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A remark on branched cyclic covers

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Abstract

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We give a simple proof of a lemma of Dellomo, which he used to calculate the inverse limit of the first homology of the branched cyclic covers of the 3-sphere, branched over a knot, and we show that the inverse limit of the higher homology is trivial.

In [3] Dellomo gives a formula for

$$\check{H}_1(\hat{\Sigma}) = \varprojlim H_1(\Sigma_n; \mathbb{Z}) ,$$

the inverse limit of the first homology of the branched cyclic covers of S^3, branched over a knot. (The index set is $\mathbb{N}$, ordered by divisibility.) A key step in his argument is the lemma in [3, Section 4], which establishes a stability result for the homology with coefficients $\mathbb{Z}/p^k\mathbb{Z}$ of such branched cyclic covers. We shall give an alternative, simpler proof of this proposition, and show that the higher homology of the inverse limit is trivial.

Let $k : S^1 \to S^3$ be a tame knot, with exterior X and group $G = \pi_1(X)$. Let X' be the infinite cyclic covering space of X. A transverse orientation for the knot determines an isomorphism of the covering group G/G' onto $\mathbb{Z}$, and hence we may view $M = H_1(X'; \mathbb{Z})$ as a module over the ring $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$. This module is $\mathbb{Z}$-torsion free and (hence) has a short free resolution over Λ, and multiplication by $t - 1$ is an automorphism [4, Chapter IV].

Let Σ_n be the n-fold branched cyclic covering of S^3, branched over k. Then $H_1(\Sigma_n; \mathbb{Z}) \approx M/(t^n - 1)M$, and multiplication by t gives the action of a generator of the covering group. (If R is any other coefficient ring, we also have

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$H_1(\Sigma_n; R) \approx M \otimes R/(t^n - 1)M \otimes R$.) Moreover, the map induced by the covering projection $\Sigma_{mn} \to \Sigma_n$ in the canonical quotient map (cf. [1, Chapter 8] or [4, Chapter VIII]).

Let p be a prime and let d_p be the dimension of M/pM as a vector space over the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Then we shall prove the following:

Lemma (Dellomo [3]). *For each exponent $k \geq 1$ there is an $n_k \geq 1$ such that for all $m \geq 1$,*

$$H_1(\Sigma_{mn_k}; \mathbb{Z}/p^k\mathbb{Z}) \approx H_1(\Sigma_{n_k}; \mathbb{Z}/p^k\mathbb{Z}) \approx (\mathbb{Z}/p^k\mathbb{Z})^{d_p} .$$

Proof. Let $0 \to \Lambda^q \xrightarrow{P} \Lambda^q \to M \to 0$ be a short free resolution of M over Λ. Then the annihilator ideal of M is principal, generated by $\alpha = \Delta_0(M)/\Delta_1(M)$, where $\Delta_0(M) = \det P$ and $\Delta_1(M)$ is the highest common factor of the $(a-1) \times (a-1)$ subdeterminants of P [4, p. 31]. Let $\pi : \Lambda \to \Lambda/p\Lambda = \mathbb{F}_p[t, t^{-1}]$ be the homomorphism which reduces coefficients modulo (p). Then $\pi(\Delta_0(M)) \neq 0$, since $\Delta_0(M)(1) = \pm 1$ [4, p. 42]. After multiplying $\pi(\Delta_0(M))$ by a power of t if necessary, we may assume that it is a polynomial with nonzero constant term; the dimension of M/pM over $\mathbb{F}_p$ is then the degree of $\pi(\Delta_0(M))$. Since $\pi(\det P) \neq 0$ we obtain a short free resolution for M/pM over $\Lambda/p\Lambda$ by reducing the entries of P modulo (p), and so the annihilator of M/pM over $\Lambda/p\Lambda$ is generated by $\pi(\alpha)$. It follows that the annihilator of M/pM as a Λ-module is the ideal (α, p). We may now show by induction on k that the annihilator of M/p^kM as a Λ-module is the ideal (α, p^k). (For suppose θ annihilates $M/p^{k+1}M$. Then $\theta M \subseteq p^{k+1}M \subseteq p^kM$, so by the hypothesis of induction $\theta = \alpha\beta + p^k\gamma$ for some β,γ in Λ. Hence $\gamma p^kM \subseteq p^{k+1}M$. Since M is $\mathbb{Z}$-torsion free, $\gamma M \subseteq pM$ and so $\gamma = \alpha\rho + p\sigma$ for some ρ,σ in Λ. Thus $\theta = \alpha(\beta + \rho p^k) + p^{k+1}\sigma$ is in (α, p^{k+1}).)

All the roots of $\pi(\alpha)$ (in some algebraic closure of $\mathbb{F}_p$) are roots of unity; let h be the lowest common multiple of their orders. We may assume that $\pi(\alpha)$ is a polynomial of degree $d \leq d_p$ and so the roots of $\pi(\alpha)$ have multiplicity at most d, which is less than p^d. Therefore

$$\pi(\alpha) \quad \text{divides} \quad (t^h - 1)^{p^d} = (t^{hp^d} - 1) .$$

Let $n = hp^d$. Then for any $m \geq 1$ we have that $\pi(\alpha)$ divides $t^{nm} - 1$, and so $t^{nm} - 1$ annihilates M/pM. Therefore,

$$\begin{aligned} H_1(\Sigma_{nm}; \mathbb{Z}/p\mathbb{Z}) &\approx (M/pM)/(t^{nm-1})(M/pM) = M/pM \\ &\approx (\mathbb{Z}/p\mathbb{Z})^{d_p} \quad \text{for all } m \geq 1 . \end{aligned}$$

Now since $\pi(\alpha)$ divides $t^n - 1$, there is some λ in Λ such that α divides $t^n - 1 + p\lambda$. Therefore, α also divides $(t^n - 1 + p\lambda)^{p^k}$, which equals $t^{np^k} - 1 + p^k\mu$ for some μ in Λ, at least if p is odd. When $p = 2$ we observe instead that α divides

$$\begin{aligned}&(t^{n2^{k+1}} - 1)(t^n - 1 + 2\lambda)^{2^{k+1}}\\ &= (t^{n2^{k+1}} - 1)(t^{n2^{k+1}} + 1 + 2^k\mu)\\ &= t^{n2^{k+2}} - 1 + 2^k\nu\end{aligned}$$

for some μ, ν in Λ. For $k \geq 1$ let $n_k = np^k$ if p is odd, and let $n_k = n2^{k+2}$ if $p = 2$. Then for any $m \geq 1$ we have that $t^{mn_k} - 1$ is in the ideal (α, p^k), and so annihilates M/p^kM. It follows as before that

$$H_1(\Sigma_{mn_k}; \mathbb{Z}/p^k\mathbb{Z}) \approx M/p^kM \quad \text{for all } m \geq 1\,.$$

Since M is a $\mathbb{Z}$-torsion free abelian group, $M/p^kM \approx (\mathbb{Z}/p^k\mathbb{Z})^e$ for some exponent e, and on reduction modulo (p) we find that $e = d_p$. □

Using his lemma and a proposition on limits of surjective inverse systems of finitely generated abelian groups, Dellomo proves that

$$\check{H}_1(\hat{\Sigma}) = \mathbb{Z}^{2s} \oplus \prod_p (\hat{\mathbb{Z}}_p)^{d_p - 2s}\,,$$

where $2s = \max\{\beta_1(\Sigma_m) \mid m \text{ in } \mathbb{N}\}$ is the number of roots of the Alexander polynomial of k which are roots of unity, and where $\hat{\mathbb{Z}}_p$ is the additive group of p-adic integers. (For almost all primes p, d_p is the degree of the Alexander polynomial.)

We may also ask what are the higher homology groups

$$\check{H}_i(\hat{\Sigma}) = \varprojlim H_i(\Sigma_n; \mathbb{Z})\,,$$

for $i = 2,3$. Since $H_3(\Sigma_n; \mathbb{Z}) \approx \mathbb{Z}$ and the map from Σ_{mn} to Σ_n has degree m,

$$\check{H}_3(\hat{\Sigma}) = \varprojlim (\mathbb{Z} \xrightarrow{m} \mathbb{Z}) = 0\,.$$

In fact, $\check{H}_2(\hat{\Sigma})$ is also 0. For let X_n be the n-fold (unbranched) cyclic cover of X. Then the inclusion of X_n into $\Sigma_n = X_n \cup D^2 \times S^1$ induces an isomorphism from $H_2(X_n; \mathbb{Z})$ to $H_2(\Sigma_n; \mathbb{Z})$, by excision. From the Wang sequence for the projection of X' onto X_n we see that $H_2(X_n; \mathbb{Z})$ may be identified with $K_n = \ker(t^n - 1 : M \to M)$, and the map induced by the projection of X_{mn} onto X_n is multiplication by $(t^{mn} - 1)/(t^n - 1)$. Since M is a noetherian Λ-module the increasing sequence of submodules $K_{n!}$ stabilizes. Hence there is an N such that $K_{mN} = K_N$ for all $m \geq 1$. Moreover, K_N is a finitely generated $\mathbb{Z}$-torsion free Λ-module which is annihilated by $t^N - 1$, and so is a finitely generated free abelian group. The map from K_{mN} to K_N is given by multiplication by

$$(t^{mN} - 1)/(t^N - 1) = \sum_{0 \le i < m} t^{iN} = m$$

since t^N acts as 1 on K_{mN}. Since the subset $\{mN \mid m \ge 1\}$ is cofinal in $\mathbb{N}$ it follows that

$$\check{H}_2(\hat{\Sigma}) = \varprojlim_m \{K_N \xrightarrow{m} K_N\} = 0 .$$

Remark. Similar calculations apply to the homology of branched cyclic covers of simple higher-dimensional knots.

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