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Second order q-difference equations solvable by factorization method

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Abstract

By solving an infinite nonlinear system of q-difference equations one constructs a chain of q-difference operators. The eigenproblems for the chain are solved and some applications, including the one related to q-Hahn orthogonal polynomials, are discussed. It is shown that in the limit $q \to 1$ the present method corresponds to the one developed by Infeld and Hull.

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Keywords: q-difference equations; Factorization method; Discretization

1. Introduction

The discretization of the ordinary differential equations is an important and necessary step toward finding their numerical solutions. In place of the standard discretization based on the arithmetic progression, one can use a not less efficient q-discretization related to geometric progression. This alternative method leads to q-difference equations, which in the limit $q \to 1$ correspond to the original differential equations. The theory of q-difference equations and the related q-special functions theory have a long history (see e.g., [12]). During the last two decades they have been reviewed because of the great success of the theory of quantum groups.

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The other crucial way of solving ordinary differential equations is based on the factorization method first used by Darboux [10]. Later the method was rediscovered many times, in particular by the founders of quantum mechanics, see [23,11], while studying the Schrödinger equation. We refer to [18] for an exhaustive presentation of the factorization method. In [16], which is now considered to be fundamental, Infeld and Hull summarized the quantum mechanical applications of the method. Fixing an infinite system of Riccati type equations they have constructed a chain of second order differential operators and proposed some method of solving corresponding eigenproblems.

This paper uses the formalism of the factorization method developed in [14] based on generalized difference calculus. Other approaches to the factorization method in discrete case may be found, e.g., in [21,3,2,7,8,13,1].

We construct the chain (71) of second order q-difference operators by solving an infinite nonlinear q-difference system. This chain depends on a freely chosen function and a finite number of real parameters. In Section 3 we find a family of eigenvectors for the operators of (71). In Section 4 it is shown that q-Hahn orthogonal polynomials, which are q-deformation of the classical orthogonal polynomials, form the family of solutions obtained by our method. Other examples of solutions obtained by the factorization of q-difference equations are presented in Section 5. Finally, passing to the limit $q \to 1$ in (116), (117) we obtain some families of solutions for second order differential equations.

2. Factorized chain of the second order q-difference operators

In this section we shall consider the sequence of the second order q-difference unbounded operators

$$\mathbf{H}_k = Z_k(x)\partial_q Q^{-1}\partial_q + W_k(x)\partial_q + V_k(x), \quad k \in \mathbb{N} \cup \{0\}, \; 0 < q < 1, \tag{1}$$

acting in the Hilbert spaces $\mathscr{H}_k$. By definition $\mathscr{H}_k$ consists of the complex valued functions $\psi : [a, b]_q \to \mathbb{C}$ defined on the q-interval

$$[a, b]_q := \{q^n a : n \in \mathbb{N} \cup \{0\}\} \cup \{q^n b : n \in \mathbb{N} \cup \{0\}\} \tag{2}$$

and square-integrable, i.e. $\langle\psi|\psi\rangle_k < +\infty$, with respect to the scalar products

$$\langle\psi|\varphi\rangle_k := \int_a^b \overline{\psi(x)}\varphi(x)\varrho_k(x)\, \mathrm{d}_q x. \tag{3}$$

Let us recall (see [12]) that by definition the q-derivative is

$$\partial_q \psi(x) = \frac{\psi(x) - \psi(qx)}{(1-q)x} \tag{4}$$

and the q-integral on the q-interval $[a, b]_q$ is given by

$$\int_a^b \psi(x)\, \mathrm{d}_q x := \sum_{n=0}^{\infty} (1-q)q^n (b\psi(q^n b) - a\psi(q^n a)). \tag{5}$$

If $a = 0$ and $b = \infty$ then

$$\int_0^\infty \psi(x)\, \mathrm{d}_q x := \lim_{n\to\infty} \int_0^{q^{-n}} \psi(x)\, \mathrm{d}_q x = \sum_{n=-\infty}^{\infty} (1-q)q^n \psi(q^n). \tag{6}$$

In the case if $a=-\infty$ and $b=\infty$

$$\int_{-\infty}^{\infty} \psi(x)\, \mathrm{d}_q x := \lim_{n\to\infty} \int_{-q^{-n}}^{q^{-n}} \psi(x)\, \mathrm{d}_q x = \sum_{n=-\infty}^{\infty} (1-q)q^n(\psi(q^n)+\psi(-q^n)). \tag{7}$$

In the limit $q \to 1$ the above definitions correspond to their counterparts in standard calculus. It will be assumed that $b \neq q^n a$, for all $n \in \mathbb{Z}$, because in the opposite case the Hilbert space is finite dimensional and this case will not be discussed in the paper.

Let $\mathscr{D}([a,b]_q)$ be the set of functions on $[a,b]_q$ with finite support. It is clear that $\mathscr{D}([a,b]_q) \subset \mathscr{H}_k$ and is dense. Moreover, all domains of $\mathbf{H}_k$ contain $\mathscr{D}([a,b]_q)$.

The scalar products (3) are defined by the weight functions $\varrho_k : [a,b]_q \to \mathbb{R}$, which are related by the recursion relations

$$\varrho_{k-1} = \eta_k \varrho_k \tag{8}$$

and

$$\varrho_{k-1} = Q(B_k \varrho_k), \tag{9}$$

where η_k, B_k are real valued functions on $[a,b]_q$ and the operator Q is defined by the formula

$$Q\varphi(x) = \varphi(qx). \tag{10}$$

For the sake of consistency we need to add the conditions

$$Q(B_k\varrho_k) = \eta_k\varrho_k \tag{11}$$

on the functions η_k and B_k. Additionally, we impose the boundary conditions

$$B_k(a)\varrho_k(a) = B_k(b)\varrho_k(b) = 0. \tag{12}$$

If we introduce the functions

$$A_k(x) := \frac{B_k(x)-\eta_k(x)}{(1-q)x}, \tag{13}$$

we can rewrite the formula (11) in the form of q-Pearson equation [22]

$$\partial_q(B_k\varrho_k) = A_k\varrho_k. \tag{14}$$

In the limit $q \to 1$, Eq. (14) corresponds to the Pearson equation which is important for the theory of classical orthogonal polynomials [9].

We say that the operators $\mathbf{H}_k$ admit a factorization if

$$\mathbf{H}_k = \mathbf{A}_k^*\mathbf{A}_k + a_k \tag{15}$$

and

$$\mathbf{H}_k = d_{k+1}^{-1}(\mathbf{A}_{k+1}\mathbf{A}_{k+1}^* + a_{k+1}), \tag{16}$$

where the annihilation operators $\mathbf{A}_k : \mathscr{H}_k \to \mathscr{H}_{k-1}$ are of the form

$$\mathbf{A}_k = \partial_q + f_k \tag{17}$$

and f_k are real valued functions on the set $[a,b]_q$. The adjoint operators $\mathbf{A}_k^*: \mathscr{H}_{k-1} \to \mathscr{H}_k$, called the creation operators, are given by

$$\mathbf{A}_k^* = (\partial_q + f_k)^* = B_k(-\partial_q Q^{-1} + f_k) - A_k(1 + (1-q)xf_k). \tag{18}$$

The derivation of the formula (18) is given in Appendix A. Note that both domains of $\mathbf{A}_k$, $\mathbf{A}_k^*$ contain $\mathscr{D}([a,b]_q)$. It follows from (15) that the real valued functions Z_k, W_k and V_k are related to f_k, B_k, A_k by the formulas

$$Z_k = -B_k Q^{-1}(1 + (1-q)id\, f_k), \tag{19}$$

$$W_k = B_k f_k - A_k(1 + (1-q)id\, f_k) - q^{-1} B_k Q^{-1}(f_k), \tag{20}$$

$$V_k = -B_k \partial_q (Q^{-1}(f_k)) - A_k f_k (1 + (1-q)id\, f_k) + B_k f_k^2 + a_k. \tag{21}$$

Necessary and sufficient conditions for the consistency of factorization formulas (15) and (16) are

$$\eta_{k+1}(x) = g_k(x)\eta_k(q^{-1}x), \tag{22}$$

$$\varphi_{k+1}(x) = \frac{d_{k+1}}{g_k(x)}\, \varphi_k(q^{-1}x), \tag{23}$$

$$\alpha_k(x) - \frac{g_k(qx)}{d_{k+1}}\, \alpha_k(qx) = \left(\frac{q^2 d_{k+1} B_k(qx) - g_k(q^2x) B_k(q^2x)}{(1-q)^2 q^3 x^2} + d_{k+1} a_k - a_{k+1} \right) \frac{g_k(qx)}{\mathrm{d}_{k+1}^2}, \tag{24}$$

where we have introduced the additional notations

$$g_k(x) := \frac{B_{k+1}(x)}{B_k(x)}, \tag{25}$$

$$\varphi_k(x) := f_k(x) + \frac{1}{(1-q)x}, \tag{26}$$

$$\alpha_k(x) := \varphi_k^2(x)\eta_k(x). \tag{27}$$

The detailed derivation of these formulas is given in Appendix B and in [14].

Relations (22), (23) and (25), (27) allow us to express the functions B_k, η_k, φ_k and α_k by the initial data B_0, η_0, φ_0 and α_0:

$$B_k(x) = g_{k-1}(x) g_{k-2}(x) \dots g_0(x) B_0(x), \tag{28}$$

$$\eta_k(x) = g_{k-1}(x) g_{k-2}(q^{-1}x) \dots g_0(q^{-k+1}x)\eta_0(q^{-k}x), \tag{29}$$

$$\varphi_k(x) = \frac{d_k \dots d_1}{g_{k-1}(x) \dots g_0(q^{-k+1}x)}\, \varphi_0(q^{-k}x), \tag{30}$$

$$\alpha_k(x) = \frac{(d_k \dots d_1)^2}{g_{k-1}(x) \dots g_0(q^{-k+1}x)}\, \alpha_0(q^{-k}x). \tag{31}$$

Substituting (28)–(31) into condition (24) we obtain the infinite sequence of the nonlinear q-difference equations

$$\begin{aligned}&\alpha_0(x) - d_{k+1}\frac{G_{k+1}(x)}{G_k(qx)}\alpha_0(qx)\\ &\quad = G_{k+1}(x)\left(d_{k+1}a_k - a_{k+1}\right.\\ &\qquad \left. + \frac{q^2 d_{k+1} g_{k-1}(q^{k+1}x)\ldots g_0(q^{k+1}x)B_0(q^{k+1}x) - g_k(q^{k+2}x)\ldots g_0(q^{k+2}x)B_0(q^{k+2}x)}{(1-q)^2 q^{2k+3}x^2}\right),\end{aligned} \tag{32}$$

where

$$G_k(x) := \frac{g_{k-1}(q^k x)\ldots g_0(qx)}{(d_k \ldots d_1)^2} \quad \text{for } k \in \mathbb{N}, \tag{33}$$

$$G_0(x) := 1, \tag{34}$$

for the functions α_0, B_0 and g_k for $k \in \mathbb{N} \cup \{0\}$.

One sees from (28)–(31) that the sequence of functions g_k, $k \in \mathbb{N}$, satisfying (32) defines the chain of q-difference operators (1) if the first element $\mathbf{H}_0$ of the chain is given. So, the problem of construction of the factorized chain given by (15) and (16) is equivalent to solving of the system of functional equations (32).

Let us now present the limit behaviour of the formulas obtained above when the parameter q tends to 1. It is easy to see that the set $[a, b]_q$ becomes the interval $[a, b]$ in the limit $q \to 1$ and the scalar product turns to be

$$\langle\psi|\varphi\rangle_k = \int_a^b \overline{\psi(x)}\varphi(x)\varrho_k(x)\,\mathrm{d}x, \tag{35}$$

where the weight function $\varrho_k(x)$ satisfies Pearson equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varrho_k B_k) = \varrho_k A_k, \tag{36}$$

with the boundary conditions (12). For $q \to 1$ the operator Q goes to the identity operator and $\partial_q \underset{q\to 1}{\longrightarrow} \mathrm{d}/\mathrm{d}x$. In the limiting case the annihilation and creation operators are of the form

$$\mathbf{A}_k = \frac{\mathrm{d}}{\mathrm{d}x} + f_k, \tag{37}$$

$$\mathbf{A}_k^* = B_k\left(-\frac{\mathrm{d}}{\mathrm{d}x} + f_k\right) - A_k \tag{38}$$

and the operators $\mathbf{H}_k$ are given by

$$\mathbf{H}_k = -B_k\frac{\mathrm{d}^2}{\mathrm{d}x^2} - A_k\frac{\mathrm{d}}{\mathrm{d}x} + (f_k^2 - f_k')B_k - f_k A_k + a_k. \tag{39}$$

The q-difference equation (1) tends to the differential equation

$$\left(Z_k(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}+W_k(x)\,\frac{\mathrm{d}}{\mathrm{d}x}+V_k(x)\right)\psi_k(x)=\lambda_k\psi_k(x), \tag{40}$$

where the coefficients are given by

$$Z_k(x)=-B_k(x), \tag{41}$$

$$W_k(x)=-A_k(x), \tag{42}$$

$$V_k(x)=(f_k^2(x)-f_k'(x))B_k(x)-f_k(x)A_k(x)+a_k. \tag{43}$$

The recurrence transformations (22), (23) for $q\to 1$ tend to

$$B_{k+1}=d_{k+1}B_k, \tag{44}$$

$$A_{k+1}=d_{k+1}\left(A_k-\frac{\mathrm{d}}{\mathrm{d}x}\,B_k\right). \tag{45}$$

The sequence of q-difference equations (24) tends to the sequence of non-linear differential equations

$$B_k(f_{k+1}^2-f_k^2+f_{k+1}'+f_k')-A_k(f_{k+1}-f_k)+2B_k'f_{k+1}-A_k'+B_k''=a_k-\frac{a_{k+1}}{d_{k+1}}, \tag{46}$$

$k\in\mathbb{N}\cup\{0\}$. Eq. (46) for $B_k(x)\equiv 1$ and $A_k(x)\equiv 0$ was considered in many papers (see [16,18–20,24,25]), but nevertheless for these differential-difference equations there is no complete theory. One of the methods for solving (46) is to look for the solutions in the form of infinite series

$$f_k=\sum_{i\in\mathbb{Z}}\tilde{f}_i(x)k^i \tag{47}$$

and obtain in this way the conditions on the function $\tilde{f}_i(x)$. The case of solutions given by the finite series were considered by Infeld and Hull [16]. The classification of all factorizable one-dimensional problems is still an open question.

Now, we come back to the general case. Regarding the extreme nonlinearity of system (32), the possibility to solve it is rather out of question. Therefore, we shall restrict ourselves to the subcase

$$g_k(x):=d_{k+1}q^{\gamma}\quad\text{for } \gamma\in\mathbb{R} \tag{48}$$

and consider system (32), which is reduced now to

$$\begin{aligned}\alpha_0(x)-q^{\gamma}\alpha_0(qx)&=\frac{q^{(k+1)\gamma}}{d_{k+1}\dots d_1}\,(d_{k+1}a_k-a_{k+1})\\&\quad+q^{2(k+1)\gamma}Q^{k+1}\frac{q^{2-\gamma}B_0(x)-B_0(qx)}{(1-q)^2qx^2},\end{aligned} \tag{49}$$

as the infinite system of equations on the initial functions B_0 and α_0. Eliminating α_0 from (49) we obtain

$$(1-q)^2 q^{3-\gamma} d_1^{-1} x^2 \left(\frac{q^{k\gamma}}{d_{k+1}\dots d_1}(d_{k+1}a_k - a_{k+1}) - d_1 a_0 + a_1 \right) \\ = q^{2-\gamma} B_0(qx) - B_0(q^2 x) - q^{2k(\gamma-1)}(q^{2-\gamma} B_0(q^{k+1}x) - B_0(q^{k+2}x)), \quad k \in \mathbb{N}. \tag{50}$$

Now, we shall look for the solution of (50) in the form

$$B_0(x) = x^\delta \sum_{n\in\mathbb{Z}} b_n x^n, \tag{51}$$

where $\delta \in [0, 1)$. Substituting (51) into (50) and comparing the coefficients in front of x^n we obtain the expressions for $a_k \in \mathbb{R}$:

$$a_{k+1} = d_{k+1}\dots d_1 q^{-\gamma k} \left(-a_0 \frac{[\gamma k]}{[\gamma]} + \frac{a_1}{d_1} \frac{[\gamma(k+1)]}{[\gamma]} - q b_2 [\gamma k][\gamma(k+1)] \right), \quad k \in \mathbb{N}, \tag{52}$$

where $[\gamma] = (1-q^\gamma)/(1-q)$, and the function B_0:

$$B_0(x) = b_2 x^2 + b_1 x^{2-\gamma} + b_0 x^{2-2\gamma}, \tag{53}$$

where $b_2, b_1, b_0 \in \mathbb{R}$. From (53) and (49) we have

(i) if $\gamma \neq 0$, then

$$\alpha_0(x) = \frac{q^{\gamma+1} b_2}{(1-q)^2} + \frac{q^\gamma (d_1 a_0 - a_1)}{(1-q^\gamma) d_1} + h x^{-\gamma} + \frac{q^{1-\gamma} b_0}{(1-q)^2} x^{-2\gamma}, \tag{54}$$

where $h \in \mathbb{R}$;

(ii) if $\gamma = 0$, then

$$\alpha_0(x) = h \quad \text{and} \quad d_1 a_0 = a_1, \tag{55}$$

where $h \in \mathbb{R}$. Finally, substituting (48) into (28)–(31) we find the following transformation formulas:

$$B_k(x) = q^{\gamma k} d_k \dots d_1 B_0(x), \tag{56}$$

$$\eta_k(x) = q^{\gamma k} d_k \dots d_1 \eta_0(q^{-k} x), \tag{57}$$

$$\varphi_k(x) = q^{-\gamma k} \varphi_0(q^{-k} x), \tag{58}$$

$$\alpha_k(x) = q^{-\gamma k} d_k \dots d_1 \alpha_0(q^{-k} x), \tag{59}$$

where B_0, α_0 are given by (53), (54) and (55), respectively. The functions η_0 and $\varphi_0(x)$ are related to A_0 and α_0 by

$$\eta_0(x) = b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma} - (1-q)xA_0(x), \tag{60}$$

$$\varphi_0(x) = \sqrt{\frac{\alpha_0(x)}{\eta_0(x)}}. \tag{61}$$

At the moment, given the functions B_0, α_0, we can use (56)–(59), (13), (14), (26) and (27) in order to express the functions A_k, f_k and ϱ_k:

$$\begin{aligned} A_k(x) = q^{\gamma k} d_k \dots d_1 (q^{-k} A_0(q^{-k}x) + [-2k]b_2 x \\ + [k(\gamma-2)]b_1 x^{1-\gamma} + [2k(\gamma-1)]b_0 x^{1-2\gamma}), \end{aligned} \tag{62}$$

$$f_k(x) = q^{-\gamma k} f_0(q^{-k}x) - \frac{1-q^{k(1-\gamma)}}{(1-q)x}, \tag{63}$$

$$\varrho_k(x) = \frac{q^{-\gamma k(k+1)/2}}{d_k d_{k-1}^2 \dots d_1^k} \frac{\varrho_0(q^{-k}x)}{\prod_{n=0}^{k-1}(b_2 q^{-2n}x^2 + b_1 q^{n(\gamma-2)}x^{2-\gamma} + b_0 q^{2n(\gamma-1)}x^{2-2\gamma})} \tag{64}$$

by A_0, f_0 and ϱ_0. From conditions (13), (14), (26) and (27) we see that the functions A_0, f_0, ϱ_0 are related by

$$\varrho_0(x) = \frac{q^2 b_2 x + b_1 q^{2-\gamma} x^{1-\gamma} + b_0 q^{2(1-\gamma)} x^{1-2\gamma}}{b_2 x + b_1 x^{1-\gamma} + b_0 x^{1-2\gamma} - (1-q)A_0(x)} \varrho_0(qx), \tag{65}$$

$$\left(f_0(x) + \frac{1}{(1-q)x}\right)^2 = \frac{\alpha_0(x)}{b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma} - (1-q)xA_0(x)}. \tag{66}$$

So, further we shall assume that the function $A_0(x)/B_0(x)$ is continuous in 0. Under this assumption we obtain from (65) and (66)

$$f_0(x) = \sqrt{\frac{\alpha_0(x)}{b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma} - (1-q)xA_0(x)}} - \frac{1}{(1-q)x}, \tag{67}$$

$$\varrho_0(x) = \frac{1}{b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma}} \prod_{n=0}^{\infty} \left(Q^n \frac{1}{1 - \frac{(1-q)x(A_0(x)}{(b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma})}} \right). \tag{68}$$

This means that one finds the explicit formulas for the annihilation and creation operators

$$\mathbf{A}_k = \partial_q - \frac{1}{(1-q)x} + q^{-\gamma k}\sqrt{\frac{\alpha_0(q^{-k}x)}{\eta_0(q^{-k}x)}}, \tag{69}$$

$$\begin{aligned} \mathbf{A}_k^* = d_k \dots d_1 \left(-q^{\gamma k}(b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma}) \left(\partial_q Q^{-1} + \frac{1}{(1-q)x} \right) \right. \\ \left. + \sqrt{\alpha_0(q^{-k}x)\eta_0(q^{-k}x)} \right) \end{aligned} \tag{70}$$

and from this the explicit expression for $\mathbf{H}_k$

$$\mathbf{H}_k = d_k \ldots d_1 \left(-(1-q)q^{-1}x^3(b_2 + b_1 x^{-\gamma} + b_0 x^{-2\gamma}) \sqrt{\frac{\alpha_0(q^{-(k+1)}x)}{\eta_0(q^{-(k+1)}x)}}\, \partial_q Q^{-1} \partial_q \right.$$
$$+ \left(-q^{-1}x^2(b_2 + b_1 x^{-\gamma} + b_0 x^{-2\gamma}) \sqrt{\frac{\alpha_0(q^{-(k+1)}x)}{\eta_0(q^{-(k+1)}x)}} + \sqrt{\alpha_0(q^{-k}x)\eta_0(q^{-k}x)} \right) \partial_q$$
$$+ \frac{b_2 + b_1 x^{-\gamma} + b_0 x^{-2\gamma}}{(1-q)^2} \left(q^{1+k\gamma} - (1-q)x \sqrt{\frac{\alpha_0(q^{-(k+1)}x)}{\eta_0(q^{-(k+1)}x)}} \right)$$
$$+ q^{-\gamma k}\alpha_0(q^{-k}x) - \frac{1}{(1-q)x}\sqrt{\eta_0(q^{-k}x)\alpha_0(q^{-k}x)}$$
$$\left. - q^{-\gamma(k-1)} \left(a_0 \frac{[\gamma(k-1)]}{[\gamma]} - \frac{a_1}{d_1}\frac{[\gamma k]}{[\gamma]} + q b_2 [\gamma(k-1)][\gamma k] \right) \right), \tag{71}$$

which depend only on a function A_0.

The chains of operators $\mathbf{A}_k$, $\mathbf{A}_k^*$ and $\mathbf{H}_k$ appearing in (69), (70) and (71) in the limit $q \to 1$ are given by

$$\mathbf{A}_k = \frac{\mathrm{d}}{\mathrm{d}x} + f_0(x) + \frac{k(\gamma-1)}{x}, \tag{72}$$

$$\mathbf{A}_k^* = d_k \ldots d_1 \left(B_0(x) \left(-\frac{\mathrm{d}}{\mathrm{d}x} + f_0(x) + \frac{k(\gamma-1)}{x} \right) - A_0(x) + k \frac{\mathrm{d}}{\mathrm{d}x} B_0(x) \right), \tag{73}$$

$$\mathbf{H}_k = d_k \ldots d_1 \left(-B_0(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (A_0(x) - kB_0'(x)) \frac{\mathrm{d}}{\mathrm{d}x} \right.$$
$$+ \left(f_0^2(x) - f_0'(x) + \frac{2k(\gamma-1)}{x} f_0(x) + \frac{k(\gamma-1)(k(\gamma-1)+1)}{x^2} \right) B_0(x)$$
$$\left. - \left(f_0(x) + \frac{k(\gamma-1)}{x} \right)(A_0(x) - kB_0'(x)) - a_0(k-1) + \frac{a_1}{d_1}k - b_2\gamma^2 k(k-1) \right), \tag{74}$$

where

$$B_k(x) = d_k \ldots d_1 B_0(x), \tag{75}$$

$$A_k(x) = d_k \ldots d_1 \left(A_0(x) - k \frac{\mathrm{d}}{\mathrm{d}x} B_0(x) \right), \tag{76}$$

$$f_k(x) = f_0(x) + k(\gamma-1)\frac{1}{x}, \tag{77}$$

$$\varrho_k(x) = \frac{1}{d_k d_{k-1}^2 \ldots d_1^k} \frac{\varrho_0(x)}{B_0^k(x)} \tag{78}$$

and the functions B_0, f_0 and ϱ_0 have the form

$$B_0(x) = b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma}, \tag{79}$$

$$f_0(x) = \begin{cases} \dfrac{-b_2(\gamma+1)x + \frac{d_1a_0-a_1}{\gamma d_1}x - b_1\tilde{h}x^{1-\gamma} - b_0(1-\gamma)x^{1-2\gamma} + A_0(x)}{2(b_2x^2 + b_1x^{2-\gamma} + b_0x^{2-2\gamma})} & \text{for } \gamma \neq 0, \\ -\dfrac{\tilde{\alpha}}{2}\dfrac{1}{x} + \dfrac{A_0(x)}{2(b_2+b_1+b_0)x^2} & \text{for } \gamma = 0, \end{cases} \tag{80}$$

$$\varrho_0(x) = \frac{1}{B_0(x)}\, \mathrm{e}^{\int_0^x (A_0(t)/B_0(t))\,\mathrm{d}t}. \tag{81}$$

Summing up we see that the construction presented above gives us the nontrivial chain of Hamiltonians (71) parameterized by the freely chosen function A_0 and the real parameters b_0, b_1, b_2, h, a_0, a_1, γ and d_k, $k \in \mathbb{N}$.

3. Eigenvalue problem for the chain of operators

We shall be interested in solving the eigenvalue problems

$$\mathbf{H}_k\psi_k = \lambda_k\psi_k \quad \text{for } k \in \mathbb{N} \cup \{0\}. \tag{82}$$

If the operators $\mathbf{H}_k$ admit the factorization given by (15) and (16), then the eigenvalue (82) is equivalent to the two equations

$$\mathbf{A}_k^*\mathbf{A}_k\psi_k = (\lambda_k - a_k)\psi_k, \tag{83}$$

$$\mathbf{A}_{k+1}\mathbf{A}_{k+1}^*\psi_k = (d_{k+1}\lambda_k - a_{k+1})\psi_k. \tag{84}$$

From (83) and (84) one gets

$$\mathbf{H}_{k+1}\mathbf{A}_{k+1}^*\psi_k = d_{k+1}a_k\mathbf{A}_{k+1}^*\psi_k \tag{85}$$

if

$$\mathbf{H}_k\psi_k = a_k\psi_k \tag{86}$$

or equivalently, if

$$\mathbf{A}_k\psi_k = 0. \tag{87}$$

Let us remark here that from (84) it follows that $\mathbf{A}_{k+1}^*\psi_k \in \mathscr{H}_{k+1}$. The formulas (84) show also that the application of $\mathbf{A}_{k+1}$ to $\mathbf{A}_{k+1}^*\psi_k$ turns it back to the eigenvector of $\mathbf{H}_k$ proportional to the eigenvector ψ_k. Therefore, in the case when $\lambda_k = a_k$ the eigenvalue problem (82) is reduced to Eq. (87) which is a first rank q-difference equation, i.e.

$$\psi_k(x) = \frac{q^{\gamma k}}{(1-q)x}\sqrt{\frac{\eta_0(q^{-k}x)}{\alpha_0(q^{-k}x)}}\psi_k(qx), \tag{88}$$

		$A_0(x)$	ξ_k
$\gamma>0$	$b_0\neq 0$	$x^{1-2\gamma}A(x)$	$-(\gamma-1)k-\frac{1}{2}\log_q\left(q^{\gamma-1}-(1-q)q^{\gamma-1}\frac{A(0)}{b_0}\right)$
	$b_0=0$ $b_1\neq 0$ $h\neq 0$	$x^{1-\gamma}A(x)$	$-(\gamma-1)k-\frac{1}{2}\log_q\left(\frac{b_1-(1-q)A(0)}{(1-q)^2h}\right)$
	$b_0=b_1=h=0$ $b_2\neq 0$ $b_2 \neq \frac{(1-q)(a_1-d_1a_0)}{[\gamma]qd_1}$	$xA(x)$	$-(\gamma-1)k-\frac{1}{2}\log_q\left(\frac{b_2-(1-q)A(0)}{q^{\gamma+1}b_2+\frac{(1-q)q^{\gamma}(d_1a_0-a_1)}{[\gamma]d_1}}\right)$
$\gamma=0$		$xA(x)$	$k-\frac{1}{2}\log_q\left(\frac{b_2+b_1+b_0-(1-q)A(0)}{(1-q)^2\alpha}\right)$
$\gamma<0$	$b_2\neq 0$	$xA(x)$	-$(\gamma-1)k-\frac{1}{2}\log_q\left(\frac{b_2-(1-q)A(0)}{q^{\gamma+1}b_2+\frac{(1-q)q^{\gamma}(d_1a_0-a_1)}{[\gamma]d_1}}\right)$
	$b_2=0$ $b_1\neq 0$ $h\neq 0$ $d_1a_0=a_1$	$x^{1-\gamma}A(x)$	$-(\gamma-1)k-\frac{1}{2}\log_q\left(\frac{b_1-(1-q)A(0)}{(1-q)^2h}\right)$
	$b_2=b_1=h=0$ $b_0\neq 0$ $d_1a_0=a_1$	$x^{1-2\gamma}A(x)$	$-(\gamma-1)k-\frac{1}{2}\log_q\left(q^{\gamma-1}-(1-q)q^{\gamma-1}\frac{A(0)}{b_0}\right)$

Fig. 1. Table of the forms of the function A_0 and the parameter ξ_k.

where B_0 and α_0 are given by (53) and (54)–(55), respectively. By applying the iteration method to (88) we find the solution

$$\psi_k(x)=x^{\xi_k}\prod_{n=0}^{\infty}\frac{q^{\xi_k+\gamma k}}{(1-q)q^n x}\sqrt{\frac{\eta_0(q^{n-k}x)}{\alpha_0(q^{n-k}x)}}, \tag{89}$$

where admissible choices of the real parameter ξ_k and function A_0 are presented in Fig. 1. $A(x)$ is to be an arbitrary analytic function. Now, let us answer the question of when the solution ψ_k of (89) belongs to the Hilbert space $\mathscr{H}_k$. In order to do this we observe that

$$(|\psi_k|^2\varrho_k)(x)=\frac{q^{2\gamma k}}{(1-q)^2x^2}\frac{B_0(qx)}{\alpha_0(q^{-k}x)}(|\psi_k|^2\varrho_k)(qx). \tag{90}$$

Eq. (90) can be written for $\gamma=0$ in the form

$$(|\psi_k|^2\varrho_k)(x)=\frac{q^2(b_2+b_1+b_0)}{(1-q)^2\alpha}(|\psi_k|^2\varrho_k)(qx), \tag{91}$$

and for $\gamma\neq 0$ in the form

$$\begin{aligned}(|\psi_k|^2\varrho_k)(x) &= \frac{q^{1-\gamma}(b_2(qx)^{2\gamma}+b_1(qx)^{\gamma}+b_0)}{q^{2\gamma}\left(b_2+\frac{(1-q)^2}{(1-q^{\gamma})}\frac{(d_1a_0-a_1)}{qd_1}\right)(q^{-k}x)^{2\gamma}+(1-q)^2q^{\gamma-1}h(q^{-k}x)^{\gamma}+b_0}\\ &\quad\times(|\psi_k|^2\varrho_k)(qx).\end{aligned} \tag{92}$$

We also observe that the function $|\psi_k|^2\varrho_k$ does not depend on $A_0(x)$. Using iteration method we obtain the classes of solutions of (90) described in the following proposition.

Proposition 1. *For the solutions to Eq.* (90), *the following cases hold*:

1. *For* $\gamma = 0$ *we have*

$$(|\psi_k|^2 \varrho_k)(x) = x^r, \tag{93}$$

   where $q^{-r} = q^2(b_2 + b_1 + b_0)/(1-q)^2\alpha$.

2. *For* $\gamma \neq 0$ *we have following possibilities*:

   (i) *If* $b_0 \neq 0$, $b_2 \neq 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)} \frac{(d_1 a_0 - a_1)}{q d_1} \neq 0$, *then*

$$(|\psi_k|^2 \varrho_k)(x) = x^{\gamma-1} \frac{\left(\frac{(qx)^\gamma}{x_1}; q^\gamma\right)_\infty \left(\frac{(qx)^\gamma}{x_2}; q^\gamma\right)_\infty}{\left(\frac{(q^{-k}x)^\gamma}{y_1}; q^\gamma\right)_\infty \left(\frac{(q^{-k}x)^\gamma}{y_2}; q^\gamma\right)_\infty}. \tag{94}$$

   (ii) *If* $b_0 \neq 0$, $b_2 \neq 0$, $h \neq 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)} \frac{(d_1 a_0 - a_1)}{q d_1} = 0$, *then*

$$(|\psi_k|^2 \varrho_k)(x) = x^{\gamma-1} \frac{\left(\frac{(qx)^\gamma}{x_1}; q^\gamma\right)_\infty \left(\frac{(qx)^\gamma}{x_2}; q^\gamma\right)_\infty}{\left(\frac{(q^{-k}x)^\gamma}{y_1}; q^\gamma\right)_\infty}. \tag{95}$$

   (iii) *If* $b_0 \neq 0$, $b_2 \neq 0$, $h = 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)} \frac{(d_1 a_0 - a_1)}{q d_1} = 0$, *then*

$$(|\psi_k|^2 \varrho_k)(x) = x^{\gamma-1} \left(\frac{(qx)^\gamma}{x_1}; q^\gamma\right)_\infty \left(\frac{(qx)^\gamma}{x_2}; q^\gamma\right)_\infty. \tag{96}$$

   (iv) *If* $b_0 = 0$, $b_1 \neq 0$, $b_2 \neq 0$, $h \neq 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)} \frac{(d_1 a_0 - a_1)}{q d_1} \neq 0$, *then*

$$(|\psi_k|^2 \varrho_k)(x) = x^r \frac{\left(\frac{(qx)^\gamma}{x_1}; q^\gamma\right)_\infty}{\left(\frac{(q^{-k}x)^\gamma}{y_1}; q^\gamma\right)_\infty}, \tag{97}$$

   where $q^{-r} = \left|\frac{q^{2+\gamma(k-1)} b_1}{(1-q)^2 h}\right|$.

   (v) *If* $b_0 = 0$, $b_1 \neq 0$, $b_2 \neq 0$, $h \neq 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)} \frac{(d_1 a_0 - a_1)}{q d_1} = 0$, *then*

$$(|\psi_k|^2 \varrho_k)(x) = x^r \left(\frac{(qx)^\gamma}{x_1}; q^\gamma\right)_\infty, \tag{98}$$

   where $q^{-r} = \left|\frac{q^{2+\gamma(k-1)} b_1}{(1-q)^2 h}\right|$.

(vi) *If* $b_0 = h = 0$, $b_1 \neq 0$, $b_2 \neq 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)}\frac{(d_1a_0-a_1)}{qd_1} \neq 0$, *then*

(a)

$$(|\psi_k|^2\varrho_k)(x) = x^r \frac{\left(\frac{(qx)^\gamma}{x_1}; q^\gamma\right)_\infty}{(-(q^{-k}x)^\gamma; q^\gamma)_\infty(-q^\gamma(q^{-k}x)^{-\gamma}; q^\gamma)_\infty}, \tag{99}$$

where $q^{-r} = \dfrac{q^{k\gamma+1}b_1}{q^{2\gamma}\left(b_2 + \frac{(1-q)^2}{(1-q^\gamma)}\frac{(d_1a_0-a_1)}{qd_1}\right)} > 0;$

(b)

$$(|\psi_k|^2\varrho_k)(x) = x^r \frac{\left(\frac{(qx)^\gamma}{x_1}; q^\gamma\right)_\infty}{((q^{-k}x)^\gamma; q^\gamma)_\infty(q^\gamma(q^{-k}x)^{-\gamma}; q^\gamma)_\infty}, \tag{100}$$

where $-q^{-r} = \dfrac{q^{k\gamma+1}b_1}{q^{2\gamma}\left(b_2 + \frac{(1-q)^2}{(1-q^\gamma)}\frac{(d_1a_0-a_1)}{qd_1}\right)} < 0.$

(vii) *If* $b_0 = b_1 = 0$, $b_2 \neq 0$, $h \neq 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)}\frac{(d_1a_0-a_1)}{qd_1} \neq 0$, *then*

(a)

$$(|\psi_k|^2\varrho_k)(x) = x^r \frac{(-x^\gamma; q^\gamma)_\infty(-q^\gamma x^{-\gamma}; q^\gamma)_\infty}{\left(\frac{(q^{-k}x)^\gamma}{y_1}; q^\gamma\right)_\infty}, \tag{101}$$

where $q^{-r} = q^{2+k\gamma}b_2/(1-q)^2h > 0;$

(b)

$$(|\psi_k|^2\varrho_k)(x) = x^r \frac{(x^\gamma; q^\gamma)_\infty(q^\gamma x^{-\gamma}; q^\gamma)_\infty}{\left(\frac{(q^{-k}x)^\gamma}{y_1}; q^\gamma\right)_\infty}, \tag{102}$$

where $-q^{-r} = \frac{q^{2+k\gamma}b_2}{(1-q)^2h} < 0.$

(viii) *If* $b_0 = b_1 = h = 0$, $b_2 \neq 0$ *and* $b_2 + \frac{(1-q)^2}{(1-q^\gamma)}\frac{(d_1a_0-a_1)}{qd_1} \neq 0$, *then*

$$(|\psi_k|^2\varrho_k)(x) = x^r, \tag{103}$$

where

$$q^{-r} = \left|\frac{q^{1-\gamma+2k\gamma}b_2}{b_2 + \frac{(1-q)^2}{(1-q^\gamma)}\frac{(d_1a_0-a_1)}{qd_1}}\right|.$$

In all the above cases x_1, x_2 *are roots of the polynomial*

$$b_2x^2 + b_1x + b_0 = 0 \tag{104}$$

and y_1, y_2 *are roots of the polynomial*

$$\left(q^{2\gamma}b_2 + (1-q)^2 \frac{q^{2\gamma-1}(d_1a_0 - a_1)}{(1-q^\gamma)d_1}\right)x^2 + (1-q)^2q^{\gamma-1}hx + b_0 = 0. \tag{105}$$

Proof. We easily obtain the subcases (i)–(iii) by iteration. The other cases are proved by calculation of the Laurent expression coefficient and application of Jacobi identity

$$\sum_{k=-\infty}^{\infty} q^{k^2}x^k = (q^2;q^2)_\infty(-qx;q^2)_\infty(-q/x;q^2)_\infty \tag{106}$$

(see [12]). □

The proposition given below classifies those function (89) which are elements of Hilbert space $\mathscr{H}_k$.

Proposition 2. *The solution* (89) *of Eq.* (87) *belongs to the Hilbert space* $\mathscr{H}_k$ *if and only if the parameters* $b_0, b_1, b_2, \alpha, h, d_1, a_0, a_1$ *and* γ *satisfy the following conditions*:

(1) $\gamma = 0$ *and* $\alpha/(b_2+b_1+b_0) < q/(1-q)^2$.
(2) $\gamma > 0$ *and one of the following conditions is fulfilled*:

- (i) $b_0 \neq 0, b_2 \neq 0$ *and* $b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) \neq 0$;
- (ii) $b_0 \neq 0, b_2 \neq 0, h \neq 0$ *and* $b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) = 0$;
- (iii) $b_0 \neq 0, b_2 \neq 0, h = 0$ *and* $b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) = 0$;
- (iv) $b_0 = 0,\ b_1 \neq 0,\ b_2 \neq 0,\ h \neq 0,\ b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) \neq 0$ *and* $h/b_1 < q^{1+\gamma(k-1)}/(1-q)^2$;
- (v) $b_0 = 0,\ b_1 \neq 0,\ b_2 \neq 0,\ h \neq 0,\ b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) = 0$ *and* $h/b_1 < q^{1+\gamma(k-1)}/(1-q)^2$;
- (vi) $b_0 = h = 0, b_1 \neq 0, b_2 \neq 0$ *and* $b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) \neq 0$;
- (vii) *in this case the solutions never belong to the Hilbert space*;
- (viii) $b_0 = b_1 = h = 0, b_2 \neq 0, b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) \neq 0$ *and* $d_1a_0 - a_1/qd_1b_2 < ((1-q^\gamma)/(1-q)^2)(q^{\gamma(2k-1)} - 1))$.

The notation and classification given above are compatible with Proposition 1.

Proof. The function ψ_k belongs to the Hilbert space if

$$\int_a^b (|\psi_k|^2\varrho_k)(x)\,\mathrm{d}_qx < +\infty. \tag{107}$$

This is equivalent to

$$\sum_{n=0}^{\infty} (1-q)q^ny(|\psi_k|^2\varrho_k)(q^ny) < +\infty \tag{108}$$

for $y=a, b$. So, for the case (i) (i.e., $b_0 \neq 0, b_2 \neq 0$ and $b_2 + ((1-q)^2/(1-q^\gamma))((d_1a_0 - a_1)/qd_1) \neq 0$) we have from Proposition 1 that $|\psi_k|^2\varrho_k$ is given by (94), and we show that

$$(1-q)y^\gamma \sum_{n=0}^{\infty} q^{\gamma n} \frac{\left(\frac{(q^{n+1}y)^\gamma}{x_1}; q^\gamma\right)_\infty \left(\frac{(q^{n+1}y)^\gamma}{x_2}; q^\gamma\right)_\infty}{\left(\frac{(q^{n-k}y)^\gamma}{y_1}; q^\gamma\right)_\infty \left(\frac{(q^{n-k}y)^\gamma}{y_2}; q^\gamma\right)_\infty} < +\infty. \tag{109}$$

From the identity

$$(q^n a; q)_\infty = \frac{(a; q)_\infty}{(a; q)_n}, \tag{110}$$

where

$$(a; q)_\infty = (1-a)(1-qa)\ldots, \tag{111}$$

$$(a; q)_n = (1-a)(1-qa)\ldots(1-q^{(n-1)}a), \tag{112}$$

we obtain the conditions equivalent to (109)

$$(1-q)y^\gamma \frac{\left(\frac{(qy)^\gamma}{x_1}; q^\gamma\right)_\infty \left(\frac{(qy)^\gamma}{x_2}; q^\gamma\right)_\infty}{\left(\frac{(q^{-k}y)^\gamma}{y_1}; q^\gamma\right)_\infty \left(\frac{(q^{-k}y)^\gamma}{y_2}; q^\gamma\right)_\infty}$$

$$\times \sum_{n=0}^{\infty} q^{\gamma n} \frac{\left(\frac{(q^{-k}y)^\gamma}{y_1}; q^\gamma\right)_n \left(\frac{(q^{-k}y)^\gamma}{y_2}; q^\gamma\right)_n}{\left(\frac{(qy)^\gamma}{x_1}; q^\gamma\right)_n \left(\frac{(qy)^\gamma}{x_2}; q^\gamma\right)_n} < +\infty. \tag{113}$$

Those conditions are fulfilled for $\gamma > 0$. The proofs of the other cases are similar to the one above. □

Finally, let us come back to the general situation and observe that (85), (86) and (87) imply that the function

$$\psi_k^n(x) := \mathbf{A}_k^* \ldots \mathbf{A}_{k-n+1}^* \psi_{k-n}(x), \quad n = 1, \ldots, k, \tag{114}$$

is an eigenvector of the operator $\mathbf{H}_k$ with the eigenvalue

$$\lambda_k^n = d_k d_{k-1} \ldots d_{k-n+1} a_{k-n} \tag{115}$$

if ψ_{k-n} is the eigenvector of $\mathbf{H}_{k-n}$ with eigenvalue a_{k-n}. Moreover, one comes back to the eigensubspace $\mathbb{C}\psi_{k-n}$ acting on $\mathbb{C}\psi_k^n$ by the annihilation operators $\mathbf{A}_{k-n+1}, \ldots$ and $\mathbf{A}_k$. The above described procedures can be illustrated by a lattice of points in the (k, n) plane (Fig. 2).

The eigenfunctions of the operator $\mathbf{H}_k$ given by (89) and (114) in the limit $q \to 1$ tend to

$$\psi_k(x) = x^{-k(\gamma-1)} e^{-\int_0^x f_0(t)\,dt}, \tag{116}$$

$$\psi_k^n(x) = \mathbf{A}_k^* \ldots \mathbf{A}_{k-n+1}^* x^{-(k-n)(\gamma-1)} e^{-\int_0^x f_0(t)\,dt} \quad \text{for } n = 1, 2, \ldots, k, \tag{117}$$

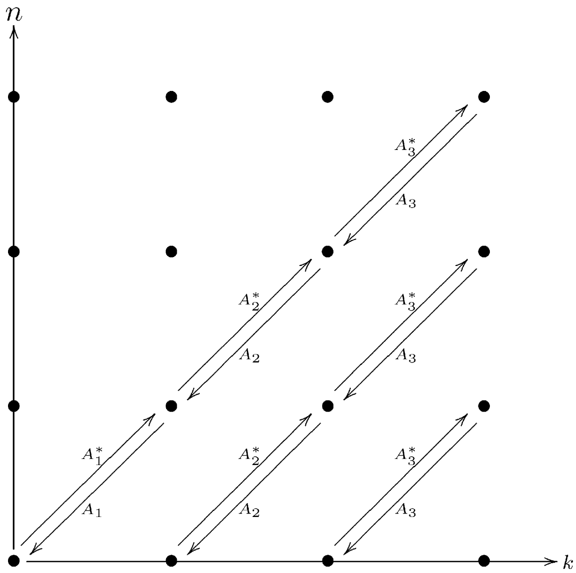

Fig. 2. Presentation of action of the operators $\mathbf{A}_k^*$.

with the eigenvalues

$$\lambda_k^n = d_k \ldots d_1 \left(-a_0(k-n-1) + \frac{a_1}{d_1}(k-n) - b_2\gamma^2(k-n)(k-n-1) \right). \tag{118}$$

The Hamiltonian (71) can be considered as a q-deformation of Schrödinger operator with known potentials. It can be shown that in the limit case $q \to 1$ by standard change of variables we can express Hamiltonian (74) as $\mathrm{d}^2/\mathrm{d}x^2 + V(x)$, where V becomes harmonic (1D or 3D), Morse, Rosen–Morse, Eckart or Poschl–Teller potential.

In the next sections we want to present some important examples, including the example of orthogonal polynomials of q-Hahn class which, in the limit $q \to 1$, gives classical orthogonal polynomials. These examples will illustrate how the factorization method works in our approach by writing down special cases of Hamiltonian (71) for some choices of the free parameters when we can find some solutions.

4. The class of q-Hahn orthogonal polynomials

We shall consider the class of q-Hahn polynomials orthogonal with respect to the measures equivalent to the Jackson measure $d_q x$.

We obtain the class of q-Hahn orthogonal polynomials when we require that the functions $f_k(x) \equiv 0$ and $d_k = q^{-1}$ for $k \in \mathbb{N}$. This is equivalent to

$$\gamma = 1, \tag{119}$$

$$B_k(x) = B_0(x) = b_2x^2 + b_1x + b_0, \tag{120}$$

$$A_0(x) = ([2]b_2 - qa_0 + q^2 a_1)x + \frac{b_1}{1-q} - (1-q)h. \tag{121}$$

We see that the functions B_k and A_0 are a second and a first order polynomials, respectively. From (62) we obtain that the function A_k is also first order polynomial

$$A_k(x) = q^{-k} A_0(q^{-k}x) + \frac{1 - Q^{-k}}{(1-q)x} B_0(x) = \tilde{a}_k x + \tilde{b}_k, \tag{122}$$

where

$$\tilde{a}_k = -q^{-2(k-1)}([2(k-1)]b_2 + q^{-1}a_0 - a_1), \tag{123}$$

$$\tilde{b}_k = \frac{b_1}{1-q} - (1-q)q^{-k}h. \tag{124}$$

Hence, the annihilation and creation operators are given by

$$\mathbf{A}_k = \partial_q, \tag{125}$$

$$\mathbf{A}_k^* = -(b_2x^2 + b_1x + b_0)\partial_q Q^{-1} - \tilde{a}_k x - \tilde{b}_k \tag{126}$$

and the Hamiltonian by

$$\begin{aligned} H_k = & -(b_2x^2 + b_1x + b_0)\partial_q Q^{-1}\partial_q - (\tilde{a}_k x + \tilde{b}_k)\partial_q \\ & + q^{-2(k-1)}(-q^{-1}a_0[k-1] + a_1[k] - b_2[k-1][k]). \end{aligned} \tag{127}$$

The eigenvalue problem for the Hamiltonian (127) is known as the q-Hahn equation [15,22]

$$(B_0(x)\partial_q Q^{-1}\partial_q + A_k(x)\partial_q)\psi_k^n = \lambda_k^n \psi_k^n. \tag{128}$$

The eigenvectors related to the eigenvalues

$$\lambda_k^0 = 0 \tag{129}$$

$$\lambda_k^n = \tilde{a}_k[n] + b_2[n][n-1]q^{-(n-1)} \tag{130}$$

are given by

$$\psi_k^0 = 1, \tag{131}$$

$$\psi_k^n = \mathbf{A}_k^* \ldots \mathbf{A}_{k-n+1}^* 1 = \prod_{i=k-n+1}^{k} (-(b_2x^2 + b_1x + b_0)\partial_q Q^{-1} - \tilde{a}_i x - \tilde{b}_i)1, \tag{132}$$

for $k \in \mathbb{N} \cup \{0\}$ and $n = 1, 2, \ldots, k$. The functions ψ_k^n (132) are polynomials. Each of the families $\{\psi_k^n\}_{n=0}^k$ is a system of polynomials orthogonal with respect to the scalar product given by Jackson's integral

$$\int_a^b \psi_k^n(x)\psi_k^m(x)\varrho_k(x)\, \mathrm{d}_q x \sim \delta_{nm}, \tag{133}$$

where the weight functions are obtained from (68)

$$\varrho_k(x) = \frac{\varrho_0(q^{-k}x)}{B_0(q^{-k+1}x)\dots B_0(x)}. \tag{134}$$

The classes of the weight functions ϱ_0 and the set of integration $[a, b]_q$ in (133) are presented in [22].

Example 1. Let us denote the roots of polynomials $B_0(x)$ and $B_0(x) - (1-q)xA_0(x)$ by x_1, x_2 and y_1, y_2, respectively. For fixed $k \in \mathbb{N} \cup \{0\}$ we shall assume the condition

$$q^k y_1 < x_1 < 0 < x_2 < q^k y_2 \tag{135}$$

valid in the generic case. After substitution

$$a_k := \frac{q^{-k-1}x_2}{y_2}, \tag{136}$$

$$b_k := \frac{q^{-k-1}x_1}{y_1}, \tag{137}$$

$$c_k := \frac{q^{-k-1}x_1}{y_2}, \tag{138}$$

$$P_n^{(k)}\left(\frac{1}{q^k y_2}x; a_k, b_k, c_k; q\right) := \psi_k^n(x) \tag{139}$$

and the change of variables $y = (1/q^k y_2)x$, we obtain from (128) the second order linear q-difference equation

$$\begin{aligned} &a_k q(y-1)(b_k y - c_k)P_n^{(k)}(qy; a_k, b_k, c_k; q) + (y - a_k q)(y - c_k q)P_n^{(k)}(q^{-1}y; a_k, b_k, c_k; q) \\ &\quad - (a_k q(y-1)(b_k y - c_k) + (y - a_k q)(y - c_k q))P_n^{(k)}(y; a_k, b_k, c_k; q) \\ &\quad\quad = q^{-n}(1-q^n)(1 - a_k b_k q^{n+1})y^2 P_n^{(k)}(y; a_k, b_k, c_k; q) \end{aligned} \tag{140}$$

for the big q-Jacobi polynomials, see [12]. The weight function (134) after the above substitution assumes the form

$$\varrho_k(y) = \frac{q^k y_2 \left(\frac{y}{a_k}; q\right)_\infty \left(\frac{y}{c_k}; q\right)_\infty}{b_0^k (y; q)_\infty \left(\frac{b_k x}{c_k}; q\right)_\infty}. \tag{141}$$

The big q-Jacobi polynomials are orthogonal with respect to the scalar product with the weight function (141) on the q-interval $[c_k q, a_k q]_\infty$.

Finally, let us remark that the q-derivative ∂_q plays the role of the lowering operator $\partial_q P_n^{(k)} = P_{n-1}^{(k-1)}$ which decreases the discrete parameter k and the degree of the polynomial.

Example 2. In this case we assume that one of the roots x_1, x_2 of $B_0(x)$ is $x_2 = 1$ and $b_2 - (1-q)\widetilde{a_0} = b_1 - (1-q)\widetilde{b_0} = 0$. Then Eq. (128) reduces to the second order q-difference equation

$$q^{n-1}x_1\psi_k^n(qx) + q^n(x-1)(x-x_1)\psi_k^n(q^{-1}x) - (q^{n-1}x_1 + q^n(x-1)(x-x_1))\psi_k^n(x) \\ = (1-q^n)x^2\psi_k^n(x) \tag{142}$$

for the Al–Salam–Carlitz I polynomials, see [12]. The solutions of (4) are orthogonal with respect to the scalar product given by q-integral on the q-interval $[x_1, 1]_q$ with the weight function of the form

$$\varrho_k(x) = b_0^{-k}\left(\frac{qx}{x_1}; q\right)_\infty (qx; q)_\infty. \tag{143}$$

Let us now come back to the general case. In the limit $q \to 1$ this case gives us the Hahn equation describing the classical orthogonal polynomials

$$\left(B_0(x)\frac{\mathrm{d}}{\mathrm{d}x} + A_k(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}\psi_k^n(x) = \lambda_k^n\psi_k^n(x). \tag{144}$$

The functions B_0 and A_k are second and first order polynomials given by

$$B_k(x) = B_0(x) = b_2x^2 + b_1x + b_0, \tag{145}$$

$$A_k(x) = \tilde{a}_kx + \tilde{b}_k, \tag{146}$$

where

$$\tilde{a}_k = -2(k-1)b_2 + a_1 - a_0, \tag{147}$$

$$\tilde{b}_k = b_1(\tilde{h} - k) \tag{148}$$

(in order to obtain these formulas we demand additionally that $h = b_1q^{\tilde{h}}/(1-q)^2$ in (124)). By appropriate choice of polynomials A_0 and B_0 we obtain known families of orthogonal polynomials, for details see [17].

The eigenvectors ψ_k^n (orthogonal polynomials), in the limiting case, have the forms

$$\psi_k^0(x) = 1, \tag{149}$$

$$\psi_k^n(x) = \left(B_0(x)\frac{\mathrm{d}}{\mathrm{d}x} + A_k(x)\right)\left(B_0(x)\frac{\mathrm{d}}{\mathrm{d}x} + A_{k-1}(x)\right)\cdots\left(B_0(x)\frac{\mathrm{d}}{\mathrm{d}x} + A_{k-n+1}(x)\right)1 \tag{150}$$

and correspond to the eigenvalues

$$\lambda_k^n = \tilde{a}_kn + b_2n(n-1). \tag{151}$$

5. The case of constant weight functions

We assume that all weight functions are constant $\varrho_k(x) \equiv \text{const}$. We obtain two cases, which we consider below. One of them can be considered as some discretization of the harmonic oscillator and the other, 3D harmonic oscillator. We will write explicit formulas for Hamiltonians and eigenvalues.

5.1. q-deformation of the harmonic oscillator

Additionally, we demand that $d_k = q^{-1}$ and $b_0 = \varrho_0 = 1$ for the sake of transparency of the formulas. In this case we have

$$\gamma = 1, \tag{152}$$

$$B_k(x) = 1, \tag{153}$$

$$A_k(x) = 0, \tag{154}$$

$$f_k(x) = q^{-k} f_0(q^{-k}x), \tag{155}$$

$$\varrho_k = 1, \tag{156}$$

where

$$f_0(x) = \sqrt{\frac{q^2(q^{-1}a_0 - a_1)}{1-q} + \frac{h}{x} + \frac{1}{(1-q)^2}\frac{1}{x^2}} - \frac{1}{(1-q)x}. \tag{157}$$

Thus the annihilation and creation operators are given by

$$\mathbf{A}_k = \partial_q + q^{-k} f_0(q^{-k}x), \tag{158}$$

$$\mathbf{A}_k^* = -\partial_q Q^{-1} + q^{-k} f_0(q^{-k}x) \tag{159}$$

and Hamiltonian has the form

$$\begin{aligned}\mathbf{H}_k = &-(1 + (1-q)q^{-k-1} x f_0(q^{-k-1}x))\partial_q Q^{-1}\partial_q \\ &+ q^{-k}(f_0(q^{-k}x) - q^{-1} f_0(q^{-k-1}x))\partial_q \\ &- q^{-k}\partial_q(f_0(q^{-k-1}x)) + q^{-2k} f_0^2(q^{-k}x) + q^{-2k}(a_0 + (q^2 a_1 - a_0)[k]).\end{aligned} \tag{160}$$

We will show later that it is one of the possible discretization of harmonic oscillator. By solving Eq. (88) we find the basic state ψ_k^0 of Hamiltonian (160) in two situations.

(1) If $a_0 \neq q a_1$, then

$$\psi_k^0(x) = \frac{C_k^0}{\sqrt{\left(\frac{q^{-k}x}{x_1}; q\right)_\infty \left(\frac{q^{-k}x}{x_2}; q\right)_\infty}}, \tag{161}$$

where x_1 and x_2 are roots of the polynomial

$$(1-q)q^2(q^{-1}a_0 - a_1)x^2 + (1-q)^2 h x + 1 = 0 \tag{162}$$

and $C_k^0 \in \mathbb{R}\backslash\{0\}$.

(2) If $a_0 = qa_1$ and $h \neq 0$, then

$$\psi_k^0(x) = \frac{C_k^0}{\sqrt{(-(1-q)^2 hq^{-k}x; q)_\infty}}. \tag{163}$$

It easy to see that the operator Q^{-1} acts as follows:

$$\psi_0^0 \overset{\frac{C_1^0}{C_0^0}Q^{-1}}{\longrightarrow} \psi_1^0 \overset{\frac{C_2^0}{C_1^0}Q^{-1}}{\longrightarrow} \cdots \overset{\frac{C_k^0}{C_{k-1}^0}Q^{-1}}{\longrightarrow} \psi_k^0 \overset{\frac{C_{k+1}^0}{C_k^0}Q^{-1}}{\longrightarrow} \cdots.$$

The functions ψ_k^0 are eigenvectors of the Hamiltonians $\mathbf{H}_k$ with the eigenvalues

$$\lambda_k^0 = a_k = q^{-2k}(a_0 + (q^2 a_1 - a_0)[k]). \tag{164}$$

Similarly it is easy to show that the functions

$$\psi_k^n(x) = Q^{-k}\psi_0^n(x) \tag{165}$$

are eigenvectors of $\mathbf{H}_k$ with

$$\lambda_k^n = q^{-2k}(\lambda_0^n + (q^2 a_1 - a_0)[k]), \tag{166}$$

due to the following commutation relations:

$$q\mathbf{A}_k^* Q^{-1} = Q^{-1}\mathbf{A}_{k-1}^*, \tag{167}$$

$$\mathbf{A}_k^* Q = qQ\mathbf{A}_{k+1}^*, \tag{168}$$

$$q\mathbf{A}_k Q^{-1} = Q^{-1}\mathbf{A}_{k-1}, \tag{169}$$

$$\mathbf{A}_k Q = qQ\mathbf{A}_{k+1}. \tag{170}$$

Finally, we present in Fig. 3 the action of the operators $\mathbf{A}_k$, A_k^* and state the following:

Proposition 3. *The functions*

$$\psi_k^n(x) = \frac{1}{\sqrt{(a_0 - qa_1)^n n_q! q^{n(n-1)+k}}} Q^{n-k}\mathbf{A}_n^* \ldots \mathbf{A}_1^* \psi_0^0(x), \tag{171}$$

for $k \in \mathbb{N} \cup \{0\}$ *and* $n \in \mathbb{N} \cup \{0\}$, *where the function* ψ_0^0 *is given by* (161) *or* (163), *are the eigenvectors of Hamiltonians* (160) *corresponding to the eigenvalues*

$$\lambda_k^n = q^{-2k+n}(a_0 + (q^2 a_1 - a_0)[k-n]). \tag{172}$$

The q-deformation of the harmonic oscillator presented here is connected with the discrete q-Hermite II polynomials. In order to see this let us rewrite eigenfunctions of the Hamiltonian (160) in the form

$$\psi_k^n(x) = P_n^{(k)}(q^{-k}x)\psi_k^0(x), \tag{173}$$

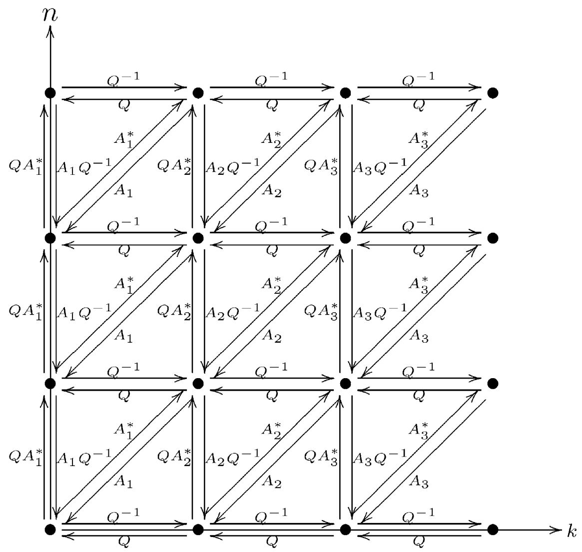

Fig. 3. Presentation of action of the operators.

where ψ_k^0 is the basic state given by formula (161), assume that $x_1 = -x_2 = i$, $a_0 - qa_1 = 1/q(1-q)$ and apply the change of variables $y = q^{-k}x$. Then the eigenproblem of this Hamiltonian reduces to the equation

$$-(1-q^n)y^2 P_n^{(k)}(y) = (1+y^2)P_n^{(k)}(qy) - (1+q+y^2)P_n^{(k)}(y) + qP_n^{(k)}(q^{-1}y). \tag{174}$$

It is an equation for the discrete q-Hermite II polynomials which are orthogonal with respect to the scalar product given by the q-integral on the q-interval $[-\infty, \infty]_q$ with the weight function

$$\varrho_k(y) = q^k(\psi_k^0)^2(y) = \frac{q^k(C_k^0)^2}{(iy;q)_\infty(-iy;q)_\infty}. \tag{175}$$

The factorization method for other q-deformations of the harmonic oscillator was developed in [4–6]. Models presented in these works are related to the continuous q-Hermite and to Stieltjes–Wigert polynomials.

It is easy to show that in the limit $q \to 1$ Hamiltonian (160) gives us the Hamiltonian of the harmonic oscillator

$$\mathbf{H}_k = -\frac{d^2}{dx^2} + \frac{(a_0-a_1)^2}{4}x^2 + \frac{a_1+a_0}{2} + (a_1-a_0)k \tag{176}$$

with eigenvectors

$$\psi_k^n(x) = \left(-\frac{d}{dx} + \frac{a_0-a_1}{2}x\right)^n e^{-(a_0-a_1/4)x^2} \quad \text{for } n \in \mathbb{N}\cup\{0\} \tag{177}$$

corresponding to the eigenvalues

$$\lambda_k^n = a_0 + (a_0-a_1)(n-k). \tag{178}$$

5.2. q-deformation of the three-dimensional isotropic harmonic oscillator

Additionally, we demand that $d_k = q^{-2}$ and $b_1 = \varrho_0 = 1$. In this case we have

$$\gamma = 2, \tag{179}$$

$$B_k(x) = 1, \tag{180}$$

$$A_k(x) = 0, \tag{181}$$

$$f_k(x) = q^{-2k} f_0(q^{-k}x) - \frac{1-q^{-k}}{(1-q)x}, \tag{182}$$

$$\varrho_k = 1, \tag{183}$$

where

$$f_0(x) = \sqrt{\frac{q^4(q^{-2}a_0 - a_1)}{1-q^2} + \frac{h}{x^2}} - \frac{1}{(1-q)x}. \tag{184}$$

The annihilation and creation operators have the form

$$\mathbf{A}_k = \partial_q + q^{-2k} f_0(q^{-k}x) - \frac{1-q^{-k}}{(1-q)x}, \tag{185}$$

$$\mathbf{A}_k^* = -\partial_q Q^{-1} + q^{-2k} f_0(q^{-k}x) - \frac{1-q^{-k}}{(1-q)x}, \tag{186}$$

and the Hamiltonians are given by the formulas

$$\begin{aligned}\mathbf{H}_k = & -(q^{-k} + (1-q)q^{-2k-1}xf_0(q^{-k-1}x))\partial_q Q^{-1}\partial_q \\ & + q^{-2k}(f_0(q^{-k}x) - q^{-1}f_0(q^{-k-1}x))\partial_q - q^{-2k}(\partial_q f_0(q^{-k-1}x)) + \frac{q^{-2k}[k][k+1]}{x^2} \\ & + q^{-4k} f_0^2(q^{-k}x) + 2q^{-3k}\frac{[k]}{x} f_0(q^{-k}x) + q^{-4k}\left(a_0 + (q^4 a_1 - a_0)\frac{[2k]}{[2]}\right).\end{aligned} \tag{187}$$

We will show that it can be considered as q-deformation of radial part of 3D isotropic harmonic oscillator.

The basic states of Hamiltonians (187) can be found as solution (88).

(1) If $a_0 \neq q^2 a_1$, then

$$\psi_k^0(x) = \frac{C_k^0}{\sqrt{\left(-\frac{q^4(q^{-2}a_0-a_1)}{(1-q^2)h} q^{-2k}x^2; q\right)_\infty}} x^{\xi_k}, \tag{188}$$

where $C_k^0 \in \mathbb{R}\backslash\{0\}$ and

$$\xi_k := -k + \log_q (1-q)\sqrt{h}. \tag{189}$$

(2) If $a_0 = q^2 a_1$, then

$$\psi_k^n(x) = C_k^0 x^{\xi_k}. \tag{190}$$

These are the eigenfunctions of the Hamiltonian corresponding to the eigenvalues

$$\lambda_k^0 = a_k = q^{-4k}\left(a_0 + (q^4 a_1 - a_0)\frac{[2k]}{[2]}\right). \tag{191}$$

Finally, we have the following proposition:

Proposition 4. *The functions*

$$\begin{aligned}\psi_k^n(x) &\\ &= \mathbf{A}_k^* \ldots \mathbf{A}_{k-n+1}^* \psi_{k-n}^0 \\ &= \prod_{i=k-n+1}^{k}\left(\frac{1}{(1-q)x}\left(-Q^{-1} + q^{-k}(1-q)\sqrt{h}\sqrt{1+\frac{q^4(q^{-2}a_0 - a_1)}{(1-q^2)h}q^{-2k}x^2}\right)\right)\psi_{k-n}^0,\end{aligned} \tag{192}$$

for $n = 1, 2, \ldots, k$, *are the eigenvectors of Hamiltonian* (187) *with the eigenvalues*

$$\lambda_k^n = q^{-2n} a_{k-n} = q^{-2(2k-n)}\left(a_0 + (q^4 a_1 - a_0)\frac{[2(k-n)]}{[2]}\right). \tag{193}$$

The q-deformation of the 3D isotropic harmonic oscillator presented here is connected with the q-Laguerre polynomials. In order to see then let us rewrite the eigenfunctions of Hamiltonian (187) in the form

$$\psi_k^n(x) = P_n^{(k)}((q^{-k}x)^2)\psi_k^0(x), \tag{194}$$

where ψ_k^0 is the basic state given by formula (188). We additionally assume that $q^4(q^{-2}a_0 - a_1)/(1 - q^2)h = 1$ and also apply the change of variables $y = (q^{-k}x)^2$. For details how to perform change of variables for chain of factorized operators see [14]. Then the eigenproblem for this Hamiltonian reduces to the q-difference equation

$$\begin{aligned}&q^{2\xi_k - 1}(1+y)P_n^{(k)}(q^2 y) - (1 + q^{2\xi_k-1}(1+y))P_n^{(k)}(y) + P_n^{(k)}(q^{-2}y) \\ &\quad = -q^{2\xi_k-1}(1-q^n)yP_n^{(k)}(y).\end{aligned} \tag{195}$$

It is the equation for the q-Laguerre polynomials which are orthogonal with respect to the scalar product given by the q-integral with the weight function

$$\varrho_k(y) = \frac{q^k y^{-(1/2)}}{1+q}(\psi_k^0)^2(y) = \frac{q^{k(2\xi_k+1)}(C_k^0)^2}{(1+q)(-y;q)_\infty}y^{\xi_k-(1/2)}. \tag{196}$$

In the limit $q \to 1$ the case considered in this subsection gives us the radial part of three-dimensional isotropic harmonic oscillator

$$\mathbf{H}_k = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(k-\tilde{h}/2)(k-(\tilde{h}/2)+1)}{x^2} + \frac{(a_0-a_1)^2}{16}x^2 - \frac{a_0-a_1}{2}\left(k+\frac{\tilde{h}}{2}\right) + \frac{3a_0+a_1}{4} \tag{197}$$

with eigenvectors

$$\psi_k^0(x) = C_k^0 x^{(\tilde{h}/2)-k} \mathrm{e}^{-((a_0-a_1)/8)x^2}, \tag{198}$$

$$\psi_k^n(x) = \prod_{i=k-n+1}^{k}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{a_0-a_1}{4}x - \frac{\tilde{h}}{2}\frac{1}{x} + \frac{i}{x}\right) x^{(\tilde{h}/2)-k} \mathrm{e}^{-((a_0-a_1)/8)x^2} \quad \text{for } n=1,\ldots,k, \tag{199}$$

corresponding to the eigenvalues

$$\lambda_k^n = a_0 + (a_1 - a_0)(k-n). \tag{200}$$

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Appendix A. Derivation of formula (18)

By the definition of the adjoint operator we have

$$\begin{aligned}
\langle Q^*\psi_k|\varphi_k\rangle_k = \langle \psi_k|Q\varphi_k\rangle_k &= \int_a^b \overline{\psi_k(x)}\varphi_k(qx)\varrho_k(x)\,\mathrm{d}_q x \\
&= \sum_{n=0}^{\infty}(1-q)q^n b\overline{\psi_k(q^n b)}\varphi_k(q^{n+1}b)\varrho_k(q^n b) \\
&\quad - \sum_{n=0}^{\infty}(1-q)q^n a\overline{\psi_k(q^n a)}\varphi_k(q^{n+1}a)\varrho_k(q^n a) \\
&\overset{m=n+1}{=} \sum_{m=1}^{\infty}(1-q)q^m bq^{-1}\overline{\psi_k(q^{m-1}b)}\varphi_k(q^m b)\varrho_k(q^{m-1}b) \\
&\quad - \sum_{m=1}^{\infty}(1-q)q^m aq^{-1}\overline{\psi_k(q^{m-1}a)}\varphi_k(q^m a)\varrho_k(q^{m-1}a).
\end{aligned}$$

In this sum the expression for $m=0$, i.e.,

$$(1-q)(b\overline{\psi_k(q^{-1}b)}\varphi_k(b)\varrho_k(b) - a\overline{\psi_k(q^{-1}a)}\varphi_k(a)\varrho_k(a)), \tag{A.1}$$

does not appear. The functions $\psi_k(x)$ and $\varphi_k(x)$ are defined on the set $\{q^n b : n \in \mathbb{N} \cup \{0\}\} \cup \{q^n a : n \in \mathbb{N} \cup \{0\}\}$ and for the other points we shall put these functions equal to zero

$$(Q^{-1}\psi)(b) := 0,$$

$$(Q^{-1}\psi)(a) := 0.$$

From Eq. (11) we obtain for $x \neq a$ and $x \neq b$ that

$$\begin{aligned}\langle Q^*\psi_k|\varphi_k\rangle_k &= \int_a^b \overline{\psi_k(q^{-1}x)}\varphi_k(x)\varrho_k(x)\,\frac{B_k(x)}{\eta_k(q^{-1}x)}\,q^{-1}\,\mathrm{d}_q x \\ &= \left\langle q^{-1}\,\frac{B_k}{Q^{-1}\eta_k}\,(Q^{-1}\psi_k)|\varphi_k\right\rangle_k.\end{aligned} \tag{A.2}$$

Similarly we have

$$\begin{aligned}\langle f\psi_k|\varphi_{k-1}\rangle_{k-1} &= \int_a^b \overline{f(x)\psi_k(x)}\varphi_{k-1}(x)\varrho_{k-1}(x)\,\mathrm{d}_q x \\ &= \int_a^b \overline{\psi_k(x)f(x)}\varphi_{k-1}(x)\eta_k(x)\varrho_k(x)\,\mathrm{d}_q x = \langle\psi_k|\overline{f}\eta_k\varphi_{k-1}\rangle_k,\end{aligned} \tag{A.3}$$

where we use Eq. (8).

Summarizing we obtain formula (18)

$$\mathbf{A}_k^* = (\partial_q + f_k)^* = B_k(-\partial_q Q^{-1} + f_k) - A_k(1+(1-q)xf_k), \tag{A.4}$$

where the operator Q^{-1} is given by

$$Q^{-1}\varphi(x) = \begin{cases}\varphi(q^{-1}x) & \text{for } x \neq a \text{ and } x \neq b, \\ 0 & \text{for } x = a \text{ or } x = b.\end{cases} \tag{A.5}$$

Appendix B. Derivation of formulas (22)–(24)

The operators of annihilation and creation given by (17), (18) can be rewritten in the form

$$\mathbf{A}_k = \partial_q + f_k = -\frac{1}{(1-q)x}\,Q + \varphi_k, \tag{B.1}$$

$$\begin{aligned}\mathbf{A}_k^* &= B_k(-\partial_q Q^{-1} + f_k) - A_k(1+(1-q)xf_k) \\ &= -\frac{B_k}{(1-q)x}\,Q^{-1} + \eta_k\varphi_k,\end{aligned} \tag{B.2}$$

where the functions φ_k, η_k are defined by (26) and (13). From the conditions (15) and (16) we have that

$$\begin{aligned}&\eta_k(qx)\varphi_k(qx) = d_k\eta_{k-1}(x)\varphi_{k-1}(x),\\&B_k(x)\varphi_k(x) = d_k B_{k-1}(x)\varphi_{k-1}(q^{-1}x),\\&\eta_k(x)\varphi_k^2(x) - d_k\eta_{k-1}(x)\varphi_{k-1}^2(x) = d_k a_{k-1} - a_k + \frac{q^2 d_k B_{k-1}(x) - B_k(qx)}{(1-q)^2 q x^2}.\end{aligned} \tag{B.3}$$

The first and second equations of (B.3) are equivalent to

$$\frac{\varphi_k(qx)}{\varphi_{k-1}(x)} = d_k\,\frac{\eta_{k-1}(x)}{\eta_k(qx)}, \tag{B.4}$$

$$\frac{\varphi_k(qx)}{\varphi_{k-1}(x)} = d_k\frac{B_{k-1}(qx)}{B_k(qx)}. \tag{B.5}$$

A simple calculation gives us

$$\eta_k(x) = \frac{B_k(x)}{B_{k-1}(x)}\,\eta_{k-1}(q^{-1}x) = g_{k-1}(x)\eta_{k-1}(q^{-1}x), \tag{B.6}$$

$$\varphi_k(x) = d_k\,\frac{B_{k-1}(x)}{B_k(x)}\,\varphi_{k-1}(q^{-1}x) = \frac{d_k}{g_{k-1}(x)}\,\varphi_{k-1}(q^{-1}x), \tag{B.7}$$

where the function $g_k(x)$ is given by (25). Substituting (B.6), (B.7) into the third relation in (B.3) we obtain finally

$$\begin{aligned}&\eta_{k-1}(x)\varphi_{k-1}^2(x) - \frac{g_{k-1}(qx)}{d_k}\,\eta_{k-1}(qx)\varphi_{k-1}^2(qx)\\&\quad = \left(d_k a_{k-1} - a_k + \frac{q^2 d_k B_{k-1}(qx) - g_{k-1}(q^2x)B_{k-1}(q^2x)}{(1-q)^2 q^3 x^2}\right)\frac{g_{k-1}(qx)}{\mathrm{d}_k^2}.\end{aligned} \tag{B.8}$$

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