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Globalization of confluent partial actions on topological and metric spaces

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Abstract

We generalize Exel's notion of partial group action to monoids. For partial monoid actions that can be defined by means of suitably well-behaved systems of generators and relations, we employ classical rewriting theory in order to describe the universal induced global action on an extended set. This universal action can be lifted to the setting of topological spaces and continuous maps, as well as to that of metric spaces and non-expansive maps. Well-known constructions such as Shimrat's homogeneous extension are special cases of this construction. We investigate various properties of the arising spaces in relation to the original space; in particular, we prove embedding theorems and preservation properties concerning separation axioms and dimension. These results imply that every normal (metric) space can be embedded into a normal (metrically) ultrahomogeneous space of the same dimension and cardinality.

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Introduction

Many extension problems in topology involve the question whether a given collection of partial maps on a space can be realized as the set of traces of a corresponding collection of total maps on some superspace. Consider, for example, the problem of constructing a homogeneous extension of a given topological (or metric) space X. A space is homogeneous (ultrahomogeneous) iff each partial homeomorphism (isometry) between two singleton (finite) subspaces extends to a global homeomorphism (isometry) [9] (cf. also [5,14,23], and [7] for ultrahomogeneous graphs). One way to look at the extension problem is to regard these partial maps as algebraic operators, so that we have a set of generators and relations for an algebra; the algebra thus generated can be expected to serve as a carrier set for the extended space. Indeed, this is precisely what happens in the constructions by Shimrat [27], Belnov [4], Okromeshko and Pestov [22], Uspenskij [29], and Megrelishvili [19–21].

Here, we pursue this concept at what may be hoped is the right level of generality: we begin by providing a generalization of Exel's notion of *partial group action* [11] to partial actions of monoids (i.e., the elements of the monoid act as partial maps on the space; cf. Definition 2.3). Partial actions of monoids are characterized in the same way as partial group actions as restrictions of global actions to arbitrary subsets. We then study properties of the *globalization* of a partial action, i.e., of the extended space which is universal w.r.t. the property that it has a global action of the original monoid. Most of the results we obtain depend on *confluence* of the partial action. Here, confluence means that the monoid and the carrier set of the globalization are given in terms of generators and relations in such a way that equality of elements can be decided by repeated uni-directional application of equations; this concept is borrowed from rewriting theory. The confluence condition is satisfied, for instance, in the case where the monoid is generated by a category whose morphisms act as partial maps on the space.

The basic construction of the globalization works in many topological categories; here, we concentrate on topological spaces on the one hand, and metric spaces on the other hand. For the topological case, we prove that, under confluence, the original space is topologically embedded in its globalization (and we provide an example which shows that this result fails in the non-confluent case). Moreover, we show that the globalization inherits its normality and dimension from the original space. Since free homogeneous extensions are globalizations for (confluent) 'singleton partial actions', this entails the corresponding results for such extensions.

The metric setting is best considered in the larger category of pseudometric spaces. Requiring confluence throughout, we prove an embedding theorem, and we show that for an important class of cases, the pseudometric globalization and the metric globalization coincide. We demonstrate that, in these cases, dimension is preserved. Furthermore, under suitable compactness assumptions, we prove existence of geodesic paths; by consequence, the globalization of a path metric space [12] is again a path metric space.

For every metric space, there exists an isometric embedding into a metrically ultrahomogeneous space of the same weight. This is a part of a recent result by Uspenskij [29], and well-known for the case of separable spaces [28] (see also [30]; for further information about Urysohn spaces, see [9,12,23,31]). We show that in many cases the metric globalization preserves the dimension. This implies that every metric space X admits a closed isometric embedding into an ultrahomogeneous metric space Z of the same dimension and cardinality. It is an open question if Z can be chosen in such a way that the weight of X is also preserved.

1. Confluently generated monoids

In preparation for the central notion of 'well-behaved' partial action, we now introduce a class of monoid presentations for which the word problem is solvable by means of head-on application of directed equations, i.e., by the classical rewriting method as used, up to now, mainly in computer science applications such as λ-calculus and automatic theorem proving [3,16] (see however [25,26] for applications to extensions of categories).

We recall that a *monoid presentation* $\langle G \mid R\rangle$ consists of a set G of *generators* and a relation $R \subset G^* \times G^*$, where G^* is the set of *words* over G, i.e., $G^* = \bigcup_{n=0}^{\infty} G^n$. Here, we explicitly insist that R is a directed relation (rather than symmetric); the elements (l, r) of R, written $l \to r$, are called *reduction rules* with *left side* l and *right side* r. Words are written either in the form $(g_n, \dots, g_1)$ or, where this is unlikely to cause confusion, simply in the form $g_n \dots g_1$. One way of describing the monoid engendered by $\langle G \mid R\rangle$ is as follows. The set G^* is made into a monoid by taking concatenation of words as multiplication, denoted as usual simply by juxtaposition; the unit is the empty word $()$. From R, we obtain a *one-step reduction* relation $\to$ on $G^* \times G^*$ by putting $w_1 l w_2 \to w_1 r w_2$ whenever $(l, r) \in R$ and $w_1, w_2 \in G^*$. Let $\overset{*}{\leftrightarrow}$ denote the equivalence relation generated by $\to$; then the monoid M described by $\langle G \mid R\rangle$ is $G^*/\overset{*}{\leftrightarrow}$.

It is well known that the word problem for monoids, i.e., the question whether or not $w_1 \overset{*}{\leftrightarrow} w_2$ for given words w_1, w_2, is in general undecidable. However, one can sometimes get a grip on the word problem by means of normal forms: a word w is called *normal* if it cannot be reduced under $\to$, i.e., if there is no word w' such that $w \to w'$ (otherwise w is called *reducible*); thus, a word is normal iff it does not contain a left side of a reduction rule. A normal word w' is called a *normal form* of a word w if $w \overset{*}{\leftrightarrow} w'$. We say that a monoid presentation is *noetherian* or *well-founded* if the relation $\to$ is well-founded, i.e., if there is no infinite sequence of reductions $w_1 \to w_2 \to \cdots$; this property guarantees existence, but not uniqueness of normal forms. However, one can characterize those cases where one does have uniqueness of normal forms. We denote the transitive and reflexive closure of $\to$ by $\overset{*}{\to}$ (reversely: $\overset{*}{\leftarrow}$); if $w \overset{*}{\to} w'$, then w' is said to be a *reduct* of w.

Proposition 1.1. *For a noetherian monoid presentation $\langle G \mid R\rangle$, the following are equivalent*:

(i) *Each word in G^* has a unique normal form.*
(ii) *Each word in G^* has a unique normal reduct.*
(iii) *Whenever $w \overset{*}{\to} s_1$ and $w \overset{*}{\to} s_2$, then there exists a* common reduct $t \in G^*$ *of (s_1, s_2), i.e., $s_1 \overset{*}{\to} t$ and $s_2 \overset{*}{\to} t$.*
(iv) *Whenever $w \to s_1$ and $w \to s_2$, then there is a common reduct of (s_1, s_2).*

This proposition is a special case of a central lemma of rewriting theory often referred to as *Newman's Lemma* (see, e.g., [16], Theorem 1.0.7.). Condition (iii) is called *confluence*, while condition (iv) is called *weak confluence*. The importance of the criterion lies in the fact that weak confluence is often reasonably easy to verify. In particular, it is enough to check weak confluence for so-called *critical pairs*, i.e., cases where left sides of reductions rules overlap. More precisely,

> *one can restrict condition* (iv) *to words w that are completely made up of the overlapping left sides of the two involved reduction rules*

(including the case that one of these left sides is contained in the other); it is easy to see that this restricted condition is equivalent to the original condition (iv). Since the proof of Proposition 1.1 is both short and instructive, we repeat it here:

Proof. (i) $\Rightarrow$ (iv) By the noetherian property, there exist normal words t_1 and t_2 such that $s_1 \overset{*}{\to} t_1$ and $s_2 \overset{*}{\to} t_2$. Then t_1 and t_2 are normal forms of w. By (i), we conclude $t_1 = t_2$.

(iv) $\Rightarrow$ (iii) We proceed by the principle of *noetherian* or *well-founded induction*, i.e., we prove the claim for w under the assumption that it holds for all proper reducts of w. We can assume w.l.o.g. that both $w \overset{*}{\to} s_1$ and $w \overset{*}{\to} s_2$ involve at least one reduction step, i.e., we have $w \to w_1' \overset{*}{\to} s_1$ and $w \to w_2' \overset{*}{\to} s_2$. By (iv), we obtain a common reduct t of (w_1', w_2'). By the inductive assumption, we obtain common reducts r_1 of (s_1, t) and r_2 of (s_2, t); again by the inductive assumption, there is a common reduct of (r_1, r_2), which is then also a common reduct of (s_1, s_2).

(iii) $\Rightarrow$ (ii) Existence follows immediately from the noetherian property. Concerning uniqueness, just observe that the existence of a common reduct of two normal words implies their equality.

(ii) $\Rightarrow$ (i) Whenever $w \to w'$, then (ii) implies that w and w' have the same normal reduct. Thus, since $\overset{*}{\leftrightarrow}$ is the equivalence relation generated by $\to$, this holds also whenever $w \overset{*}{\leftrightarrow} w'$. In particular, for normal words w and w', $w \overset{*}{\leftrightarrow} w'$ implies $w = w'$. $\square$

Definition 1.2. A noetherian monoid presentation is called *confluent* if it satisfies the equivalent conditions of Proposition 1.1 and does not contain reduction rules with left side g, where $g \in G$.

The requirement that there are no left sides consisting of a single generator can be satisfied for any noetherian monoid presentation by removing superfluous generators, since for a reduction rule with left side g, the noetherian condition implies that g cannot occur on the right side. Moreover, a noetherian monoid presentation cannot contain a reduction rule with left side $()$. Thus, in confluent monoid presentations any word with at most one letter is normal.

Example 1.3.

(i) Every monoid has a trivial confluent presentation: take all elements as generators, with reduction rules $uv \to p$ whenever $uv = p$.

(ii) The free monoid over a set G of generators trivially has a confluent presentation $\langle G \mid \emptyset \rangle$.

(iii) The free group over a set S of generators, seen as a monoid, has a confluent presentation $\langle S \oplus S^{-1} \mid R \rangle$, where $\oplus$ denotes the disjoint union and R consists of the reduction rules $ss^{-1} \to e$, $s^{-1}s \to e$ for each $s \in S$.

(iv) The free product $M_1 * M_2$ of two monoids M_1, M_2 with confluent presentations $\langle G_i \mid R_i \rangle$, $i = 1, 2$, respectively, has a confluent presentation $\langle G_1 \oplus G_2 \mid R_1 \oplus R_2 \rangle$. If M_1 and M_2 are groups, then $M_1 * M_2$ is a group, the free product of M_1 and M_2 as groups.

(v) The product $M_1 \times M_2$ of two monoids M_1, M_2 with confluent presentations $\langle G_i \mid R_i \rangle$, $i = 1, 2$, respectively, has a confluent presentation $\langle G_1 \oplus G_2 \mid R \rangle$, where R consists of all reduction rules in R_1 and R_2 and the additional reduction rules $gh \to hg$ whenever $g \in G_2$, $h \in G_1$.

(vi) Given a subset A of a monoid M that consists of left cancellable elements, the monoid M_A obtained by freely adjoining left inverses for the elements of A has a confluent presentation $\langle G \mid R \rangle$ as follows: we can assume that none of the elements of A has a right inverse (since a right inverse of a left cancellable element is already a left inverse). Then G consists of the elements of M and a new element l_a for each $a \in A$; R consists of the reduction rules for M according to (i) and the reduction rules $(l_a, au) \to (u)$ for each $a \in A$, $u \in M$. This is a special case of a construction for categories discussed in [25].

(vii) The infinite dihedral group has a confluent presentation $\langle \{a, b, b^{-1}\} \mid R \rangle$, where R consists of the reduction rules $bb^{-1} \to e$, $b^{-1}b \to e$, $aa \to e$, $ab \to b^{-1}a$, and $ab^{-1} \to ba$. (If the last reduction rule is left out, one still has a presentation of the same group, which however fails to be confluent.)

(viii) Given a category **C** [2,18], the monoid $M(\mathbf{C})$ induced by identifying all objects of **C** (see, e.g., [6]) has a presentation $\langle G \mid R \rangle$ given as follows. The set G of generators consists of all morphisms of **C**. There are two types of reduction rules: on the one hand, rules of the form $(f, g) \to (f \circ g)$ for all pairs (f, g) of composable morphisms in **C**, and on the other hand rules of the form $(id_C) \to ()$ for all objects C of **C**. This presentation satisfies the conditions of Proposition 1.1; it is turned into a confluent presentation in the stricter sense of Definition 1.2 by removing all identities from the set of generators and modifying the reduction rule associated to a pair (f, g) of morphisms to be $(f, g) \to ()$ in case $f \circ g = id$. This is a special case of the semicategory method introduced in [25].

Henceforth, we shall mostly denote elements of the monoid M presented by $\langle G \mid R \rangle$ directly as words (or composites of letters) rather than cluttering the notation by actually writing down equivalence classes of words. E.g., phrases such as 'u has normal form $g_n \cdots g_1$' means that an element $u \in M$ is represented by the normal word $(g_n, \ldots, g_1) \in G^*$. The unit element will be denoted by e.

Definition 1.4. Let M be a monoid with confluent presentation $\langle G \mid R\rangle$. An element $u \in M$ with normal form $g_n \cdots g_1$, where $g_i \in G$ for $i = 1, \dots, n$, is said to have *length* $\lg(u) = n$ (in particular, $\lg(e) = 0$). For a further $v \in M$ with normal form $v = h_m \cdots h_1$, we say that uv is normal if $g_n \cdots g_1 h_m \cdots h_1$ is normal. We denote the order on M induced by the prefix order on normal forms by $\preceq$; explicitly: we write $u \preceq p$ iff there exists v such that $p = uv$ is normal. If additionally $u \neq p$, then we write $u \prec p$. The direct predecessor $g_n \cdots g_2$ of u w.r.t. this order is denoted $\mathrm{pre}(u)$.

2. Partial actions and globalizations

Partial actions of groups have been defined and shown to coincide with the restrictions of group actions to arbitrary subsets in [11]. We recall the definition, rephrased according to [15]:

Definition 2.1. Let G be a group with unit e, let X be a set, and let α be a partial map $G \times X \to X$. We denote $\alpha(u, x)$ by $u \cdot x$, with $\cdot$ being right associative; i.e., $u \cdot v \cdot x$ denotes $u \cdot (v \cdot x)$. The map α is called a *partial action* of G on X if, for each $x \in X$,

(i) $e \cdot x = x$,
(ii) if $u \cdot x$ is defined for $u \in G$, then $u^{-1} \cdot u \cdot x = x$, and
(iii) if $u \cdot v \cdot x$ is defined, then $(uv) \cdot x = u \cdot v \cdot x$.

Here, equality is to be read as *strong* or *Kleene* equality, i.e., whenever one side is defined, then so is the other and the two sides are equal.

Concrete examples of partial group actions, including partial actions of groups of Möbius transforms, as well as further references can be found in [15].

Remark 2.2. In [15], partial actions are defined by conditions (ii) and (iii) above, and partial actions satisfying condition (i) are called *unital*. The original definition of partial actions [11] includes condition (i).

We generalize this definition to monoids as follows.

Definition 2.3. Given a set X, a *partial action* of a monoid M with unit e on X is a partial map

$$\alpha : M \times X \to X,$$

with the notation $\alpha(u, x) = u \cdot x$ as in Definition 2.1, such that

(i) $e \cdot x = x$ for all x, and
(ii) $(uv) \cdot x = u \cdot v \cdot x$ whenever $v \cdot x$ is defined.

(Again, (ii) is a strong equation.) Given two such partial actions of M on sets X_1, X_2, a map $f : X_1 \to X_2$ is called *equivariant* if $u \cdot f(x)$ is defined and equal to $f(u \cdot x)$ whenever $u \cdot x$ is defined.

We explicitly record the fact that partial monoid actions indeed generalize partial group actions:

Proposition 2.4. *The partial monoid actions of a group G are precisely its partial group actions.*

Proof. In the notation as above, let $e \cdot x = x$ for all $x \in X$. We have to show that conditions (ii) and (iii) of Definition 2.1 hold iff condition (ii) of Definition 2.3 holds.

'If': condition (iii) is immediate, since definedness of $u \cdot v \cdot x$ entails definedness of $v \cdot x$. Moreover, if $u \cdot x$ is defined, then by Definition 2.3(ii), we have $u^{-1} \cdot u \cdot x = (u^{-1}u) \cdot x = e \cdot x = x$; this establishes Definition 2.1(ii).

'Only if': the right-to-left direction of the strong equation in Definition 2.3(ii) is just Definition 2.1(iii). To see the converse direction, let $u, v \in G$, and let $v \cdot x$ and $(uv) \cdot x$ be defined; we have to show that $u \cdot v \cdot x$ is defined. By Definition 2.1(ii), $v^{-1} \cdot v \cdot x = x$, so that $(uv) \cdot v^{-1} \cdot v \cdot x$ is defined; by Definition 2.1(iii), it follows that $(uvv^{-1}) \cdot v \cdot x$ is defined, and this is $u \cdot v \cdot x$. $\square$

A partial action is equivalently determined by the partial maps

$$u : X \to X$$
$$x \mapsto u \cdot x$$

associated to $u \in M$. The domain of $u : X \to X$ is denoted $\mathrm{dom}(u)$.

Here, we are interested mainly in partial actions on spaces of some kind. E.g., we call a partial action of M on a topological space X *continuous* if the associated partial map $\alpha : M \times X \to X$ is continuous on its domain, where M carries the discrete topology, equivalently: if each of the maps $u : X \to X$ is continuous on $\mathrm{dom}(u)$. A partial action is called *closed* (*open*) if $\mathrm{dom}(u)$ is closed (open) for each $u \in M$, and *strongly closed* (*strongly open*) if, moreover, $u : X \to X$ is closed (open) on $\mathrm{dom}(u)$ for each u.

It is clear that a (total) action of M on a set Y induces a partial action on each subset $X \subset Y$. This statement has a converse:

Definition 2.5. Given a partial action of M on X, its (*universal*) *globalization* consists of a set Y with a total action of M and an equivariant map $i : X \to Y$ such that every equivariant map from X to a total action of M factors uniquely through i.

(Topological and metric globalizations are defined analogously, requiring continuity and non-expansiveness, respectively, for all involved maps.)

The globalization is easy to construct at the set level: the set Y is the quotient of $M \times X$ modulo the equivalence relation $\simeq$ generated by

$$(uv, x) \sim (u, v \cdot x) \quad \text{whenever } v \cdot x \text{ is defined} \tag{1}$$

(the generating relation $\sim$ is reflexive and transitive, but unlike in the case of groups fails to be symmetric). We denote the equivalence class of (u, x) by $[u, x]$. The action of M is defined by $u \cdot [v, x] = [uv, x]$. Moreover, $i(x) = [e, x]$. This map makes X a subset of Y:

Proposition 2.6. *The map $i : X \to Y$ defined above is injective, and the action of M on Y induces the original partial action on X.*

Proof. Define an equivalence relation ρ on $M \times X$ by

$$(u, x)\rho(v, y) \iff u \cdot x = v \cdot y,$$

where again equality is strong equality. By Definition 2.3(ii), ρ contains the relation $\sim$ defined in formula (1) above. Thus, ρ contains also the equivalence $\simeq$ generated by $\sim$; i.e., $(u, x) \simeq (v, y)$ implies the strong equation $u \cdot x = v \cdot y$. In particular, $(e, x) \simeq (e, y)$ implies $x = e \cdot x = e \cdot y = y$, so that i is injective. Moreover, it follows that $(u, x) \simeq (e, y)$ implies that $u \cdot x = y$ is defined, i.e., the restriction of the action on Y to X is the given partial action. $\square$

Thus, partial actions of monoids are precisely the restrictions of total actions to arbitrary subsets. From now on, we will identify X with $i(X)$ whenever convenient. By the second part of the above proposition, overloading the notation $u \cdot x$ to denote both the action on Y and the partial action on X is unlikely to cause any confusion.

The proof of the above proposition shows that equivalence classes of elements of X are easy to describe; however, a similarly convenient description is not generally available for equivalence classes of arbitrary (u, x)—that is, $(u, x) \simeq (v, y)$ may mean that one has to take a 'zig–zag path' from (u, x) to (v, y) that uses the generating relation $\sim$ of formula (1) both from left to right and from right to left. However, the situation is better for partial actions that have well-behaved presentations in the same spirit as confluently presented monoids.

Let α be a partial action of a monoid M on X, and let $\langle G \mid R\rangle$ be a confluent presentation of M. Then we regard the restriction of α to $G \times X$ as a collection of additional *reduction rules*, i.e., we write

$$(g, x) \to (g \cdot x) \quad \text{whenever } g \cdot x \text{ is defined for } g \in G,\ x \in X, \tag{2}$$

in addition to the reduction rules already given by R. In the same way as for monoid presentations, this gives rise to a *one-step reduction* relation $\to$ on the set $G^* \times X$, whose elements we denote in either of the two forms $(g_n, \ldots, g_1, x)$ or $g_n \cdots g_1 \cdot x$. Explicitly, we write $(g_n, \ldots, g_2, g_1, x) \to (g_n, \ldots, g_2, g_1 \cdot x)$ whenever $g_1 \cdot x$ is defined, and $w_1 \cdot x \to w_2 \cdot x$ whenever $w_1 \to w_2$ for words $w_1, w_2 \in G^*$. Moreover, we denote the transitive and reflexive hull of $\to$ and the equivalence relation generated by $\to$ on $G^* \times X$ by $\xrightarrow{*}$ and $\xleftrightarrow{*}$, respectively, and we use the terms *normal*, *normal form*, *reduct*, and *common reduct* as introduced for words in G^* in the previous section with the obvious analogous meanings for words in $G^* \times X$. Since the additional reduction rules always reduce the word length by 1, it is clear that reduction in $G^* \times X$ is also *well-founded* (or *noetherian*), i.e., that there are no infinite reduction sequences in $G^* \times X$. Thus, we have an analogue of Proposition 1.1 (with almost literally the same proof):

Proposition 2.7. *In the above notation, the following are equivalent*:

(i) *Each word in $G^* \times X$ has a unique normal form.*
(ii) *Each word in $G^* \times X$ has a unique normal reduct.*
(iii) *Whenever $w \overset{*}{\to} s_1$ and $w \overset{*}{\to} s_1$ in $G^* \times X$, then there exists a common reduct of (s_1, s_2),*
(iv) *Whenever $w \to s_1$ and $w \to s_2$ in $G^* \times X$, then there exists a common reduct of (s_1, s_2).*

In fact, the point behind all these analogies is that $(G^* \times X, \to)$ is just another example of a rewrite system, and the above proposition is another special case of Newman's Lemma. Concerning the verification of *weak confluence*, i.e., condition (iv) above, we remark that, besides checking confluence of $\langle G \mid R\rangle$, it suffices to consider cases of the form $w = g_n \cdots g_1 \cdot x$, where $g_n \cdots g_1$ is the left side of a reduction rule in R and $g_1 \cdot x$ is defined.

Definition 2.8. A partial action of a monoid M on a set X is called *confluent* if M has a confluent presentation $\langle G \mid R\rangle$ (cf. Section 1) such that the equivalent conditions of Proposition 2.7 hold for the associated reduction relation $\to$ on $G^* \times X$, and such that this reduction relation *generates* the given partial action. The latter means explicitly that, for $g_n \cdots g_1 \in G^*$,

$$(g_n \cdots g_1) \cdot x = y \quad \text{implies} \quad (g_n, \ldots, g_1, x) \overset{*}{\to} (y)$$

(the converse implication holds by the definition of partial actions).

For the sake of brevity, we shall fix the notation introduced so far (α *for the action*, X *for the space*, Y *for the globalization*, G *for the set of generators*, *etc.*) *throughout.*

By the generation condition, the quotient of $G^* \times X$ modulo the equivalence relation $\overset{*}{\leftrightarrow}$ is the universal globalization constructed above, so that we now have a way of deciding equivalence of representations for elements of the globalization outside X, namely via reduction to normal form. This will allow us to reach a good understanding of the properties of the globalization as a space.

In typical applications, a confluent partial action will often be given in terms of a monoid presentation $\langle G \mid R\rangle$ and a partial map $G \times X \to X$; in this case, the partial action of the monoid M presented by $\langle G \mid R\rangle$ is *defined* by putting $g_n \cdots g_1 \cdot x = y \iff (g_n, \ldots, g_1, x) \overset{*}{\to} (y)$. Verifying the conditions of Proposition 2.7 then guarantees that this does indeed define a partial action.

Example 2.9.

(i) A partial action of M is confluent w.r.t. the trivial confluent presentation of M (cf. Example 1.3(i)) iff, whenever $v \cdot x$, then either $(uv) \cdot x$ is defined or $(u, v \cdot x) = (uv, x)$: to see this, assume $(u, v \cdot x) \neq (uv, x)$; then $(uv) \cdot x$ is the only possible common reduct of the reducts $(u, v \cdot x)$ and (uv, x) of (u, v, x). Most of the time, this is a rather too strong property to require. In particular, if M is a group, then this holds iff, for each $v \neq e$, definedness of $v \cdot x$ implies definedness of $(uv) \cdot x$ for each u—this means that the partial action at hand is essentially just a total action on the subset $\{x \mid v \cdot x$ is defined for some $v \neq e\}$ of X.

(ii) Partial actions of the free monoid over G are always confluent w.r.t. the confluent presentation $\langle G \mid \emptyset \rangle$.

(iii) Partial actions of the free group over S are always confluent w.r.t. the confluent presentation of Example 1.3(iii).

(iv) Two confluent partial actions of monoids M_1 and M_2 on a set X, respectively, give rise to a confluent partial action of $M_1 * M_2$ on X w.r.t. the confluent presentation given in Example 1.3(iv).

(v) A total action of M on X can be extended to a confluent partial action on X of the extended monoid M_A of Example 1.3(vi) w.r.t. the confluent presentation given there (by putting $l_a \cdot (au \cdot x) = u \cdot x$ for each $a \in A$, $u \in M$, $x \in X$) iff each $a \in A$ acts injectively on X.

(vi) A partial action of the infinite dihedral group is confluent w.r.t. the confluent presentation given in Example 1.3(vii) iff
   (a) $a \cdot x$ and $ab \cdot x$ are defined whenever $b \cdot x$ is defined, and
   (b) $a \cdot x$ and $ab^{-1} \cdot x$ are defined whenever $b^{-1} \cdot x$ is defined.

(vii) A partial action of the monoid $M(\mathbf{C})$ generated by a small category $\mathbf{C}$ as in Example 1.3(viii) on a set X is confluent (w.r.t. the given confluent presentation of $M(\mathbf{C})$) iff, whenever f and g are composable morphisms in $\mathbf{C}$ and $g \cdot x$ is defined, then either $(f \circ g, x) = (f, g \cdot x)$, or $(f \circ g) \cdot x$ is defined (and hence also $f \cdot (g \cdot x)$).
In particular, this is the case if the partial action is given by a functor from $\mathbf{C}$ into the category $\mathbf{S}(X)$ of maps between subsets of X; this generalizes the preactions of groupoids considered in [19–21]. Here, we need only the simpler case that $\mathbf{C}$ is actually a subcategory of $\mathbf{S}(X)$. Explicitly, such a subcategory determines a confluent partial action of $M(\mathbf{C})$ as follows: if $f : A \to B$ is a morphism of $\mathbf{C}$, i.e., a map between subsets A and B of X, then $f \cdot x$ is defined iff $x \in A$, and in this case equal to $f(x)$. Analogously, one obtains a continuous partial action on a topological space X from a subcategory of the category $\mathbf{T}(X)$ of continuous maps between subspaces of X, etc.

Remark 2.10. Due to Example 1.3(i), it does not make sense to regard the existence of a confluent presentation as a property of a monoid; rather, a confluent presentation is considered as extra structure on a monoid. Contrastingly, the results about confluent partial actions presented below depend only on the existence of a confluent presentation; in the few places where we do make reference to the generating system in definitions, these definitions will turn out to be in fact independent of the chosen generating system by virtue of subsequently established results (see, for example, Definition 5.5 and Proposition 5.6). Thus, we mostly think of confluence of a partial action as a property; Example 2.14 will show that not all partial actions have this property.

As in the case of monoids, we usually denote the elements of Y directly by their representatives in $G^* \times X$ rather than as explicit equivalence classes. Of course, we can still represent elements of Y as pairs $(u, x) \in M \times X$. We will say that (u, x) or $u \cdot x$ is in normal form if $g_n \cdots g_1 \cdot x$ is in normal form, where $g_n \cdots g_1$ is the normal form of u; similarly, we write $u \cdot x \overset{*}{\to} v \cdot y$ if this relation holds with u and v replaced by their normal forms, etc. By the definition of confluent presentation, $g \cdot x$ is normal for $g \in G$, $x \in X$, whenever $g \cdot x$ is undefined in X. Moreover, $e \cdot x$ is always normal. We put

$$R_u = \{x \in X \mid u \cdot x \text{ is normal}\} = X \setminus \mathrm{dom}(g_1),$$

where u has normal form $g_n \cdots g_1$ (note that $R_e = X$). The action of u gives rise to a bijective map $u : R_u \to u \cdot R_u$.

Definition 2.11. An element $a \in Y$ with normal form $g_n \cdots g_1 \cdot x$ is said to have *length* $\mathrm{lg}(a) = n$. We put

$$Y_n = \bigl\{a \in Y \mid \mathrm{lg}(a) \leqslant n\bigr\}.$$

Of course, a confluent partial action is continuous iff the partial map $g : X \to X$ is continuous for each generator $g \in G$. A similar reduction holds for the domain conditions (closedness, etc.); cf. Section 3.

We finish this section by exhibiting an example of a partial action that fails to be confluent. This relies on an observation concerning the structure of the universal globalization Y of a confluent partial action.

Lemma 2.12. *Let α be confluent, and let $a = u \cdot x$ have normal form $v \cdot y$. Then $a \in w \cdot X$ whenever $v \preceq w \preceq u$ in the prefix order* (*cf. Definition* 1.4).

Proof. The reduction from (u, x) to (v, y) works by taking the normal form of u and then shifting letters from left to right according to formula (2). Thus, there must be an intermediate step of the form (w, z), which proves the claim. □

Proposition 2.13. *If α is confluent, then for every triple $(u_1, u_2, u_3) \in M^3$* (*indexed modulo* 3), *there exists $w \in M$ such that, for $i = 1, 2, 3$,*

$$u_i \cdot X \cap u_{i+1} \cdot X \subset w \cdot X$$

in Y.

Proof. Let $w_i = u_i \wedge u_{i+1}$ for $i = 1, 2, 3$. Here, $\wedge$ denotes the meet in the prefix order (cf. Definition 1.4), i.e., the largest common prefix. Now since for each i, w_i and w_{i+1} are both prefixes of u_{i+1}, they are comparable under the prefix order; i.e., the w_i form a chain. We can assume w.l.o.g. that w_1 is the largest element of this chain.

Then $w := w_1$ has the claimed property. Indeed, if $a = u_i \cdot x = u_{i+1} \cdot y$, then by confluence, a must have normal form $a = v \cdot z$, where $v \preceq u_i$ and $v \preceq u_{i+1}$. Thus, $v \preceq w_i \preceq w_1$; by Lemma 2.12, this implies $a \in w_1 \cdot X$, because we have $w_1 \preceq u_i$ or $w_1 \preceq u_{i+1}$. □

Example 2.14. Let $\mathcal{V}_4$ denote the Klein four-group $\{e, u, v, uv\}$, and let α be the partial action of $\mathcal{V}_4$ on the set $\{0, 1, 2\}$ defined by letting u, v, and uv act as partial identities defined on the domains {0}, {1}, and {2}, respectively. Then the triple $(e, u, uv) \in \mathcal{V}_4^3$ violates the property in Proposition 2.13. To see this, we show that

$$0 \notin v \cdot X \cup uv \cdot X, \quad u \cdot 1 \notin X, \quad \text{and} \quad 2 \notin u \cdot X.$$

The equivalence class of $(e, 0)$ in $M \times X$ is $\{(e, 0), (u, 0)\}$, because this set is closed under the generating relation $\sim$ of formula (1) above, so that indeed $0 \notin v \cdot X \cup uv \cdot X$; the other claims are proved similarly. Since we have

$$0 \in X \cap u \cdot X, \quad u \cdot 1 \in u \cdot X \cap uv \cdot X, \quad \text{and} \quad 2 \in uv \cdot X \cap X,$$

we have shown that there is no $w \in \mathcal{V}_4$ such that $w \cdot X$ contains all three pairwise intersections of X, $u \cdot X$, and $uv \cdot X$. Thus, α fails to be confluent.

3. Topological globalizations

We now move on to discuss universal globalizations of continuous partial actions of a monoid M on a topological space X; here, the universality is, of course, to be understood w.r.t. continuous equivariant maps. The main result of this section states essentially that globalizations of *confluent* partial actions of monoids are topological embeddings. A corresponding result for *open* partial group actions (without confluence) is established in [15] and in [1]. We shall provide an example that shows that the result fails for arbitrary partial group actions.

The universal globalization of a continuous partial action is constructed by endowing the globalization Y constructed above with the final topology w.r.t. the maps

$$u : X \to Y$$
$$x \mapsto u \cdot x,$$

where u ranges over M (i.e., $V \subset Y$ is open iff $u^{-1}[V]$ is open in X for each $u \in M$); equivalently, the topology on Y is the quotient topology induced by the map $M \times X \to Y$, where M carries the discrete topology. This ensures the desired universal property: given a continuous equivariant map $f : X \to Z$, where M acts globally (and continuously) on Z, the desired factorization $f^{\#} : Y \to Z$ exists uniquely as an equivariant map by the universal property of Y at the level of sets. In order to establish that $f^{\#}$ is continuous, it suffices to show that $f^{\#}u : X \to Z$ is continuous for each $u \in M$; but $f^{\#}u$ is, by equivariance of $f^{\#}$, the map $x \mapsto u \cdot f(x)$, hence continuous.

Under additional assumptions concerning the domains, the inclusion $X \hookrightarrow Y$ is extremely well-behaved:

Proposition 3.1. *If α is closed* (*open*)*, then the map $X \hookrightarrow Y$ is closed* (*open*)*, in particular a topological embedding.*

(The open case for partial group actions appears in [1,15].)

Proof. Let $A \subset X$ be closed (open). Then $u^{-1}[A]$ is closed (open) in $\mathrm{dom}(u)$ and hence in X for each $u \in M$; thus, A is closed (open) in Y. □

The embedding property fails in the general case:

Example 3.2. We proceed similarly as in Example 2.14. Let $\mathcal{V}_4$ denote the Klein four-group $\{e, u, v, uv\}$, and let α be the partial action of $\mathcal{V}_4$ on the closed interval $X = [-1, 1]$ defined as follows: let $\operatorname{dom}(u) = A = \{\frac{1}{2}\}$, let $\operatorname{dom}(v) = B = \{\frac{1}{n} + \frac{1}{2} \mid n \in \mathbb{N},\ n \geqslant 2\}$, and let $\operatorname{dom} uv = C = [-1, 1] \cap \mathbb{Q}$. Let u and v act as the identity on A and B, respectively, and let $(uv) \cdot x = -x$ for $x \in C$. It is easily checked that α is indeed a partial group action. As in Example 2.14, one shows that α fails to be confluent, because the triple $(1, u, uv)$ violates the property in Proposition 2.13 (alternatively, non-confluence of α can be deduced from the following and Corollary 3.4).

We claim that *the globalization $X \hookrightarrow Y$ of α fails to be a topological embedding* (which, incidentally, implies that Y fails to be Hausdorff, since X is compact and $X \hookrightarrow Y$ is injective). To see this, let U be the open set $(0, 1)$ in X. We show that U fails to be open in Y, i.e., that $V \cap X \neq U$ for each open $V \subset Y$ such that $U \subset V$; in fact, such a V always contains a negative number:

We have $u \cdot \frac{1}{2} = \frac{1}{2} \in V$, i.e., $\frac{1}{2} \in u^{-1}[V]$. Therefore the open set $u^{-1}[V] \subset X$ intersects B, i.e., we have $b \in B$ such that $(uv) \cdot b = u \cdot b \in V$. Thus, the open set $(uv)^{-1}[V] \subset X$ intersects $C \cap (0, 1]$, so that we obtain $c \in C \cap (0, 1]$ such that $(uv) \cdot c \in V$; but then $(uv) \cdot c = -c$ is a negative number.

Notice that it is not possible to repair the embedding property by just changing the topology on Y: the topology is already as large as possible (being a final lift of maps that are certainly expected to be continuous), and the failure of $X \hookrightarrow Y$ to be an embedding is due to Y having *too few* open sets. This pathology does not happen in the confluent case:

Theorem 3.3. *If α is confluent, then the map $u : R_u \to Y$ (cf. Section 2) is a topological embedding for each $u \in M$.*

Corollary 3.4. *If α is confluent, then the globalization $X \hookrightarrow Y$ is a topological embedding.*

(It is unlikely that the converse holds, i.e., that confluence is also a necessary condition for $X \hookrightarrow Y$ to be an embedding.)

Proof of Corollary 3.4. The inclusion $X \hookrightarrow Y$ is the map $e : R_e \to Y$. $\square$

Proof of Theorem 3.3. All that remains to be shown is that the original topology of R_u agrees with the subspace topology on $u \cdot R_u$ w.r.t. Y, i.e., that, whenever U is open in R_u, then there exists an open $\overline{U} \subset Y$ such that $\overline{U} \cap u \cdot R_u = u \cdot U$.

We define $\overline{U}$ as the union of a system of subsets $U_v \subset Y$ to be constructed below, indexed over all $v \in M$ such that $u \preceq v$ (this is the prefix ordering of Definition 1.4, which depends on confluence. As announced above, we reuse notation without further comments), with the following properties for each $v \succeq u$:

(i) $U_p \subset U_v$ whenever $u \preceq p \preceq v$.
(ii) $U_v \cap u \cdot R_u = u \cdot U$.

(iii) $v^{-1}[U_v]$ is open in X.
(iv) Each $a \in U_v \setminus U_{\mathrm{pre}(v)}$ has normal form $v \cdot x$ for some x.

Then certainly

$$\overline{U} \cap u \cdot R_u = u \cdot U.$$

Moreover, the properties above imply

(v) For each $v \in M$, $v \cdot x \in \overline{U}$ implies $u \preceq v$ and $v \cdot x \in U_v$.

To prove (v), let p be the minimal $p \succeq u$ w.r.t. $\preceq$ such that $v \cdot x \in U_p$. By (iv), $v \cdot x$ has normal form $p \cdot y$ for some y, so that $p \preceq v$, and hence in particular $u \preceq v$. By (i), we obtain $v \cdot x \in U_v$ as required. Now (v) enables us to show that $\bar{U}$ is open: we have to show that $v^{-1}[\overline{U}]$ is open for each $v \in M$. By (v), this set is empty in case $u \not\preceq v$. Otherwise, we have, again by (v),

$$v^{-1}[\overline{U}] = v^{-1}[U_v]$$

which is open in X by (iii).

The system (U_v) is constructed by induction over the prefix order, starting from $U_u = U$ (where '$U_{\mathrm{pre}(u)}$' is to be replaced by $\emptyset$ in (iv)). Now let $v \in M$, where $u \prec v$, have normal form $v = g_n \cdots g_1 = \mathrm{pre}(v) g_1$, and assume that the U_p are already constructed as required for $u \preceq p \prec v$. The set

$$B = \big(\mathrm{pre}(v)\big)^{-1}[U_{\mathrm{pre}(v)}]$$

is open in X by the inductive assumption. Thus, $g_1^{-1}[B]$ is open in the domain $D \subset X$ of g_1, i.e., equal to $D \cap V$, where V is open in X. Let

$$C = V \setminus D.$$

Note that, for $x \in C$, $v \cdot x$ is normal. Now U_v is defined as

$$U_v = U_{\mathrm{pre}(v)} \cup v \cdot C.$$

It is clear that this definition satisfies (i), (ii), and (iv) above. In order to verify (iii), let $x \in X$. Then $v \cdot x$ is normal and in U_v iff $x \in C$. If $v \cdot x$ is reducible, i.e., if $g_1(x)$ is defined in X, then $v \cdot x \in U_v$ iff $\mathrm{pre}(v) \cdot (g_1(x)) \in U_{\mathrm{pre}(v)}$ iff $g_1(x) \in B$. Thus,

$$v^{-1}[U_v] = C \cup g_1^{-1}[B] = (V \setminus D) \cup (V \cap D) = V,$$

which is open in X. □

Example 3.5. A very basic example of a partial action on X produces the free homogeneous space over X, as follows. The full subcategory $\mathbf{C}$ of $\mathbf{T}(X)$ spanned by the singleton subspaces induces a partial action as described in Example 2.9(vii). The presentation of the monoid $M(\mathbf{C})$ generated by $\mathbf{C}$ can be described as follows: the generators are of the form (xy), where $x, y \in X$ with $x \neq y$, and the relations are $(xy)(yz) \to (xz)$ when $x \neq z$, and $(xy)(yz) \to ()$ otherwise (thus, one may leave out the brackets and just write $xx = e$). The corresponding globalization is easily seen to be homogeneous. There are known ways to produce this homogeneous space, in particular *Shimrat's construction* [27] and the construction given by Belnov [4], who also establishes a kind of universal property for the extension. It can be checked that the spaces resulting from these constructions coincide with our globalization in this special case (see [19] for more details).

4. Preservation of topological properties

We will now investigate how topological properties of a space are or are not handed on to its globalization with respect to a continuous partial action α.

Theorem 4.1. *If α is confluent and X is a T_1-space, then Y is T_1 iff $u^{-1}[\{x\}]$ is closed in X for each $u \in M$ and each $x \in X$.*

Proof. The 'only if' direction is immediate. In order to prove the 'if' direction, we have to show that the latter condition implies that $u^{-1}[\{a\}]$ is closed in X for each $a \in Y$. Let a have normal form $v \cdot x$. Then $u \cdot y = v \cdot x$ for $y \in X$ iff we have $u = vp$ normal and $p \cdot y = x$, where p is necessarily uniquely determined. Thus, $u^{-1}[\{a\}]$ is the closed set $p^{-1}[\{x\}]$ in X if $v \preceq u$; otherwise, $u^{-1}[\{a\}]$ is empty. □

There are many typical cases in which this necessary and sufficient condition is easily seen to be satisfied, such as the following.

Corollary 4.2. *If X is T_1 and α is closed, then Y is T_1.*

Corollary 4.3. *If X is T_1 and M is a group, then Y is T_1.*

Corollary 4.4. *If X is T_1 and for each generator $g \in G$, the partial map $g : X \to X$ has finite fibres, then Y is T_1.*

(The latter corollary includes the case that all generators act injectively.)

Proof of Corollary 4.4. By induction over the length of $u \in M$, one shows that $u^{-1}[\{x\}]$ is finite and hence closed for each $x \in X$. □

For confluent actions, the domain conditions introduced in Section 2 can be reduced to the generating set G:

Proposition 4.5. *Let α be confluent. Then α is closed* (*open*) *iff* $\mathrm{dom}(g)$ *is closed* (*open*) *for each $g \in G$, and α is strongly closed* (*open*) *iff, moreover, $g : X \to X$ is closed* (*open*) *on* $\mathrm{dom}(g)$ *for each g.*

Proof. We prove only the closed case. Let $\mathrm{dom}(g)$ be closed for each $g \in G$. We show by induction over $\lg(u)$ that $\mathrm{dom}(u)$ is closed for each $u \in M$: let u have normal form

$u = g_n \cdots g_1$, so that $\mathrm{pre}(u) = g_n \cdots g_2$. Then $\mathrm{dom}(\mathrm{pre}(u))$ is closed by induction. By confluence, $u \cdot x$ is defined in X iff $u \cdot x$ reduces to some $y \in X$. Thus, we have

$$\mathrm{dom}(u) = g_1^{-1}\bigl[\mathrm{dom}\bigl(\mathrm{pre}(u)\bigr)\bigr],$$

which is closed in $\mathrm{dom}(g_1)$ and hence in X. The second claim is now trivial. □

Strong closedness is in a suitable sense 'inherited' by the globalization:

Proposition 4.6.

(i) *If α is strongly open, then $u : X \to Y$ is open for every $u \in M$.*
(ii) *If α is strongly closed and confluent, then $u : X \to Y$ is closed for every $u \in M$.*

Proof. (i) We have to show that $v^{-1}[u[U]]$ is open in X for each $v \in M$ and each open U in X. We can write this set as

$$v^{-1}\bigl[u[U]\bigr] = \bigcup_{n \in \mathbb{N}} V_{n,v},$$

where $V_{n,v}$ denotes the set of all $x \in X$ such that there exists $y \in U$ such that $(v, x) \simeq (u, y)$ is obtainable by applying the generating relation $\sim$ of formula (1) (Section 2) n times from left to right or from right to left. We show by induction over n that $V_{n,v}$ is open for each v: the base case is trivial. Now by the definition of $\sim$,

$$V_{n+1,v} = \bigcup_{\substack{p,q \in M \\ v = pq}} \bigl(q^{-1}[V_{n,p}] \cap X\bigr) \cup \bigcup_{p \in M} \bigl(p[V_{n,vp}] \cap X\bigr),$$

where the first part of the union corresponds to the first step in the derivation of $(v, x) \simeq (u, y)$ being of the form $(v, x) = (pq, x) \sim (p, q \cdot x) \in V_{n,p}$ and the second to that step being of the form $V_{n,vp} \ni (vp, z) \sim (v, p \cdot z) = (v, x)$. By the inductive assumption, the sets $V_{n,p}$ and $V_{n,vp}$ are open; hence, all components of the union are open, since all $p \in M$ have open domains and are open as partial maps $X \to X$.

(ii) The argument is analogous to the one above, noticing that thanks to confluence, all unions above can be restricted to finite ones: the derivation of $(v, x) \simeq (u, y)$ needs at most $\lg(v) + \lg(u)$ steps; in the first part of the union in the decomposition of $V_{n+1,v}$, the decompositions $v = pq$ can be restricted to be normal; and in the second part of the union, p need only range over generators that occur in the normal form of u. □

Corollary 4.7. *Let α be strongly open. Then the translation map $u : Y \to Y$ is open for every $u \in M$.*

Proof. Let U be an open subset of Y. We have to show that $u \cdot U$ is open. Represent this set as

$$u \cdot U = \bigcup_{v \in M} uv \cdot \bigl(v^{-1}[U] \cap X\bigr).$$

Now observe that each component set of the union is open. Indeed, since $v^{-1}[U] \cap X$ is open in X, Proposition 4.6(i) implies that $uv \cdot (v^{-1}[A] \cap X)$ is open in Y. □

As the following example shows, the 'closed version' of the last statement fails to be true even for confluent partial actions.

Example 4.8. Let $X = \mathbb{R}$ be the real line. For $n \in \mathbb{N}$, let $p_n : \mathbb{N} \to \mathbb{N}$ be the constant map with value n. These maps, together with the identity map on $\mathbb{N}$, form a monoid M which acts on $\mathbb{N} \subset \mathbb{R}$ and thus partially acts on X. Clearly, this partial action is strongly closed; but the translation $p_1 : Y \to Y$ of the corresponding globalization fails to be closed. Indeed, define a subset of Y as

$$A = \left\{ p_n \cdot \frac{1}{n} \;\middle|\; n \geqslant 2 \right\}.$$

Then A is closed in Y because $v^{-1}[A] \cap X$ has at most one point for every $v \in M$. However, $p_1 \cdot A$ is not closed. To see this, observe that $p_1 p_n = p_1$ and hence $p_1 \cdot A = \{p_1 \cdot \frac{1}{n} \mid n \geqslant 2\}$. The sequence of points $p_1 \cdot \frac{1}{n}$ in $p_1 \cdot A$ converges to the point $p_1 \cdot 0 = 1$, which is outside of $p_1 \cdot A$.

Remark 4.9. In the case that M is a group, closed partial actions are automatically strongly closed. Moreover, since in this case each translation $u : Y \to Y$ is a homeomorphism, the 'closed version' of Corollary 4.7 is trivially true.

We now approach the question of normality and dimension. Let Z be a topological space. Following Wallace [32], we say that X is of *dimensional type* Z (in short: $X \tau Z$) if, for each closed set $A \subset X$ and each continuous map $f : A \to Z$, there exists a continuous extension $\bar{f} : X \to Z$.

Theorem 4.10. *If α is closed and confluent, then $X \tau Z$ implies $Y \tau Z$.*

Proof. Let $A \subset Y$ be closed, and let $\psi : A \to Z$ be a continuous map. In order to define the required extension $\bar{\psi} : Y \to Z$, we construct a sequence of continuous functions $\psi_n : Y_n \to Z$ (cf. Section 3) such that each ψ_n extends the restriction $\psi|_{A \cap Y_n}$ and each ψ_{n+1} extends ψ_n. We then obtain $\bar{\psi}$ as the union of the ψ_n.

Y_0 is just X. Since $A \cap X$ is closed in X, we can choose ψ_1 as an extension of $\psi|_{A \cap X}$ to X.

Now assume that we have constructed the sequence up to n. We define auxiliary functions $\lambda_u : B_u \to Z$, where B_u is closed in X, for each $u \in M$ such that $\lg(u) \leqslant n$ as follows: let u have normal form $g_k \cdots g_1$, and let D be the (closed) domain of g_1. The set B_u is the union $D \cup u^{-1}[A]$ (hence closed), and λ_u is defined by

$$\lambda_u(x) = \begin{cases} \psi_n(u \cdot x), & \text{if } u \cdot x \in Y_n, \quad \text{and} \\ \psi(u \cdot x), & \text{if } u \cdot x \in A. \end{cases}$$

By assumption on ψ_n, λ_u is well-defined. It is continuous on D and on $u^{-1}[A]$, hence continuous, since both these sets are closed.

Since $X \tau Z$, each λ_u has a continuous extension $\kappa_u : X \to Z$. We put

$$\psi_{n+1}(u \cdot x) = \kappa_u(x)$$

for each $u \in M$ with $\lg(u) \leqslant n$ and each $x \in X$. Since $\lg(a) \leqslant n$ for any $a \in Y_{n+1}$ that admits more than one such representation $a = u \cdot x$, ψ_{n+1} is well-defined. It is continuous for fixed u, which implies overall continuity by definition of the topology on Y; finally, it extends $\psi|_{A\cap Y_{n+1}}$ and ψ_n by construction. □

Corollary 4.11. *If α is closed and confluent and X is normal* (*and has* $\dim(X) = n$), *then Y is normal* (*and has* $\dim(Y) = n$).

Proof. First note that Y is a T_1-space by virtue of Corollary 4.2. Now use Theorem 4.10 and well-known characterizations of normality (for $Z = [0, 1]$) and dimension (for $Z = S_n$) in terms of dimensional type. □

If α is not closed then we cannot in general expect the preservation of basic topological properties, such as for instance T_2, in Y (or, in fact, in any other globalizations):

Example 4.12. Let $h : O \to O$ be an autohomeomorphism of an open subset O of X. Suppose that sequences (x_n) and (y_n) in O both converge to the same point in $X \setminus O$, and that $(h(x_n))$ and $(h(y_n))$ converge to points c and d in $X \setminus O$, respectively. If X admits a *Hausdorff* extension $X \hookrightarrow Z$ such that h extends to a global map on Z, then $c = d$: in Z, we have

$$c = \lim h(x_n) = h(\lim x_n) = h(\lim y_n) = \lim h(y_n) = d.$$

It follows that Y cannot be Hausdorff for any (even very good) X that has such a subspace O with c and d distinct. As a concrete example, take $X = \mathbb{Z} \cup \{\infty, -\infty\}$, $O = \mathbb{Z}$, (x_n) and (y_n) the sequences of positive even and odd numbers, respectively, and $h(n) = n$ if n is even, $h(n) = -n$ otherwise. (By way of contrast, observe that, by Corollary 4.3, the globalization w.r.t. the group generated by h *is* T_1.)

This example shows in particular that the abstract globalization problem of [8, p. 294] in general fails to have a Hausdorff solution.

5. Non-expansive partial actions

We will now move on from topology into the realm of metrics and pseudometrics.

Definition 5.1. A *weak pseudometric* space is a pair (X, d), where $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$ is a symmetric distance function that satisfies the triangle inequality and $d(x, x) = 0$ for each $x \in X$. A *pseudometric* space is a weak pseudometric space (X, d) such that $d(x, y) < \infty$ for all x, y. A weak pseudometric space is called *separated* if $d(x, y) = 0$ implies $x = y$. (Thus, a metric space is a separated pseudometric space.)

We will denote all distance functions by d (and the space (X, d) just by X) where this is unlikely to cause confusion. A function f between weak pseudometric spaces is called *non-expansive* if $d(f(x), f(y)) \leqslant d(x, y)$ for all x, y.

We denote the categories of weak pseudometric, pseudometric, and metric spaces with non-expansive maps as morphisms by **wPMet**, **PMet**, and **Met**, respectively.

A partial action of a monoid M on a weak pseudometric space X is called *non-expansive* if the partial map $u : X \to X$ is non-expansive on its domain (as a subspace of X) for each $u \in M$. Note here that both **PMet** and **Met** are closed under subspaces in **wPMet**.

Since **wPMet** is a topological category [2], globalizations can be constructed in the same way as for topological partial actions by means of final lifts: in general, given weak pseudometric spaces Y_i, $i \in I$, and a family of maps $f_i : Y_i \to X$ into some set X, the final lift of $\mathcal{S} = (Y_i, f_i)_I$ is the largest weak pseudometric on X (w.r.t. the pointwise order on real-valued functions) that makes all the f_i non-expansive maps. Explicitly, given points x and y in X, an *$\mathcal{S}$-path* π from x to y of *length* n is a sequence $((i_1, x_1, y_1), \ldots, (i_n, x_n, y_n))$, $n \geqslant 1$, such that $x_j, y_j \in Y_{i_j}$, $j = 1, \ldots, n$, $f_{i_1}(x_1) = x$, $f_{i_j}(y_j) = f_{i_{j+1}}(x_{j+1})$ for $j = 1, \ldots, n-1$, and $f_{i_n}(y_n) = y$. The associated *path length* is

$$\sum_{j=1}^{n} d_j(x_j, y_j).$$

In case $x \neq y$, the distance of x and y is easily seen to be given as the infimum of the path length, taken over all $\mathcal{S}$-paths from x to y (in particular, the distance is ∞ if there is no such path); otherwise the distance is, of course, 0. If the f_i are jointly surjective (which they are in the case we are interested in), then there is always a trivial $\mathcal{S}$-path from x to x, so that the case $x = y$ does not need special treatment. Due to the triangle inequality, it suffices to consider paths $((i_j, x_j, y_j))$ where (i_j, y_j) is always different from (i_{j+1}, x_{j+1}).

Now given a partial action α on a weak pseudometric space X, we construct the underlying set of the free globalization Y as in Section 3 (as for topological spaces, we shall keep the notation α, X, Y, etc. throughout). It is easy to see that free globalizations of partial actions on weak pseudometric spaces (i.e., reflections into the full subcategory spanned by the total actions in the category of partial actions) are, as in the topological case, given as final lifts of the family $\mathcal{S}$ of maps

$$u : X \to Y,$$

where u ranges over M. For the sake of clarity, we denote the distance function on Y thus defined by D.

For the remainder of this section, we shall assume that α is confluent.

Under this condition, one may further restrict the paths to be taken into consideration: in general, we may write an $\mathcal{S}$-path π from a to b $(a, b \in Y)$ in the form

$$u_1 \cdot x_1, u_1 \cdot y_1 \overset{*}{\leftrightarrow} u_2 \cdot x_2, \ldots, u_{n-1} \cdot y_{n-1} \overset{*}{\leftrightarrow} u_n \cdot x_n, u_n \cdot y_n$$

(in short: (u_j, x_j, y_j)), where $u_1 \cdot x_1 = a$ and $u_n \cdot y_n = b$. Denote by $D(\pi)$ the corresponding path length $\sum_{j=1}^{n} d(x_j, y_j)$. By definition, $D(a, b) = \inf D(\pi)$ where π runs over all possible paths. Recall that $D(a, b) = \infty$ iff there is no path from a to b. We say that π is *geodesic* if $D(a, b) = D(\pi)$.

There are two additional *assumptions* we may introduce:

(i) *For each $j = 1, \ldots, n$, at least one of $u_j \cdot x_j$ and $u_j \cdot y_j$ is in normal form.*
Indeed, if u_j has normal form $g_k \cdots g_1$ and both x_j and y_j are in the domain of g_1, then we obtain a shorter path replacing (u_j, x_j, y_j) by $(g_k \cdots g_2, g_1 \cdot x_j, g_1 \cdot y_j)$ (since g_1 is non-expanding).

(ii) *For each* $j = 1, \dots, n-1$, *at most one of* $u_j \cdot y_j$ *and* $u_{j+1} \cdot x_{j+1}$ *is normal.*
By the above, we may assume $(u_j, y_j) \neq (u_{j+1}, x_{j+1})$. But both these pairs represent the same point of Y, which has only one normal form.

We will henceforth consider only $\mathcal{S}$-paths that are *reduced* according to these assumptions. We denote the transitive closure of the one-step reduction $\to$ by $\overset{+}{\to}$ (reversely: $\overset{+}{\leftarrow}$); i.e., $\overset{+}{\to}$ is like $\overset{*}{\to}$ except that we require that at least one reduction step takes place. If $u_j \cdot y_j$ is reducible and $u_{j+1} \cdot x_{j+1}$ is normal then necessarily $u_j \cdot y_j \overset{+}{\to} u_{j+1} \cdot x_{j+1}$, which we will indicate in the notation for paths; similarly if $u_j \cdot y_j$ is normal and $u_{j+1} \cdot x_{j+1}$ is reducible.

The 'normality patterns' that occur in reduced paths are restricted in a rather amusing way:

Lemma 5.2. *Every reduced path from* $a \in Y$ *to* $b \in Y$ *has one of the following forms*:

(A1) $n, r \overset{+}{\to} \cdots \overset{+}{\to} n, r$;
(A2) $r, n \overset{+}{\leftarrow} \cdots \overset{+}{\leftarrow} r, n$;
(A3) n, n;
(A4) $n, r \overset{+}{\to} \cdots \overset{+}{\to} n, r \overset{+}{\to} n, n$;
(A5) $n, n \overset{+}{\leftarrow} r, n \overset{+}{\leftarrow} \cdots \overset{+}{\leftarrow} r, n$;
(A6) $n, r \overset{+}{\to} \cdots \overset{+}{\to} n, r \overset{+}{\to} n, n \overset{+}{\leftarrow} r, n \overset{+}{\leftarrow} \cdots \overset{+}{\leftarrow} r, n$;
(A7) $n, r \overset{+}{\to} \cdots \overset{+}{\to} n, r \overset{*}{\leftrightarrow} r, n \overset{+}{\leftarrow} \cdots \overset{+}{\leftarrow} r, n$,

where 'n' and 'r' mean that the corresponding term of the path is normal or reducible, respectively. (*Patterns such as* $n, r \overset{+}{\to} \cdots \overset{+}{\to} n, r$ *are to be understood as 'one or more occurrences of* n, r*'.*)

Proof. If the path does not contain either of the patterns n, n and $r \overset{*}{\leftrightarrow} r$, then it must be of one of the forms (A1) and (A2). The occurrence of n, n in some place determines the entire pattern due to restrictions (i) and (ii) above, so that the path has one of the forms (A3)–(A6). Similarly, a path that contains the pattern $r \overset{*}{\leftrightarrow} r$ must be of the form (A7). □

A first consequence of this lemma is that every space is a subspace of its globalization:

Lemma 5.3. *Let* $x, y \in X$. *Then* $((e, x, y))$ *is the only reduced path from* x *to* y.

Proof. Since $e \cdot z$ is in normal form for all $z \in X$, any reduced path from x to y must have form (A3) of Lemma 5.2 (all other forms either begin with the pattern n, r or end with r, n). □

Theorem 5.4. *The embedding* $X \hookrightarrow Y$ *of a weak pseudometric space into its free globalization is isometric.*

Proof. Immediate from Lemma 5.3. □

Of course, we are mainly interested in *metric* globalizations. Now any weak pseudometric space has a separated reflection obtained by identifying points with distance zero. If X is a separated space, then the separated reflection $\overline{Y}$ of Y is the free *separated* globalization of X, and X is isometrically embedded in $\overline{Y}$, since its points have positive distances in Y and are hence kept distinct in $\overline{Y}$. We will see below (Theorem 5.11) that working with the separated reflection is unnecessary for closed partial actions. Finiteness of distances is, on the one hand, more problematic since there is no universal way to transform a weak pseudometric space into a pseudometric space. On the other hand, finiteness of distances is preserved in most cases:

Definition 5.5. α is called *nowhere degenerate* if $\mathrm{dom}(g) \neq \emptyset$ for each $g \in G$.

Proposition 5.6. *If X is a non-empty pseudometric space, then Y is pseudometric iff α is nowhere degenerate.*

Proof. If α is nowhere degenerate, then there exists, for each $y \in Y$, a path from y to some $x \in X$; hence, there is a path between any two points of Y, so that the infimum defining the distance function on Y is never taken over the empty set and hence never infinite. If, conversely, $\mathrm{dom}(g) = \emptyset$ for some $g \in G$, then there is no reduced path (and hence no path at all) from x to $g \cdot x$ for $x \in X$, so that $D(x, g \cdot x) = \infty$. Indeed, assume that π is such a path. Since both $e \cdot y$ and, by assumption on g, $g \cdot y$ are normal for all $y \in X$, the normality pattern of π as in Lemma 5.2 can neither begin with n, r nor end with r, n. Thus, π must be of the form (A3), which is impossible since $\mathrm{dom}(g) = \emptyset$ implies $g \neq e$. □

Remark 5.7. Another approach to the problem of infinite distances is to consider only spaces of diameter at most 1 and put $D(x, y) = 1$ for $x, y \in Y$ in case there is no path from x to y.

Observation 5.8. Let $a, b \in Y$ have normal forms $a = u \cdot x$ and $b = v \cdot y$, and let π be a reduced path from a to b. If π is of the form (A2) or (A5) of Lemma 5.2, then necessarily $u \preceq v$, and if π is of the form (A1) or (A4), then $v \preceq u$. Clearly, if π is of the form (A3) then $u = v$. Thus, if u and v are incomparable under $\preceq$ then π must be of the form (A6) or (A7).

Lemma 5.9. *Let $a, b \in Y$ have normal forms $a = u \cdot x$ and $b = v \cdot y$, where u has normal form $g_k \cdots g_1$.*

(i) *If $D(a, b) < d(x, \mathrm{dom}(g_1))$, then $u \preceq v$.*
(ii) *If $u = v$ then*

$$\min\bigl\{d(x, y), d\bigl(x, \mathrm{dom}(g_1)\bigr) + d\bigl(y, \mathrm{dom}(g_1)\bigr)\bigr\} \leqslant D(a, b) \leqslant d(x, y).$$

Proof. Let π be a reduced path from a to b.

(i) π cannot have a normality pattern of the form $n, r \xrightarrow{+} \cdots$, since in that case, the first step of the path would already contribute at least $d(x, \mathrm{dom}(g_1))$ to $D(a, b)$. Hence, π must be of one of the forms (A2), (A3), or (A5) of Lemma 5.2. By the observation above, this implies $u \preceq v$.

(ii) π must have one of the forms (A3), (A6), or (A7) of Lemma 5.2. In the case (A3), $D(\pi) = d(x, y)$. In the cases (A6) and (A7), the normality pattern of π is of the form $n, r \overset{+}{\rightarrow} \cdots \overset{+}{\leftarrow} r, n$. Therefore $D(\pi) \geqslant d(x, \mathrm{dom}(g_1)) + d(y, \mathrm{dom}(g_1))$. This proves the first inequality; the second follows from the fact that $u : X \to Y$ is non-expansive. □

We say that a function $\phi : E \to L$ between pseudometric spaces is *locally isometric* if for every $x \in E$ there exists $\varepsilon > 0$ such that ϕ isometrically maps the ε-ball $B(x, \varepsilon)$ in E onto the ε-ball $B(\phi(x), \varepsilon)$ in L. Clearly, E is separated iff $\phi(E)$ is separated. Every locally isometric injective map is a topological embedding.

Proposition 5.10. *If α is closed, then*

(i) *$D(u \cdot x, v \cdot y) = 0$ implies $u = v$ for normal forms $u \cdot x$, $v \cdot y$.*
(ii) *The set $\bigcup_{u \preceq v} v \cdot R_v$ is open for each u.*
(iii) *Each Y_k (in particular, $Y_0 = X$) is closed in Y.*
(iv) *The subspace $Y_{k+1} \setminus Y_k$ is a topological sum $\bigcup_{\mathrm{lg}(u)=k+1} u \cdot R_u$ of disjoint subsets $u \cdot R_u$.*
(v) *For every $u \in M$ the bijective function $u : R_u \to u \cdot R_u$ is locally isometric (and, hence, a homeomorphism).*

Proof. (i) Let u have normal form $g_n \cdots g_1$. Then $D(u \cdot x, v \cdot y) = 0 < d(x, \mathrm{dom}\, g_1)$ by closedness, so that $u \preceq v$ by Lemma 5.9(i). Analogously, $v \preceq u$.

(ii) Let $u \in M$, and let a have normal form $p \cdot x$ (i.e., $a \in p \cdot R_p$) for some $u \preceq p$ with normal form $p = g_n \cdots g_1$. Put $\varepsilon = d(x, \mathrm{dom}(g_1))$. By closedness, $\varepsilon > 0$. By Lemma 5.9(i), the ε-neighbourhood of a is contained in $\bigcup_{p \preceq v} v \cdot R_v$ and hence in $\bigcup_{u \preceq v} v \cdot R_v$, which proves the latter set to be open.

(iii) The complement of Y_k is a union of sets $\bigcup_{u \preceq v} v \cdot R_v$.

(iv) Disjointness is clear, and by (ii), each set $u \cdot R_u$ with $\mathrm{lg}(u) = k + 1$ is open in $Y_{k+1} \setminus Y_k$, since $u \cdot R_u = (\bigcup_{u \preceq v} v \cdot R_v) \cap (Y_{k+1} \setminus Y_k)$.

(v) Let $u = g_k \cdots g_1$ be normal, and let $x \in R_u = X \setminus \mathrm{dom}(g_1)$. Since α is closed, $\varepsilon := d(x, \mathrm{dom}(g_1)) > 0$. By Lemma 5.9(ii), the bijective function $u : R_u \to u \cdot R_u$ isometrically maps the ε-ball $B(x, \varepsilon)$ onto the ε-ball $B(u \cdot x, \varepsilon)$ in $u \cdot R_u$. □

As an immediate consequence, we obtain the announced separatedness result:

Theorem 5.11. *If α is closed and X is separated, then Y is separated.*

Proof. Let $u \cdot x$ and $v \cdot y$ be normal forms in Y with $D(u \cdot x, v \cdot y) = 0$. Then $u = v$ by Proposition 5.10(i); therefore $x, y \in R_u$. By Proposition 5.10(v), $D(u \cdot x, u \cdot y) = 0$ implies $d(x, y) = 0$ and hence $x = y$. □

Remark 5.12. The converse of the above theorem holds if X is complete: assume that Y is separated, let $g \in G$, and let (x_n) be a convergent sequence in $\mathrm{dom}(g)$; we have to show that $x = \lim x_n$ is in $\mathrm{dom}(g)$. Now $(g \cdot x_n)$ is a Cauchy sequence in X, hence by assumption convergent; let $z = \lim g \cdot x_n$. For every n, we have a path

$$e \cdot z,\ e \cdot (g \cdot x_n) \overset{+}{\leftarrow} g \cdot x_n,\ g \cdot x$$

from z to $g \cdot x$. The associated path length is $d(z, g(x_n)) + d(x_n, x)$, which converges to 0 as $n \to \infty$. Hence, $D(z, g \cdot x) = 0$, so that $z = g \cdot x$ by separatedness; this implies that $g \cdot x$ is defined in X as required.

Example 5.13. Even for closed partial actions of groupoids on metric spaces, the metric globalization does not in general induce the topology of the topological globalization of Section 3. Take, for instance, $X = [0, 1]$. The full subcategory of $\mathbf{M}(X)$ spanned by all singleton subspaces induces a partial action α as described in Example 2.9(vii) (cf. also Example 3.5). The universal *topological* globalization Y of α is not even first countable: as in Example 3.5, denote the map $\{x\} \to \{y\}$ by (yx) for $x \neq y$ in X. Then we have a subspace Z of Y formed by all points of the form x or $(y0) \cdot x$. The space Z is homeomorphic to the quotient space obtained by taking one base copy of [0, 1] and uncountably many copies of [0, 1] indexed over the base copy, and then identifying for each $a \in [0, 1]$ the 0 in the ath copy with the point a in the base copy. In particular, already Z fails to be first countable.

Theorem 5.14. *If X is a metric space and α is closed and nowhere degenerate, then Y is a metric space. Moreover,* $\dim(Y) = \dim(X)$.

Proof. By Theorem 5.11 and Proposition 5.6, Y is a metric space.

It remains to be shown that $\dim(X) = \dim(Y)$. Now $Y = \bigcup_{n \in \mathbb{N}} Y_n$ where, by Proposition 5.10, each Y_n is a closed subset of Y. Therefore, by the standard *countable sum theorem*, it suffices to show that $\dim(Y_n) \leqslant \dim(X)$ for every n. We proceed by induction. The case $n = 0$ is trivial, since $Y_0 = X$. We have to show that $\dim(Y_{n+1}) \leqslant \dim(X)$ provided that $\dim(Y_n) \leqslant \dim(X)$. The idea is to use the following result of Dowker [10].

Lemma 5.15 (Dowker). *Let Z be a normal space, and let Q be a closed subspace of Z such that* $\dim(Q) \leqslant k$. *Then* $\dim(Z) \leqslant k$ *if and only if every closed subspace $A \subset Z$ disjoint from Q satisfies* $\dim(A) \leqslant k$.

We apply this lemma to the closed subspace Y_n of Y_{n+1}. By the induction hypothesis, $\dim(Y_n) \leqslant \dim(X)$. We have to show that $\dim(A) \leqslant \dim(X)$ for every closed subset A of Y_{n+1} which is disjoint from Y_n, i.e., $A \subseteq Y_{n+1} \setminus Y_n$. By Proposition 5.10(iv), A is a topological sum $\bigcup_{\mathrm{lg}(u)=n+1} A_u$ of disjoint subspaces $A_u := A \cap u \cdot R_u$. Each A_u is a subspace of $u \cdot R_u$. Therefore, by Proposition 5.10(v), A_u is homeomorphic to a subspace of X. Since the dimension is hereditary (for arbitrary, not necessarily closed subspaces) in perfectly normal (e.g., metrizable) spaces, we have $\dim(A_u) \leqslant \dim(X)$. Thus, $\dim(A) \leqslant \dim(X)$. By Dowker's result this yields $\dim(Y_{n+1}) \leqslant \dim(X)$. □

Remark 5.16. One application of Theorems 5.4 and 5.14 is to obtain all sorts of metric gluing constructions. A simple example of this is Theorem 2.1 of [5], which states that *given metric spaces* X_1 *and* X_2 *with intersection* $Z = X_1 \cap X_2$ *such that* Z *is closed both in* X_1 *and in* X_2 *and the metrics of* X_1 *and* X_2 *agree on* Z*, there exists a metric on* $X_1 \cup X_2$ *which agrees with the given metrics on* X_1 *and* X_2*, respectively*. Using our results, this can be seen as follows: let G be the free group with a single generator u (i.e., $G \cong \mathbb{Z}$), let $X_1 + X_2 = X_1 \times \{1\} \cup X_2 \times \{2\}$ be the disjoint union of X_1 and X_2, and let a partial action of G on $X_1 + X_2$ be defined by $u \cdot (x, 1) = (x, 2)$ (and $u^{-1} \cdot (x, 2) = (x, 1)$) for $x \in Z$. This partial action is closed and, by Example 2.9(iii), confluent. In the globalization Y, we find the set $X_1 \cup X_2$ represented as $W = (u \cdot X_1) \cup X_2$, and the metric on W agrees with the respective metrics on X_1 and X_2, since the maps $f_1 : X_1 \to W$ and $f_2 : X_2 \to W$ defined by $f_1(x) = u \cdot (x, 1)$ and $f_2(y) = (y, 2)$ are isometries.

In standard terminology, some of the above results can be summed up as follows:

Theorem 5.17. *Let* Γ *be a set of partial non-expansive maps* (*isometries*) *with non-empty closed domain of a metric space* X*. Then there exists a closed isometric embedding* $X \hookrightarrow Y$ *into a metric space* Y *such that all members of* Γ *can be extended to global non-expansive maps* (*isometries*) *of* Y *and such that, moreover,* $\dim(Y) = \dim(X)$ *and* $|Y| \leqslant |X| \cdot |\Gamma| \cdot \aleph_0$.

Proof. Γ generates a subcategory (a subgroupoid, if all members of Γ are partial isometries) $\mathbf{C}$ of the category $\mathbf{M}(X)$ of metric subspaces of X; the set of morphisms of $\mathbf{C}$ has cardinality at most $|\Gamma| \cdot \aleph_0$. The inclusion $\mathbf{C} \hookrightarrow \mathbf{M}(X)$ induces a closed non-expansive nowhere degenerate partial action α on X as described in Example 2.9(vii). By Theorem 5.14 and Proposition 5.10(iii), the globalization of X w.r.t. α has the desired properties. □

By iterating the construction above, we can improve, in part, the known result [29][1] that every metric space X can be embedded into a metrically ultrahomogeneous space Z:

Theorem 5.18. *For every metric space* X *there exists an isometric closed embedding* $X \hookrightarrow Z$ *into a metrically ultrahomogeneous space* Z *such that* $\dim(Z) = \dim(X)$ *and* $|Z| = |X|$.

Proof. Start with the set Γ containing all partial isometries between finite subspaces of X *and* all global isometries of X (here, Γ is already a subcategory of $\mathbf{M}(X)$). Let Z_1 be the corresponding globalization according to the above theorem and iterate this process; the direct limit Z_∞ of the resulting ascending chain of metric spaces $X \hookrightarrow Z_1 \hookrightarrow Z_2 \hookrightarrow \cdots$ is an ultrahomogeneous space. Moreover, each inclusion is closed and $\dim(Z_n) = \dim(X)$ for all n. Hence, the inclusion $X \hookrightarrow Z_\infty$ is closed, and by the countable sum theorem, $\dim(Z_\infty) = \dim(X)$. A more careful choice of global isometries will guarantee that $|Z| = |X|$. □

[1] Uspenskij shows that it can be assumed that the weight is preserved and that the isometry group of X (endowed with the pointwise topology) is topologically embedded into the isometry group of Z (but this construction does not preserve dimension).

Remark 5.19. Topological versions of Theorems 5.17 and 5.18, with 'metric' replaced by 'normal' and 'metrically ultrahomogeneous' by 'topologically ultrahomogeneous', can be derived using Corollary 4.11 (see also [19–21]).

The global metric D on Y is in some respects easier to handle in case M is a group. Since the elements of M act as isometries and hence $D(u \cdot x, v \cdot y) = D(x, u^{-1}v \cdot y)$ for all $u, v \in M$ and all $x, y \in X$, it suffices to consider distances of the form $D(x, u \cdot y)$. Thus, the calculation of distances can be simplified:

Proposition 5.20. *Let M be a group. Let $u, v \in M$, let $g_k \cdots g_1$ be the normal form of $u^{-1}v$, and let $x, y \in X$. Then*

$$D(u \cdot x, v \cdot y) = \inf\left(d(y, x_1) + \sum_{i=1}^{k} d\big(g_i(x_i), x_{i+1}\big) \right),$$

where x_i ranges over $\operatorname{dom}(g_i)$ *for $i = 1, \ldots, k$ and $x_{k+1} = x$.*

Proof. As explained above, we need only calculate the distance from $a := u^{-1}v \cdot y$ to the point $x \in X$.

Since $e \cdot z$ is normal for all $z \in X$, a reduced path π from a to x cannot end with the normality pattern r, n, so that (excluding the trivial case (A3)) π must have one of the forms (A1) or (A4) of Lemma 5.2. Thus, π is determined by a subdivision $s_r \cdots s_1$ of $(g_k, \ldots, g_1)$ into non-empty words s_i and a selection of elements $x_i \in \operatorname{dom}(s_i^*)$, $i = 1, \ldots, r$; putting $x_{r+1} = x$, we can write the corresponding path length as

$$d(y, x_1) + \sum_{i=1}^{r} d\big(s_i^*(x_i), x_{i+1}\big).$$

Now observe that one subdivision of $(g_k, \ldots, g_1)$ is that into k one-element subwords $s_i = (g_i)$. Selecting elements $x_i \in \operatorname{dom}(s_i^*) = \operatorname{dom}(g_i)$, $i = 1, \ldots, k$, defines a (not necessarily reduced) path; call such paths *elementary paths*. It is easy to see that any reduced path π gives rise to an elementary path $\bar{\pi}$ such that $D(\pi) = D(\bar{\pi})$, and the lengths of elementary paths are exactly the sums given in the formula of the statement. □

A further rather immediate consequence of Lemma 5.2 is the existence of geodesic paths under suitable compactness assumptions:

Definition 5.21. Let $u \in M$ have normal form $g_k \cdots g_1$, $k \geqslant 0$. u is called a *C-element* if $\operatorname{dom}(g_i)$ is compact for $i = 1, \ldots, k$. A partial action is *compact* if $\operatorname{dom}(f)$ is compact for every morphism f.

Clearly, α is compact iff every $u \in M$ is a C-element.

Theorem 5.22. *Let X be a weak pseudometric space. If u and v are C-elements and $a = u \cdot x$, $b = v \cdot y$ are normal, then there exists a geodesic from a to b. In particular, if α is compact then there exists a geodesic for every pair of elements in Y.*

Proof. It suffices to show that, for each of the forms listed in Lemma 5.2, there exists a path which realizes the infimum among all reduced paths of that form. We treat only the case (A7); the other cases are analogous (and, mostly, easier).

A reduced path $((u_j, x_j, y_j))$ from a to b of the form (A7) is determined by a choice of a sequence $(u_1, \dots, u_k)$ such that

$$u = u_1 \succ \cdots \succ u_r \quad \text{and} \quad u_{r+1} \prec \cdots \prec u_k = v$$

for some $1 \leqslant r \leqslant k-1$, and a choice of elements $y_i \in \mathrm{dom}(g_1^i)$, $i = 1, \dots, r$ and $x_i \in \mathrm{dom}(g_1^i)$, $i = r+1, \dots, k$, where u_i has normal form $g_{s_i}^i \cdots g_1^i$. Obviously, there are only finitely many choices of $(u_1, \dots, u_k)$, so that it suffices to show that, given such a choice, the infimum among the corresponding paths is realized by some choice of elements as described. This follows by a standard compactness argument: the $\mathrm{dom}(g_1^i)$ are compact, and the path length depends continuously on the choice of the x_i and y_i. □

Corollary 5.23. *Let α be compact. If X is a* path space, *i.e., if the distance between any two points is the infimum of the lengths of all curves joining the points* [12], *then so is Y.*

6. Conclusion and outlook

We have demonstrated how a simple set-theoretic construction of globalizations for partial actions of monoids can be applied to topological and metric spaces, and we have shown that the resulting extensions are surprisingly well-behaved, provided that the partial action is confluent. In particular, we have shown that, in both cases, the original space is embedded in its extension, and that, under natural assumptions, important properties such as dimension, normality, or path metricity are preserved. Classical homogenization results arise as special cases of our construction. The main tool has been the application of rewriting theory in order to gain better control of the globalization.

Open questions include preservation of further topological and metric properties by the globalization, as well as the extension of the method to other categories. This includes categories used in general topology such as uniform spaces or, more generally, nearness spaces [13], as well as, in the realm of distance functions, the category of approach spaces [17], but also structures of a more analytical nature such as measurable maps (of mm-spaces [12,24]), smooth maps, or conformal maps.

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