North-Holland

A MIN–MAX RELATION FOR THE PARTIAL q-COLOURINGS OF A GRAPH. PART II: BOX PERFECTION

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This paper examines extensions of a min–max equality (stated in C. Berge, Part I) for the maximum number of nodes in a perfect graph which can be q-coloured.

A system L of linear inequalities in the variables $\underline{x}$ is called TDI if for every linear function $\underline{c}\underline{x}$ such that $\underline{c}$ is all integers, the dual of the linear program: maximize $\{\underline{c}\underline{x}: \underline{x}$ satisfies $L\}$ has an integer-valued optimum solution or no optimum solution. A system L is called box TDI if L together with any inequalities $\underline{l} \leqslant \underline{x} \leqslant \underline{u}$ is TDI. It is a corollary of work of Fulkerson and Lovász that: where A is a 0–1 matrix with no all-0 column and with the 1-columns of any row not a proper subset of the 1-columns of any other row, the system $L(G) = \{A\underline{x} \leqslant \underline{1},\ \underline{x} \geqslant \underline{0}\}$ is TDI if and only if A is the matrix of maximal cliques (rows) versus nodes (columns) of a perfect graph. Here we will describe a class of graphs in a graph-theoretic way, and characterize them as the graphs G for which the system $L(G)$ is box TDI. Thus we call these graphs box perfect. We also describe some classes of box perfect graphs.

1. Introduction

Consider a graph H for which every induced subgraph G of H satisfies the following min–max equalities for every positive integer q (equivalently, for every positive integer $q < \omega(G)$, the maximum size of a clique in G).

$$\begin{cases} (1.1)\ \ \text{maximum}\{|S|: S \subseteq V(G);\ \forall \text{ clique } C \text{ in } G,\ |S \cap C| \leqslant q\} \\ = \\ (1.2)\ \ \text{minimum}\{q\,|\mathscr{K}| + |V(G) - \bigcup \mathscr{K}|: \mathscr{K} \text{ is a set of cliques in } G\}. \end{cases} \tag{1}$$

($V(G)$ denotes the node-set of G. In this paper, cliques need not be maximal.)

Restricting q to be 1 in the above would say H is perfect. Also then, (using the Perfect Graph Theorem [23, 24]) a set S as in (1.1) is the same as a set which can be partitioned into no more than q stable sets; that is, a partial q-colouring. Thus:

For a perfect graph G,

$$\begin{cases} (1.1) \\ = \\ (1.3)\ \ \text{maximum}\{|S|: S \subseteq V(G),\ S \text{ is a partial } q\text{-colouring in } G\}. \end{cases}$$

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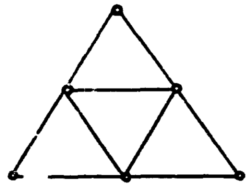

Fig. 1.

It is clear that:
For any graph G,

$$\begin{cases}(1.2)\\ =\\ (1.4)\ \text{minimum}\left\{\sum_{C\in\mathscr{C}}\min\{q,|C|\}:\ \mathscr{C}\text{ is a covering of } G \text{ by cliques}\right\}.\end{cases}$$

Note that the $\mathscr{K}$ of (1.2) and the $\mathscr{C}$ of (1.4) may be taken to be node-disjoint. Thus, for perfect graphs, the equality (1) is the same as (4) in ([3], this volume) which says $(1.3)=(1.4)$. Lovász [25] called a graph H q-perfect if for each induced subgraph G of H, $(1.3)=(1.2)$.

Greene [19] gave the graph of Fig. 1 to show that not all perfect graphs satisfy (1). It does not satisfy (1) for $q=2$. The Dilworth–Greene–Kleitman min–max theorem ([8], [20]) says that comparability graphs satisfy (1) for all q. Greene's min–max theorem [19] and a more general theorem proved independently by Edmonds and Giles [9] say that cocomparability graphs satisfy (1) for all q.

Lovász [24] proved that the substitution operation preserves perfection: if H and K are disjoint graphs and v is a node of H, then to *substitute* K for v in H, join each node of K to each neighbour of v, and delete v. Two important special cases of substitution are joined and unjoined duplication: to create *m joined duplicates* of node v, substitute a clique of m nodes for v; to create *m unjoined duplicates* of v, substitute a stable set of m nodes for v. Note that creating 0 joined or unjoined duplicates of node v corresponds to deleting v, and thus taking an induced subgraph is a special case of duplication.

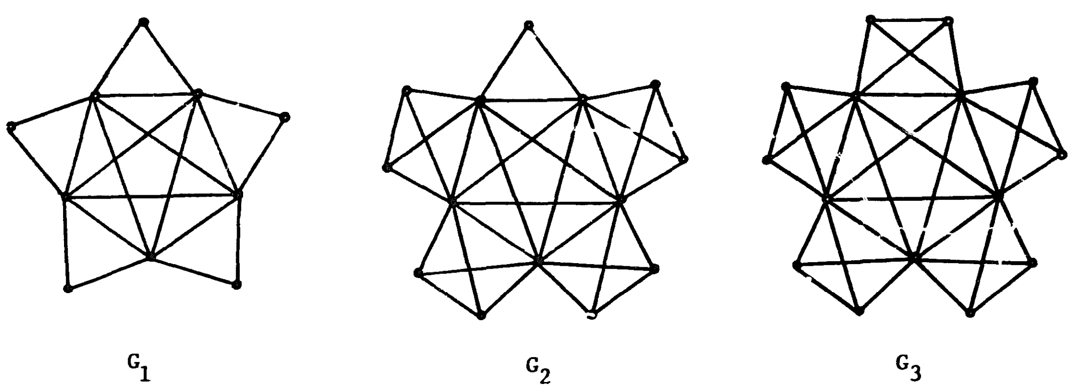

Fig. 2.

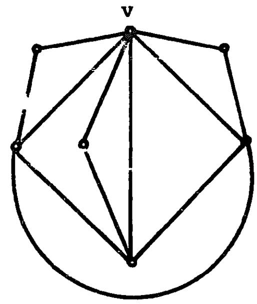

Fig. 3.

Creating joined duplicates need not preserve min–max (1): The graphs G_1 and G_2 in Fig. 2 satisfy (1) for all induced subgraphs and all q, but G_3 does not satisfy (1) for $q = 3$. (See Section 6 for more examples and proofs.)

Jean Fonlupt pointed out that creating unjoined duplicates need not preserve min–max (1). The graph in Fig. 3 satisfies (1) for all induced subgraphs and all q, but if node v is replaced by two unjoined duplicates, the new graph does not satisfy (1) for $q = 2$.

Let us examine the effect of creating joined or unjoined duplicates in G on the min–max equality (1).

For each $v \in V(G)$, let a_v be a non-negative integer. Replace each $v \in V(G)$ by a set of a_v joined duplicates to get a new graph G'—that is, substitute a clique of size a_v for v. G' satisfies (1) if and only if:

$$\left\{\begin{array}{l} (2.1)\ \text{maximum}\left\{\sum_{v \in V(G)} x_v : \forall \text{ clique } C \text{ in } G,\ \sum_{v \in C} x_v \leq q; \right. \\ \qquad\qquad\qquad\qquad \left. \forall v \in V(G),\ 0 \leq x_v \leq a_v,\ x_v \text{ integer} \right\} \\ = \\ (2.2)\ \text{minimum}\left\{q\,|\mathscr{K}| + \sum_{v \notin \bigcup \mathscr{K}} a_v : \mathscr{K} \text{ is a set of cliques in } G\right\}. \end{array}\right. \tag{2}$$

Since creating joined duplicates need not preserve min–max (1), equivalently if a graph satisfies (1) for all induced subgraphs and all q, it need not satisfy (2). The graph of Fig. 4 does not satisfy (2) for $q = 3$ and the a_v's as shown.

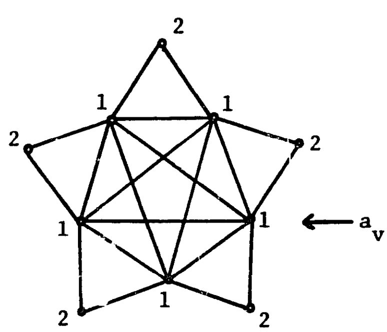

Fig. 4.

For each $v \in V(G)$, let w_v be a non-negative integer. Replace each $v \in V(G)$ by a set of w_v unjoined duplicates to get a new graph G'—that is, substitute a stable set of size w_v for v. G' satisfies (1) if and only if:

$$\left\{\begin{array}{l} \text{(3.1) maximum}\left\{\sum_{v\in S} w_v : S \subseteq V(G);\ \forall \text{ clique } C \text{ in } G,\ |S\cap C| \leq q\right\} \\ = \\ \text{(3.2) minimum}\left\{q \sum_{\substack{\text{cliques}\\ C}} y_C + \sum_{v\in V(G)} y_v : \forall v \in V(G),\ \sum_{v\in C} y_C + y_v \geq w_v;\right. \\ \qquad\qquad \forall \text{ cliques } C,\ y_C \geq 0,\ y_C \text{ integer}; \\ \qquad\qquad \left.\forall v \in V(G),\ y_v \geq 0,\ y_v \text{ integer}\right\}. \end{array}\right. \tag{3}$$

Where $q=1$, Fulkerson [14] called a graph *pluperfect* if it satisfies (3) for every non-negative integer-valued $\underline{w} = (w_v : v \in V(G))$.

We now look at a unification of (2) and (3). A graph G is called *box perfect* if for every positive integer q, and all non-negative integer-valued $\underline{w} = (w_v : v \in V(G))$ and $\underline{a} = (a_v : v \in V(G))$, the following min-max equality holds:

$$\left\{\begin{array}{l} \text{(4.1) maximum}\left\{\sum_{v\in V(G)} w_v x_v : \forall \text{ clique } C \text{ in } G,\ \sum_{v\in C} x_v \leq q;\right. \\ \qquad\qquad \left.\forall v \in V(G),\ 0 \leq x_v \leq a_v,\ x_v \text{ integer}\right\} \\ = \\ \text{(4.2) minimum}\left\{q \sum_{\substack{\text{cliques}\\ C}} y_C + \sum_{v\in V(G)} a_v y_v :\right. \\ \qquad\qquad \forall v \in V(G),\ \sum_{v\in C} y_C + y_v \geq w_v; \\ \qquad\qquad \forall \text{ cliques } C,\ y_C \geq 0,\ y_C \text{ integer}; \\ \qquad\qquad \left.\forall v \in V(G),\ y_v \geq 0,\ y_v \text{ integer}\right\}. \end{array}\right. \tag{4}$$

Where $w_v = 1\ \forall v \in V(G)$, (4) is (2). Where $a_v = 1\ \forall v \in V(G)$, (4) is (3). Where $w_v = 1$ and $a_v = 1$, $\forall v \in V(G)$, (4) is (1).

In Section 4, we will prove that box perfect graphs are precisely the graphs for which a certain system of linear inequalities is box totally dual integral.

Note that if G is box perfect then so is any induced subgraph of G: choose $a_v = 0$ (or $w_v = 0$) for v not in the induced subgraph.

2. Box perfection and joined and unjoined duplicates

For a fixed $\underline{w} = (w_v : v \in V(G))$ and $\underline{a} = (a_v : v \in V(G))$, let $G_1(\underline{w}, \underline{a})$ be the graph obtained from G by substituting a stable set S_v of size w_v for each

$v \in V(G)$, and then substituting a clique of size a_v for each $u \in S_v$. Then:

$$(4) \text{ holds for } G \Leftrightarrow (1) \text{ holds for } G_1(\underline{w}, \underline{a}) \tag{5}$$

If G is box perfect, it turns out that it is not necessary to substitute a clique of the same size for each node of S_v in order to conclude that (1) holds. In Section 5 we will prove:

Theorem 1. *Creating unjoined duplicates preserves box perfection.*

Corollary 1. *G is box perfect* $\Leftrightarrow$
Any graph obtained from G by first creating unjoined duplicates and then creating joined duplicates satisfies (1).

Alternatively, for a fixed $\underline{w}$ and $\underline{a}$, let $G_2(\underline{w}, \underline{a})$ be the graph obtained from G by substituting a clique C_v of size a_v for each $v \in V(G)$, and then substituting a stable set of size w_v for each $u \in C_v$. In general, $G_1(\underline{w}, \underline{a}) \neq G_2(\underline{w}, \underline{a})$. However:

$$(4) \text{ holds for } G \Leftrightarrow (1) \text{ holds for } G_2(\underline{w}, \underline{a}). \tag{6}$$

Similar to before, if G is box perfect, it turns out not to be necessary to substitute a stable set of the same size for each node of C_v in order to conclude that (1) holds. In Section 5 we will prove;

Theorem 2. *Creating joined duplicates preserves box perfection.*

Corollary 2. *G is box perfect.* $\Leftrightarrow$
Any graph obtained from G by first creating joined duplicates and then creating unjoined duplicates satisfies (1).

Corollary 3. *G is box perfect.* $\Leftrightarrow$
Any graph obtained from G by creating a series of joined and/or unjoined duplicates satisfies (1).

3. Classes of box perfect graphs

Theorem 3. *The following classes of graphs are box perfect.*
(i) *Comparability graphs* [4].
(ii) *Cocomparability graphs* [9, 4].
(iii) *Graphs whose clique-node incidence matrix is totally unimodular.*
(vi) *p-Comparability graphs* [4], *defined immediately below.*

A *p-comparability graph* is a graph which arises in the following way: start with a digraph G that has a set $T \subseteq V(G)$, $|T| \leqslant p$, such that every edge of G is in a

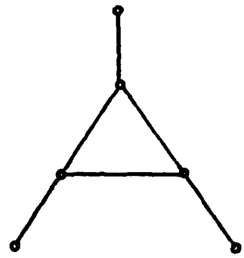

Fig. 5.

dicircuit, and every dicircuit of G intersects T exactly once; add the chords of every dicircuit, delete T, and make all edges undirected. 1-comparability graphs are precisely comparability graphs [4].

It was proved in [4] that p-comparability graphs (and thus comparability graphs), and cocomparability graphs are box perfect. These proofs were based on our Coflow Polyhedron Theorem ([4, 5]) which gives strong min–max properties for any digraph. In particular, this provides new proofs of the Greene–Kleitman Theorem and Greene's Theorem.

It also follows from the Greene–Kleitman Theorem and Corollary 1 that comparability graphs are box perfect since creating joined or unjoined duplicates preserves being a comparability graph. Similarly, it follows from Greene's Theorem and Corollary 1 that cocomparability graphs are box perfect. The Edmonds–Giles Theorem says cocomparability graphs are box perfect.

Proof of (iii). It is easy to see that if the clique-node incidence matrix A of graph G is totally unimodular, then G is box perfect: For a postive integer q, let $\underset{\sim}{q}$ be a vector of all q's. Then it is well known [22] that for integer-valued vectors $\underset{\sim}{w}$ and $\underset{\sim}{a}$, the following linear program (7) and its dual have integer-valued optimum solutions. By the linear programming duality theorem, the optimum objective values of a linear program and its dual are equal. This is precisely (4).

$$\begin{cases} \text{maximize } \underset{\sim}{w}\underset{\sim}{x} \text{ subject to} \\ A\underset{\sim}{x} \leqslant \underset{\sim}{q} \\ \underset{\sim}{0} \leqslant \underset{\sim}{x} \leqslant \underset{\sim}{a}. \quad \square \end{cases} \tag{7}$$

We comment that if G is box perfect, its complement need not be. The graph of Fig. 5 is the line-graph of a bipartite graph, and hence box perfect, but its complement, the graph of Fig. 1, does not satisfy (1) for $q = 2$.

4. Total dual integrality

Let A be a matrix, $\underset{\sim}{d}$ and $\underset{\sim}{c}$ vectors of constants, and $\underset{\sim}{x}$ a vector of variables. A system, $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$, of linear inequalities with rational A and $\underset{\sim}{d}$ is called *totally dual* *integral* (TDI) if the dual of the linear program: maximize $\{\underset{\sim}{c}\underset{\sim}{x}: A\underset{\sim}{x} \leqslant \underset{\sim}{d}\}$ has an integer-valued optimum solution for every integer-valued $\underset{\sim}{c}$ such that it has an optimum solution [9].

TDI systems are interesting because of the following result.

The TDI Theorem (Edmonds and Giles, [9]). *If $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$ is a totally dual integral system with integer-valued $\underset{\sim}{d}$, then for any $\underset{\sim}{c}$ such that maximum$\{\underset{\sim}{c}\underset{\sim}{x}: A\underset{\sim}{x} \leqslant \underset{\sim}{d}\}$ exists, there is an integer-valued optimum solution $\underset{\sim}{x}^*$.*

Thus a TDI system with integer-valued $\underset{\sim}{d}$ provides the following integer min–max equality for any integer-valued $\underset{\sim}{c}$ for which either the min or the max exists.

$$\begin{cases} \text{maximum}\{\underset{\sim}{c}\underset{\sim}{x}: A\underset{\sim}{x} \leqslant \underset{\sim}{d},\ \underset{\sim}{x} \text{ integer-valued}\} \\ = \\ \text{minimum}\{\underset{\sim}{y}\underset{\sim}{d}: \underset{\sim}{y}A = \underset{\sim}{c},\ \underset{\sim}{y} \geqslant \underset{\sim}{0},\ \underset{\sim}{y} \text{ integer-valued}\}. \end{cases} \tag{8}$$

A system, $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$, of linear inequalities is called *box totally dual integral* (*box TDI*), if it together with any upper and lower bounds on the individual variables is TDI; that is, if for any $\underset{\sim}{l}, \underset{\sim}{u} \in (\mathbb{Q} \cup \{\pm\infty\})^n$, the system $\begin{cases} A\underset{\sim}{x} \leqslant \underset{\sim}{d} \\ \underset{\sim}{l} \leqslant \underset{\sim}{x} \leqslant \underset{\sim}{u} \end{cases}$ is TDI. $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$ is called *upper box TDI* if it together with any upper bounds on the variables is TDI; that is, if for any $\underset{\sim}{u} \in (\mathbb{Q} \cup \{+\infty\})^n$, the system $\begin{cases} A\underset{\sim}{x} \leqslant \underset{\sim}{d} \\ \underset{\sim}{x} \leqslant \underset{\sim}{u} \end{cases}$ is TDI.

Groflin [21] proved that $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$ is box TDI if and only if for any subset J of the variables, and any values $u_j \in \mathbb{Q}$ for $j \in J$, the system

$$\begin{cases} A\underset{\sim}{x} \leqslant \underset{\sim}{d} \\ x_j = u_j \text{ for } j \in J \end{cases} \tag{9}$$

is TDI. Also, if $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$ is TDI, so is any system obtained by changing some of the inequalities to equations (for a proof, see [28], Theorem 22.2). It follows that if $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$ is upper box TDI, it is also box TDI, and thus box TDI and upper box TDI are equivalent.

We will consider the following system of clique inequalities and non-negativity constraints for graph G.

$$\begin{cases} (10.1)\ \forall \text{ clique } C \text{ in } G,\ \sum_{v \in C} x_v \leqslant 1; \\ (10.2)\ \forall v \in V(G),\ x_v \geqslant 0. \end{cases} \tag{10}$$

For our discussion here it does not matter if we consider only maximal cliques in (10) or all cliques.

Theorem 4. *G is perfect.*

$\Leftrightarrow$ The system (10) of clique inequalities and non-negativity constraints is TDI.

Proof. The "if" part of this theorem follows by the TDI Theorem. The "only if" part is immediate by Lovasz's theorem that creating unjoined duplicates preserves perfection (and thus for perfect graphs, (3) holds for $q = 1$). □

Theorem 4 motivates us to study graphs for which the system (10) is box TDI, which we will now show are the box perfect graphs, defined earlier in Section 1.

Theorem 5. *G is box perfect.*

$\Leftrightarrow$ The system (10) of clique inequalities and non-negativity constraints is box TDI.

Lemma. *If $\underset{\sim}{y}$ is an optimum solution to the dual of the linear program maximize$\{\underset{\sim}{c}\underset{\sim}{x}: A\underset{\sim}{x} \leqslant \underset{\sim}{d}\}$, and r is a positive rational then $\underset{\sim}{y}$ is an optimum solution to the dual of the linear program maximize $\{\underset{\sim}{c}\underset{\sim}{x}: A\underset{\sim}{x} \leqslant r\underset{\sim}{d}\}$. Thus if $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$ is TDI, so is $A\underset{\sim}{x} \leqslant r\underset{\sim}{d}$.* □

Proof of Theorem 5. Suppose G is box perfect. We will show that (10) is upper box TDI, and hence box TDI. We must show that for an integer-valued $\underset{\sim}{c}$, the dual of the linear program (11) below has an integer-valued optimum solution.

$$\begin{cases} \text{maximize} \sum_{v \in V(G)} c_v x_v \text{ subject to} \\ \forall \text{ clique } C, \sum_{v \in C} x_v \leqslant 1; \\ \forall v \in V(G),\ 0 \leqslant x_v \leqslant u_v. \end{cases} \tag{11}$$

We may assume that $c_v \geqslant 0$ and $0 \leqslant u_v < \infty$, $\forall v$. Let q be a positive integer so that for each u_v, qu_v is an integer. Then for this q, for $w_v = c_v$, and $a_v = qu_v$, (4) holds. By the lemma, the $\underset{\sim}{y}$ of (4.2), which is integer-valued, is an optimum solution to the dual of (11).

Now suppose that the system (10) is box TDI. We must show that for a positive integer q, and non-negative integer-valued $\underset{\sim}{a}$ and $\underset{\sim}{w}$, (4) holds. The system:

$$\begin{cases} \forall \text{ clique } C, \sum_{v \in C} x_v \leqslant 1; \\ \forall v \in V(G),\ 0 \leqslant x_v \leqslant a_v/q, \end{cases}$$

is TDI. Thus by the lemma, so is the system:

$$\begin{cases} \forall \text{ clique } C, \sum_{v \in C} x_v \leqslant q; \\ \forall v \in V(G),\ 0 \leqslant x_v \leqslant a_v. \end{cases} \tag{12}$$

Then the min–max (8) where $\underset{\sim}{c} = \underset{\sim}{w}$ and $A\underset{\sim}{x} \leqslant \underset{\sim}{d}$ is (12) is the same as (4). □

5. Proofs of Theorems 1 and 2

Theorem 1. *Creating unjoined duplicates preserves box perfection.*

Proof. Assume G is box perfect. Consider the graph G' obtained from G by creating an unjoined duplicate t of $u \in V(G)$. We will show that G' is box perfect by showing that (1) holds for $G_1'(\underline{w}, \underline{a})$ for all non-negative integer-valued $\underline{w}$ and $\underline{a}$. Fix $\underline{w}$ and $\underline{a}$, and let H' denote the graph obtained from G' by substituting for each node $v \in V(G')$ a stable set $S_v = \{v_i: i = 1, \ldots, w_v\}$ of size w_v. It suffices now to show that (2) holds for H' with the upper bound $a_{v_i} = a_v$ for each $v_i \in S_v$.

If $a_t = a_u$, then (2) holds for H' since (1) holds for $G_1(\underline{w}', \underline{a}')$ where $w_u' = w_u + w_t$, $w_v' = w_v$ for $v \in V(G) - u$, $a_v' = a_v$ for $v \in V(G)$. Thus without loss of generality, $0 < a_t < a_u$.

We will use the fact that (2) holds in each of the following instances:

$$H \equiv H' - S_t \text{ (i.e. } H \text{ is obtained from } G \text{ as } H' \text{ was from } G'\text{) with the same upper bounds } a_v \text{ as } H'. \tag{13}$$

$$H' \text{ with the upper bounds on the nodes } u_i \in S_u \text{ lowered from } a_u \text{ to } a_t, \text{ but the other upper bounds unchanged.} \tag{14}$$

If in some optimum $\underline{x}$ for (13), $x_{u_i} \geq a_t$ for some $u_i \in S_u$, then setting $x_{t_j}' = a_t$ for $t_j \in S_t$, and $x_p' = x_p$ for $p \in V(H') - S_t$, and taking $\mathscr{K}' =$ any optimum $\mathscr{K}$ for (13), it is clear that $\underline{x}'$ and $\mathscr{K}'$ satisfy (2) for H' with the given upper bounds. Thus we may assume that

$$\text{for every optimum } \underline{x} \text{ for (13), } x_{u_i} < a_t, \text{ for all } u_i \in S_u. \tag{15}$$

Note that every $\underline{x}$ feasible for (14) is feasible for H' with the given upper bounds. If for some optimum $\mathscr{K}$ for (14), at least w_u members of $S_u \cup S_t$ are in $\bigcup \mathscr{K}$, then we can assume that all members of S_u are in $\cup \mathscr{K}$, and then this $\mathscr{K}$ and any optimum $\underline{x}$ for (14) satisfy (2) for H' with the given upper bounds. Thus we may assume that for every optimum $\mathscr{K}$ for (14) fewer than w_u members of $S_u \cup S_t$ are in $\bigcup \mathscr{K}$. Let $\mathscr{K}^*$ be an optimum $\mathscr{K}$ for (14) such that no node of S_t is in $\bigcup \mathscr{K}$. Let $\underline{x}^*$ be an optimum $\underline{x}$ for (14). By complementary slackness, since some node u_i of S_u is not in $\bigcup \mathscr{K}$, $x_{u_i}^* = a_t$. It is easily seen that $\underline{x}^*$ restricted to H and $\mathscr{K}^*$ are optimum for (13). But this contradicts (15). □

Theorem 2. *Creating joined duplicates preserves box perfection.*

We comment that Theorem 2 also follows from Edmonds and Giles' Theorem ([9, 10, 11]) that duplicating variables preserves box total dual integrality [4].

Proof of Theorem 2. This proof is similar to the proof of Theorem 1.

Assume G is box perfect. Consider the graph G' obtained from G by creating a joined duplicate t of $u \in V(G)$. We will show that G' is box perfect by showing that (1) holds for $G_2'(\underline{w}, \underline{a})$ for all non-negative integer-valued $\underline{w}$ and $\underline{a}$. Fix $\underline{w}$ and $\underline{a}$, and let H' denote the graph obtained from G' by substituting for each node $v \in V(G')$ with a clique $C_v = \{v_i: i = 1, \ldots, a_v\}$ of size a_v. It suffices now to show that (3) holds for H' with the weights $w_{v_i} = w_v$ for each $v_i \in C_v$.

If $w_t = w_u$, then (3) holds for H' since (1) holds for $G_1(\underline{w}', \underline{a}')$ where $a_u' = a_u + a_t$, $a_v' = a_v$ for $v \in V(G) - u$, $w_v' = w_v$ for $v \in V(G)$. Thus without loss of generality, $0 < w_t < w_u$.

We will use the fact that (3) holds in each of the following instances:

$$H \equiv H' - C_t \text{ (i.e. } H \text{ is obtained from } G \text{ as } H' \text{ was from } G'\text{) with the same weights } w_v \text{ as } H'. \tag{16}$$

$$H' \text{ with the weights on the nodes } u_i \in C_u \text{ lowered from } w_u \text{ to } w_t, \text{ but the other weights unchanged.} \tag{17}$$

If in some optimum $\underline{y}$ for (16), $y_{u_i} \leq w_u - w_t$, for some $u_i \in C_u$; that is, $\sum_{u_i \in C} y_C \geq w_t$; then setting $y'_{C \cup C_t} = y_C$ for cliques C in $H' - C_t$ with $u_i \in C$; $y'_C = y_C$ for cliques C in $H' - C_t$ with $u_i \notin C$; $y'_{t_j} = 0$ for $t_j \in C_t$; and $y'_p = y_p$ for $p \in V(H') - C_t$; and letting S' be some optimum S for (16), it is clear that S' and $\underline{y}'$ satisfy (3) for H' with the given weights. Thus we may assume that:

$$\text{for every optimum } \underline{y} \text{ for (16), } y_{u_i} > w_u - w_t > 0, \text{ for all } u_i \in C_u. \tag{18}$$

If for some optimum S for (17), at least a_u members of $C_u \cup C_t$ are in S, we can assume all members of C_u are in S, and then this S together with any optimum $\underline{y}$ for (17) with y_{u_i} increased to $y_{u_i} + (w_u - w_t)$ for each $u_i \in C_u$ satisfy (3) for H' with the given weights. Thus we may assume that for every optimum S for (17), fewer than a_u members of $C_u \cup C_t$ are in S. Let S^* be an optimum S for (17) such that no node of C_t is in S^*. Let $\underline{y}^*$ be an optimum $\underline{y}$ for (17). By complementary slackness, since some node u_i of C_u is not in S^*, $y^*_{u_i} = 0$. It is easily seen that S^* and $\underline{y}^*$ restricted to H are optimum for (16). But this contradicts (18). □

6. An infinite class of graphs which satisfy (1) for all induced subgraphs and all q, but which are not box perfect.

An *m-trampoline* ($m \geq 3$) is the graph G with $2m$ nodes

$$V(G) = \{v_i: 0 \leq i \leq m-1\} \cup \{u_i: 0 \leq i \leq m-1\}$$

and

$$E(G) = \{(v_i, u_i): 0 \leq i \leq m-1\} \cup \{(v_i, u_{i+1 (\mathrm{mod}\, m)}): 0 \leq i \leq m-1\} \\ \cup \{(u_i, u_j): 0 \leq i < j \leq m-1\}.$$

A *trampoline* is a graph which is an m-trampoline for some m. A trampoline is called *odd* or *even* according to whether m is odd or even. The m 3-cliques $\{u_i, v_i, u_{i+1(\mathrm{mod}\, m)}\}$, $0 \leq i \leq m-1$, of the m-trampoline are called the *outer* *triangles.* The graph of Fig. 1 is the 3-trampoline. The graph G_1 of Fig. 2 is the 5-trampoline.

Proposition 1 [4]. *All the trampolines except the* 3-*trampoline satisfy* (1) *for all induced subgraphs and all* q.

Proposition 2 [4]. *None of the odd trampolines are box perfect: The* $(2q-1)$-*trampoline does not satisfy* (4) *where* $a_{u_i}=1$, $a_{v_i}=q-1$, *and* $w_{u_i}=w_{v_i}=1$, $0\leqslant i\leqslant m-1$.

Proof of Proposition 1. In [4], it was shown that any induced subgraph F of an m-trampoline G, except G itself, is a p-comparability graph, and hence satisfies (1).

Assuming $m\geqslant 4$, Table 1 give an S and $\mathscr{K}$ satisfying (1) for G, for all values of $q<m$. □

Table 1

q	S	$\mathscr{K}$
1	$v_0, v_1, \ldots, v_{m-1}$	the outer triangles
2	$v_0, v_1, \ldots, v_{m-1}, u_0, u_2$	the m-clique
$q=3,\ldots,m-1$	$v_0, v_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{q-1}$	the m-clique

Proof of Proposition 2. We will display an $\underset{\sim}{x}$ feasible for (4.1) except that it is not integer-valued, and a $\underset{\sim}{y}$ feasible for (4.2) except that it is not integer-valued, such that the objective values of this $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are equal, but are not an integer.

$$\left.\begin{aligned} x_{u_i}&=\tfrac{1}{2}\\ x_{v_i}&=q-1\end{aligned}\right\}0\leqslant i\leqslant 2q-2 \quad (=m-1)$$

$y_C=\frac{1}{2}$ for each of the $2q-1$ outer triangles

$y_C=0$ for any other clique

$$\left.\begin{aligned} y_{u_i}&=0\\ y_{v_i}&=\tfrac{1}{2}\end{aligned}\right\}0\leqslant i\leqslant 2q-2$$

$$\sum_{i=0}^{2q-2} x_{u_i}+\sum_{i=0}^{2q-2} x_{v_i}=(2q-1)(\tfrac{1}{2})+(2q-1)(q-1)=2q^2-2q+\tfrac{1}{2}$$

$$q\sum y_C+\sum_{i=0}^{2q-2} y_{u_i}+\sum_{i=0}^{2q-2}(q-1)y_{v_i}=q(2q-1)(\tfrac{1}{2})+0+(2q-1)(q-1)(\tfrac{1}{2})$$

$$=2q^2-2q+\tfrac{1}{2}.\quad \square$$

7. Balanced graphs

A 0–1 matrix is called *balanced* if it has no odd order submatrix with exactly two 1's in each row and column. A graph is *balanced* if its incidence matrix of maximal cliques versus nodes is balanced. Fulkerson et al. [16] proved that if matrix A is balanced, and $\underline{d}$ and $\underline{a}$ are non-negative integer-valued vectors, then the following linear program (19) and its dual have integer-valued optimum solutions:

$$\begin{cases} \text{maximize } \underline{1} \cdot \underline{x} \text{ subject to} \\ A\underline{x} \leq \underline{d} \\ \underline{0} \leq \underline{x} \leq \underline{a} \end{cases} . \tag{19}$$

Where A is the incidence matrix of maximal cliques versus nodes of balanced graph G and $\underline{d}$ is a vector of all q's, this implies that (2) holds for G. The graph of Fig. 3 is balanced but not box perfect.

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