
 44765.4 -3E-015  62 rrrrr
 46487.3 -1E-014  63 rrrr
 48252.8  2E-016  64 rbbbbbbrrrrrrrrrr
 50062.3  4E-016  65 rrrbbbbbbbbbbbbbbbbbbbbbbbbrbbbbbrb
 51916.6 -6E-016  66 rrrrrr
 53816.1 -1E-015  67 rrrbbrbrb
 55761.2 -2E-015  68 rrrrr
 57752.4 -1E-015  69 rrrr
 59790.2  7E-016  70 rrrrbbbbbbbbbbbbbbbbbbbbbbbbrr
 61874.9 -8E-016  71 rrrr
 64006.7  7E-016  72 rrrrbbbbbbbbbbbbbbbbbbbbbbbbbbrb
 66185.9 -6E-013  73 rrrbbbb
 68412.6 -1E-015  74 rrrrbbbbbbbbbbbbbbbbbbbbbbbbrb
 70687.0 -2E-015  75 rrrr
 73009.0 -7E-013  76 rrrr
 75378.6 -3E-013  77 rrr
 77795.5 -4E-013  78 rrrr
 80259.4 -1E-014  79 rrr
 82769.9 -3E-013  80 rrrr
 85326.5  3E-016  81 rrbbbbbbbbbbbbbbbbbbbbbbbbrb
 87928.3 -2E-013  82 rrrr
 90574.6 -1E-013  83 rrr
 93264.3 -2E-013  84 rrrr
 95996.2 -1E-012  85 rrrbr
 98768.9 -1E-013  86 rrrr
101580.7  1E-015  87 rrrbbbbbbbbbbbbbbbbbbbbbbbbbbrb
104429.9 -1E-013  88 rrrr
107314.4 -8E-016  89 rrrbbbbbbbbbbbbbbbbbbbbbbbbrb
110232.0 -9E-014  90 rrrr
```

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113179.9 -6E-016  91 rrrbbbbbbbbbbbbbbbbbbbbbbbbbrb
116155.5 -8E-014  92 rrrr
119155.7 -2E-015  93 rrrr
122177.1 -7E-014  94 rrrr
125216.0 -3E-015  95 rrrbbbrb
128268.4 -6E-014  96 rrrr
131330.1 -9E-016  97 rrrr
134396.5 -4E-014  98 rrrr
137462.7 -3E-015  99 rrrr
140523.5 -3E-014 100 rrrr
143573.3 -2E-014 101 rrrr
146606.3 -2E-014 102 rrrr
149616.5 -9E-014 103 rrrr
152597.2 -9E-015 104 rrrr
155542.0 -4E-013 105 rrrr
158443.8 -3E-015 106 rrrr
161295.4 -6E-013 107 rrrrbbb
164089.6 -1E-015 108 rrrr
166818.7 -2E-015 109 rrrrr
169475.1  1E-015 110 rrrbbrb
172051.1 -8E-016 111 rrrrbrb
174538.8 -3E-013 112 rrr
176930.4  1E-015 113 rrrrbrb
179218.2  0E+000 114 rrrbrb
181394.5 -1E-014 115 rrrrr
183451.7 -5E-014 116 rrrr
185382.7 -6E-016 117 rrrrrr
187180.3 -1E-012 118 rrrrbbbbb
188837.9  2E-016 119 rrrrrr
190349.2 -4E-014 120 rrrrr
191708.4 -2E-015 121 rrrrrrr
192910.0 -9E-014 122 rrrrrr
193949.3 -3E-013 123 rbbbbrrrr
194822.0 -2E-013 124 rbbbbrrrrr
195524.5 -2E-015 125 rbbrbrrrrrrrr
196054.1 -2E-015 126 rbbbbrrrrrrbrb
196408.3 -7E-015 127 bbbbbrrrrrrrr
196585.8 -2E-015 128 bbrbbrrrrrbbrb
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