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The explicit forms and zeros of the Bergman kernel function for Hartogs type domains ☆

Heungju Ahn [a], Jong-Do Park [b,*]

[a] *Department of Mathematics, Pohang University of Science and Technology San 31, Hyoja-dong, Namgu, Pohang, Kyungbuk, 790-784, Republic of Korea*

[b] *School of Mathematics, Korea Institute for Advanced Study, Hoegiro 85, Dongdaemun-gu, Seoul 130-722, Republic of Korea*

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Abstract

We define the Cartan–Hartogs domain, which is the Hartogs type domain constructed over the product of bounded Hermitian symmetric domains and compute the explicit form of the Bergman kernel for the Cartan–Hartogs domain using the virtual Bergman kernel. As the main contribution of this paper, we show that the main part of the explicit form of the Bergman kernel is a polynomial whose coefficients are combinations of Stirling numbers of the second kind. Using this observation, as an application, we give an algorithmic procedure to determine the condition that their Bergman kernel functions have zeros.

Keywords: Bergman kernel; Virtual Bergman kernel; Bounded symmetric domain; Hartogs domain; Routh–Hurwitz theorem; Stirling number of the second kind

1. Preliminaries

It has been a central problem in complex analysis to classify bounded domains in $\mathbb{C}^n$ up to biholomorphic equivalence. The Riemann mapping theorem tells us that each proper simply connected domain in $\mathbb{C}$ is biholomorphic to the unit disk $\{z \in \mathbb{C}\colon |z| < 1\}$. In higher dimensional

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* Corresponding author.
E-mail addresses: heungju@gmail.com (H. Ahn), jdpark@kias.re.kr, jongdopark@gmail.com (J.-D. Park).

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case, however, Poincaré gave a heuristic proof that two simply connected domains

$$\{(z_1,\dots,z_n)\in\mathbb{C}^n\colon |z_1|^2+\cdots+|z_n|^2<1\}$$

and

$$\{(z_1,\dots,z_n)\in\mathbb{C}^n\colon |z_1|<1,\dots,|z_n|<1\}$$

are not biholomorphically equivalent.

In 1920s, in order to develop the theory to deal with the classification problem, Stefan Bergman [1] introduced the Bergman kernel function $K_\Omega(z,w)$ for a domain Ω as follows: Let $L_a^2(\Omega)$ be the space of holomorphic square-integrable functions on Ω. For any $z\in\Omega$, $\Phi_z : L_a^2(\Omega)\to\mathbb{C}$ defined by $\Phi_z(f)=f(z)$ is a bounded linear functional on $L_a^2(\Omega)$. By Riesz representation theorem, there exists an element $K_z(\cdot)\in L_a^2(\Omega)$ such that $\Phi_z(f)=\langle f(\cdot),K_z(\cdot)\rangle$, namely

$$f(z)=\int\limits_\Omega f(w)\overline{K_z(w)}\,dV(w)$$

for all $f\in L_a^2(D)$. Define the *Bergman kernel function* $K_\Omega(z,w):=\overline{K_z(w)}$ for Ω. Equivalently, it can be represented as the infinite sum

$$K_\Omega(z,w)=\sum_{j=0}^{\infty}\phi_j(z)\overline{\phi_j(w)},$$

where $\{\phi_j\colon j=0,1,2,\dots\}$ is an orthonormal basis of $L_a^2(\Omega)$. It is defined for arbitrary domains, but it is hard to obtain concrete representations for the Bergman kernel except for special cases like a Hermitian ball or polydisk. The following problem arises naturally.

Problem A. Find an explicit form of the Bergman kernel for a given domain.

If Ω is a bounded symmetric domain of classical type, Hua [18] computed the explicit forms of the Bergman kernel using the holomorphic automorphism group. From the 1970s, the Bergman kernel for non-symmetric domains has been obtained explicitly. For the Thullen domain $\{(z,\zeta)\in\mathbb{C}^2\colon |z|^2+|\zeta|^{2p}<1\}$ with $p>0$ the Bergman kernel was investigated by Bergman [1], D'Angelo [4,5]. In particular, Mostow and Siu [22] used the explicit form for $p=7$ to construct a compact Kähler surface of negative sectional curvature, whose universal covering is not biholomorphic to the Hermitian unit ball in $\mathbb{C}^2$. Recently there are many interesting results about the explicit forms of the Bergman kernel for various domains. For the minimal ball, its Bergman kernel was obtained in [24]. The Bergman kernel for the complex ellipsoid was expressed in term of Appell hypergeometric functions in [12] and for $\{(z,\zeta)\in\mathbb{C}^2\colon |z|^4+|\zeta|^4<1\}$ in terms of elementary functions by the second author in [25]. In 2005, Edigarian and Zwonek [7] obtained the explicit form of the Bergman kernel for the symmetrized polydisk.

The problem for classifying a class of domains whose Bergman kernels are zero-free has been a well-known open problem in several complex variables ever since Lu Qi-Keng [20] raised the question related to the existence of Bergman representative coordinates. More explicitly, it was conjectured that if Ω is simply connected, then its Bergman kernel $K_\Omega(z,w) \neq 0$ for all $z, w \in \Omega$. To find an explicit form of the Bergman kernel may give the most efficient way to determine whether the Bergman kernel has zeros. Thus the following has been also a matter of great interest.

Problem B. Determine whether the Bergman kernel for a domain is zero-free. In such cases, a domain is called a *Lu Qi-Keng domain*.

In the complex plane, only a simply connected domain is a Lu Qi-Keng domain [28]. Since the Bergman kernel for a bounded symmetric domain is a negative power of a certain polynomial, it has no zeros anywhere. In 1996, Boas [2] proved that bounded Lu Qi-Keng domains of holomorphy in $\mathbb{C}^n$ form a nowhere dense subset of all bounded domains of holomorphy. Since then the concrete forms of non-Lu Qi-Keng domains have been found in the various classes of domains in $\mathbb{C}^n$. The minimal ball [26] with $n \geqslant 4$ and the symmetrized polydisks [23] with $n \geqslant 3$ are not Lu Qi-Keng domains. For the complex ellipsoids, see [3,35].

Let Ω be a bounded domain in $\mathbb{C}^n$ and $p : \Omega \to (0,\infty)$ a continuous function. Define *Hartogs type domains* by

$$\hat{\Omega}_m := \bigl\{(z,\zeta) \in \Omega \times \mathbb{C}^m\colon\ \|\zeta\|^2 < p(z)\bigr\}, \tag{1.1}$$

where $\|\cdot\|$ is the standard Hermitian norm on $\mathbb{C}^m$. It is well known that $\hat{\Omega}_m$ is pseudoconvex if and only if Ω is pseudoconvex and $-\log p$ is plurisubharmonic, and convex if and only if Ω is convex and p is concave. Ligocka [19] studied the properties of holomorphic functions on Hartogs type domains (1.1) to obtain the regularity of weighted Bergman projections on pseudoconvex domains. In [3] the inflation property (2.4) for (1.1) was used to construct complex ellipsoids whose Bergman kernel has zeros.

In this paper we solve Problems A and B for Hartogs type domains $\hat{\Omega}_m$ when Ω is the product of bounded symmetric domains and p is also the product of positive powers of generic norms. Following Hua [18], we list Cartan's bounded symmetric domain D of four classical types and two exceptional ones and define the corresponding generic norm $N_D(z,w)$ at $(z,w) \in D \times D$ of six types:

(1) Type I ($1 \leqslant m \leqslant n$): V is the space of $m \times n$ complex matrices.

$$D = \bigl\{z \in V\colon\ I - z\bar{z}^t > 0\bigr\}, \qquad N_D(z,w) = \det\bigl(I - z\overline{w}^t\bigr).$$

Here if the square matrix z is positive definite, then we write $z > 0$ and I is the identity matrix.

(2) Type II: V is the space of symmetric complex matrices.

$$D = \{z \in V\colon\ I - z\bar{z} > 0\}, \qquad N_D(z,w) = \det(I - z\overline{w}).$$

(3) Type III: V is the space of skew symmetric complex matrices.

$$D = \{z \in V\colon\ I + z\bar{z} > 0\}, \qquad N_D(z,w)^2 = \det(I + z\overline{w}).$$

(4) Type IV: $\mathbb{C}^n \supset D \ni z = (z_1, \dots, z_n)$ satisfies

$$1 - 2q(z, \bar{z}) + \left|q(z, z)\right| > 0, \quad q(z, \bar{z}) < 1,$$
$$N_D(z, w) = 1 - 2q(z, \overline{w}) + q(z, z)q(\overline{w}, \overline{w}),$$

where $q(z, w) = \sum_{j=1}^{n} z_j w_j$.

There are two more exceptional domains of dimension 27 and 16. We call them Type V and Type VI, respectively. For the precise definitions of these exceptional domains, see [10].

We define the *Cartan–Hartogs domain* $\hat{\Omega}_m$ as

$$\Omega = \Omega_1 \times \cdots \times \Omega_q, \qquad p(z_1, \dots, z_q) = \prod_{l=1}^{q} N_{\Omega_l}(z_l, z_l)^{\mu_l}, \quad \mu_l > 0 \tag{1.2}$$

where $z_l \in \Omega_l$, Ω_l is one of symmetric domains of six types, and $N_{\Omega_l}(z_l, z_l)$ is the corresponding generic norm of Ω_l $(1 \leqslant l \leqslant q)$. We solve Problem A in Theorem 2.5 and Problem B in Section 5.

In Section 2, we give an explicit form of the Bergman kernel for the Cartan–Hartogs domain using well-known *inflation principle* and *virtual Bergman kernel* initiated by Boas, Fu, and Straube [3] and systematically developed by Roos [27]. More precisely, we compute the Bergman kernel K_{Ω, p^k} with weights p^k $(k \in \mathbb{N})$ of Ω and the virtual Bergman kernel, which is just a summing up of $(k+m)!\, K_{\Omega, p^{k+m}} r^k / k!$'s, $k = 0, 1, 2, \dots$.

In Section 3, we introduce the Routh–Hurwitz criterion, which tells the condition that a polynomial has no zeros in the closed right half-plane. We will use this criterion to determine whether the Bergman kernel for a given Cartan–Hartogs domain has zeros or not.

Section 4 is the main part of this paper. The key part of the Bergman kernel of the Cartan–Hartogs domain, after some fractional linear transformation, has the form

$$\begin{aligned} H_m(\zeta) &= \sum_{j_1=0}^{n_1} \cdots \sum_{j_q=0}^{n_q} \prod_{l=1}^{q} c(\mu_l, j_l) \sum_{j=\max\{j_1, \dots, j_q\}}^{j_1+\cdots+j_q} d_j^{j_1, \dots, j_q} (m+1)_j \left(\zeta + \frac{1}{2}\right)^j \\ &= a_0 \zeta^n + a_1 \zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n, \end{aligned}$$

where n_j is the dimension of Ω_j and $n = n_1 + \cdots + n_q$. Moreover, we develop an efficient algorithm to represent the coefficients a_j in terms of the known numerical constants $c(\mu_l, j_l)$ and $d_j^{j_1, \dots, j_q}$ so that we tell whether the Bergman kernel function for a given Cartan–Hartogs domain has zeros or not. This representation can be done, since we have eventually found the relation between $c(\mu, j)$ and the Stirling numbers of the second kind.

In the last section we present plenty of examples that contain previously known results [6,32] and more. These examples show how efficient our algorithm developed in Section 4 is. Here, in order to decide whether the Bergman kernel function for the Cartan–Hartogs domain has zeros, we also give a quite satisfactory condition (Theorem 5.8) that suggests the relation between the dimension of the base domain Ω and the fiber dimension m. Also, when each μ_j, $j = 1, \dots, q$, in (1.2) is sufficiently large, we give a complete condition that the Bergman kernels for these Cartan–Hartogs domains have zeros. For example, the Bergman kernel function for

$$\{(z_1, z_2, \zeta) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^m\colon \|\zeta\|^2 < (1 - |z_1|^2)^{\mu_1}(1 - |z_2|^2)^{\mu_2}\}$$

has no zeros if and only if $\mu_1\mu_2 \leqslant 2$ when $m = 1$; $(\mu_1 - 1)(\mu_2 - 1) \leqslant 3$ when $m = 2$; when $m \geqslant 3$, the Bergman kernel function is zero-free for every $\mu_1, \mu_2 > 0$.

Ending this section, we review histories concerning with the explicit computation of the Bergman kernel and the zeros of the Bergman kernel. To find the explicit formula of the Bergman kernel for some particular domains has been extensively studied by Boas, Fu, and Straube [3], D'Angelo [4,5], Francsics and Hanges [12], Fujita [13,14], Gong and Zheng [16], Hua [18], Yin [30,31], Youssfi [34]. Especially, when $q = 1$ in (1.2), Roos, Lu, and Yin [33] obtained explicit formulas of the Bergman kernels for a class of nonhomogeneous domains. Also the conditions for the Bergman kernel function on some Hartogs domains to have zeros have been studied by Yin [32], Demmad-Abdessameud [6] when $q = 1$ and $n \leqslant 4$, and the second author [29] when $q = 2$. In [8,9], Engliš constructed a family of Hartogs type domains $\hat{\Omega}_m$ whose Bergman kernel has zeros when Ω is a bounded strongly convex domain and p does not satisfy the reverse Schwarz inequality $|\tilde{p}(z, w)|^2 \geqslant p(z)p(w)$, where $\tilde{p}$ is an extension of p to $\Omega \times \Omega$ with $\tilde{p}(z, z) = p(z)$. On the contrary, the product of generic norms, $p(z)$ in (1.2) of this paper satisfies the reverse Schwarz inequality [17].

2. Virtual Bergman kernel for the product of symmetric domains

In this section we first introduce the numerical invariants, the rank r and the multiplicities a and b of symmetric domains. From this we define the *Hua integral* of D,

$$\int_D N_D(z, z)^\mu \omega(z), \quad \text{for } \mu > -1,$$

where $\omega(z)$ is the standard volume form.

Define the numerical invariants r, a, b, and the genus $g = 2 + a(r - 1) + b$ of D. The list of numerical invariants of each symmetric domain of six types is the following:

(1) Type I. $r = m$, $a = 2$, $b = n - m$, $g = m + n$.
(2) Type II. $r = n$, $a = 1$, $b = 0$, $g = n + 1$.
(3) Type III. $r = [n/2]$, $a = 4$, $b = b_i$, $g = 2(n - 1)$, $i = 1, 2$, where $b_1 = 0$ if n is even and $b_2 = 4$ if n is odd.
(4) Type IV. $r = 2$, $a = n - 2$, $b = 0$, $g = n$.
(5) Type V. $r = 3$, $a = 8$, $b = 0$, $g = 18$.
(6) Type VI. $r = 2$, $a = 6$, $b = 4$, $g = 12$.

(For the details, see [10].) Using these invariants, we define the *Hua polynomial* with respect to D as

$$\chi_D(s) = \chi(s) = \prod_{j=1}^{r} \left(s + 1 + (j - 1)\frac{a}{2}\right)_{1+b+(r-j)a},$$

where $(s)_k$ denotes the Pochhammer symbol or raising factorial

$$(s)_k = s(s+1)\cdots(s+k-1) = \frac{\Gamma(s+k)}{\Gamma(s)}, \quad k \geqslant 1,$$

and as a convention, we define $(s)_0 = 1$. Note that the degree of the Hua polynomial is $r + rb + r(r-1)a/2$, which is equal to the complex dimension of D. Now, to compute the weighed Bergman kernel of a symmetric domain D, we consider the L^2 subspace with weight function $N_D(z,z)^s$ $(s > -1)$ of the space $H(D)$ of holomorphic functions in D

$$H^{(s)}(D) = \left\{ f \in H(D)\colon \int_D \left|f(z)\right|^2 N(z,z)^s \omega(z) < \infty \right\}$$

and its reproducing kernel (this is the precise notion of the weighted Bergman kernel) is given [11] by

$$K_{D,N^s}(z,w) = \frac{1}{\int_D N(z,z)^s \omega(z)} N(z,w)^{-g-s}. \tag{2.3}$$

Also the following fact on the Hua integral is known (for the details, see [18,27]).

Lemma 2.1. *Let D be a symmetric domain and N its generic norm. Then for any $s > 0$ we have*

$$\int_D N(z,z)^s \omega(z) = \frac{\chi(0)}{\chi(s)} \int_D \omega(z),$$

where χ is the Hua polynomial.

Corollary 2.2. *The reproducing kernel of $H^{(s)}(D)$ (the Bergman kernel with weight N^s) is*

$$K_{D,N^s}(z,w) = \frac{\chi(s)}{\chi(0)} \cdot \frac{K(z,w)}{N(z,w)^s},$$

where $K(z,w) = K_{D,N^0}(z,w)$ is the Bergman kernel for D.

Before computing the virtual Bergman kernel for Ω, we recall the definition of Ω in (1.2). In this section, we always denote the point $z \in \Omega$ by $z = (z_1, \ldots, z_q)$, where $z_l \in \Omega_l$, $1 \leqslant l \leqslant q$, and fix $\mu = (\mu_1, \ldots, \mu_q)$, $\mu_l > 0$, $1 \leqslant l \leqslant q$. Hereunder we introduce the notion of *virtual Bergman kernel* for (Ω, p) and give the complete formula of virtual Bergman kernel. If the Bergman kernel $\hat{K}_1$ for $\hat{\Omega}_1$ is given by

$$\hat{K}_1\big((z,\zeta),(w,\eta)\big) = L_1(z,w;\zeta\bar{\eta})$$

for $(z,\zeta),(w,\eta) \in \Omega \times \mathbb{C}$, then the Bergman kernel $\hat{K}_m$ for $\hat{\Omega}_m$ is

$$\hat{K}_m\big((z,\zeta),(w,\eta)\big) = \frac{1}{m!} \frac{\partial^{m-1}}{\partial r^{m-1}} L_1(z,w;r)\bigg|_{r=\langle\zeta,\eta\rangle}, \quad (z,\zeta),(w,\eta) \in \hat{\Omega}_m \tag{2.4}$$

(for the proof, see [3] or [19]). By virtue of this inflation principle, it is enough to compute L_1. In fact, L_1 has the following series expansion [19,27]:

$$L_1(z,w;r)=\sum_{k=0}^{\infty}(k+1)K_{\Omega,p^{k+1}}(z,w)r^k.$$

From this observation, the following concept arises naturally.

Definition 2.3. (See [27].) Let Ω be a domain and $p:\Omega\to(0,\infty)$ a continuous function on Ω. Then *the virtual Bergman kernel* $L_0(z,w;r)$ for (Ω,p) is defined by

$$L_0(z,w;r)=L_{\Omega,p}(z,w;r)=\sum_{k=0}^{\infty}K_{\Omega,p^k}(z,w)r^k.$$

Then one can see the following property of the virtual Bergman kernel,

$$\hat{K}_1\big((z,\zeta),(w,\eta)\big)=L_1(z,w;\zeta\bar{\eta}),\quad \text{with } L_1(z,w;r)=\frac{\partial}{\partial r}L_0(z,w;r). \tag{2.5}$$

By Corollary 2.2, we have the following relation between the Bergman kernel $K_{\Omega,p}$ for Ω with a weight function p and the Bergman kernel K_{Ω_l} of Ω_l,

$$K_{\Omega,p}(z,w)=\prod_{l=1}^{q}\frac{\chi_l(\mu_l)}{\chi_l(0)}\frac{K_{\Omega_l}(z_l,w_l)}{N_{\Omega_l}(z_l,w_l)^{\mu_l}},$$

where χ_l is the Hua polynomial of a symmetric domain Ω_l. Note that since $p^k=N_{\Omega_1}(z_1,z_1)^{k\mu_1}\cdots N_{\Omega_q}(z_q,z_q)^{k\mu_q}$, we have

$$K_{\Omega,p^k}(z,w)=\prod_{l=1}^{q}\frac{\chi_l(k\mu_l)}{\chi_l(0)}\frac{K_{\Omega_l}(z_l,w_l)}{N_{\Omega_l}(z_l,w_l)^{k\mu_l}}.$$

Then the virtual Bergman kernel $L_0^{(\mu)}$ for (Ω,p) has the following form,

$$\begin{aligned}L_0^{(\mu)}(z,w;r)&=\sum_{k=0}^{\infty}\prod_{l=1}^{q}\frac{\chi_l(k\mu_l)}{\chi_l(0)}\frac{K_{\Omega_l}(z_l,w_l)}{N_{\Omega_l}(z_l,w_l)^{k\mu_l}}r^k\\&=\prod_{l=1}^{q}K_{\Omega_l}(z_l,w_l)\left(\sum_{k=0}^{\infty}\prod_{l=1}^{q}\frac{\chi_l(k\mu_l)}{\chi_l(0)}\left(\frac{r}{\prod_{l=1}^{q}N_{\Omega_l}(z_l,w_l)^{\mu_l}}\right)^k\right).\end{aligned}$$

It is convenient to write

$$F(t)=\sum_{k=0}^{\infty}\prod_{l=1}^{q}\frac{\chi_l(k\mu_l)}{\chi_l(0)}t^k.$$

Now we want to represent the function $F(t)$ as a closed form. To do this, we regard the Hua polynomial $\chi_l(k\mu_l)$ of the symmetric domain Ω_l as a polynomial in k-variable. Let n_l be the degree of $\chi_l(k\mu_l)$. Note that n_l is the complex dimension of Ω_l.

Definition 2.4. Let χ_l be the Hua polynomial of a symmetric domain Ω_l.

(i) For any $\mu_l > 0$, define $c(\mu_l, j_l)$ by

$$\frac{\chi_l(k\mu_l)}{\chi_l(0)} = \sum_{j_l=0}^{n_l} c(\mu_l, j_l)(k+1)_{j_l}. \tag{2.6}$$

(ii) For any j_l with $0 \leqslant j_l \leqslant n_l$ for $l = 1, \dots, q$, define $d_j^{j_1,\dots,j_q}$ by

$$\prod_{l=1}^{q}(k+1)_{j_l} = \sum_{j=\max\{j_1,\dots,j_q\}}^{j_1+\cdots+j_q} d_j^{j_1,\dots,j_q}(k+1)_j. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned}
F(t) &= \sum_{k=0}^{\infty}\left(\sum_{j_1=0}^{n_1}\cdots\sum_{j_q=0}^{n_q}\prod_{l=1}^{q} c(\mu_l, j_l)(k+1)_{j_l}\right)t^k \\
&= \sum_{j_1=0}^{n_1}\cdots\sum_{j_q=0}^{n_q}\prod_{l=1}^{q} c(\mu_l, j_l)\left(\sum_{k=0}^{\infty}\prod_{l=1}^{q}(k+1)_{j_l}t^k\right) \\
&= \sum_{j_1=0}^{n_1}\cdots\sum_{j_q=0}^{n_q}\prod_{l=1}^{q} c(\mu_l, j_l)\left(\sum_{j=\max\{j_1,\dots,j_q\}}^{j_1+\cdots+j_q} d_j^{j_1,\dots,j_q}\left(\sum_{k=0}^{\infty}(k+1)_j t^k\right)\right) \\
&= \sum_{j_1=0}^{n_1}\cdots\sum_{j_q=0}^{n_q}\prod_{l=1}^{q} c(\mu_l, j_l)\left(\sum_{j=\max\{j_1,\dots,j_q\}}^{j_1+\cdots+j_q} d_j^{j_1,\dots,j_q}\frac{j!}{(1-t)^{j+1}}\right).
\end{aligned}$$

Thus we have simplified form of virtual Bergman kernel for (Ω, p) as following

$$L_0^{(\mu)}(z, w; r) = \prod_{l=1}^{q} K_{\Omega_l}(z_l, w_l) F\left(\frac{r}{\prod_{l=1}^{q} N_{\Omega_l}(z_l, w_l)^{\mu_l}}\right).$$

By (2.4) and (2.5), the Bergman kernel for the Hartogs domain $\hat{\Omega}_m$ is

$$\begin{aligned}
\hat{K}_m\big((z,\zeta),(w,\eta)\big) &= \frac{1}{m!}\prod_{l=1}^{q} K_{\Omega_l}(z_l, w_l)\frac{\partial^m}{\partial r^m} F\left(\frac{r}{\prod_{l=1}^{q} N_{\Omega_l}(z_l, w_l)^{\mu_l}}\right)\bigg|_{r=\langle\zeta,\eta\rangle} \\
&= \frac{1}{m!}\prod_{l=1}^{q} K_{\Omega_l}(z_l, w_l)\prod_{l=1}^{q} N_{\Omega_l}(z_l, w_l)^{-m\mu_l} F^{(m)}(t),
\end{aligned}$$

where $F^{(m)}(t)$ is evaluated at $t = \langle \zeta, \eta \rangle / \prod_{l=1}^{q} N_{\Omega_l}(z_l, w_l)^{\mu_l}$ for m times derivatives of $F(t)$. Thus we obtain the solution to Problem A as follows.

Theorem 2.5. *Let $((z,\zeta),(w,\eta)) \in \hat{\Omega}_m \times \hat{\Omega}_m$. Then the Bergman kernel for $\hat{\Omega}_m$ is given by*

$$\hat{K}_m\big((z,\zeta),(w,\eta)\big) = \frac{1}{m!}\prod_{l=1}^{q} \frac{K_{\Omega_l}(z_l, w_l)}{N_{\Omega_l}(z_l, w_l)^{m\mu_l}} F^{(m)}\left(\frac{\langle \zeta, \eta \rangle}{\prod_{l=1}^{q} N_{\Omega_l}(z_l, w_l)^{\mu_l}}\right),$$

where

$$F^{(m)}(t) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_q=0}^{n_q} \prod_{l=1}^{q} c(\mu_l, j_l) \sum_{j=\max\{j_1,\dots,j_q\}}^{j_1+\dots+j_q} d_j^{j_1,\dots,j_q} \frac{(j+m)!}{(1-t)^{j+m+1}}.$$

Example 2.6. Using Theorem 2.5, we obtain the explicit form of the Bergman kernel for $\{(z_1,z_2,\zeta) \in \mathbb{C}^3\colon |\zeta|^2 < (1-|z_1|^2)^{\mu_1}(1-|z_2|^2)^{\mu_2}\}$. In this case, Ω is a polydisk, i.e., $\Omega = \{(z_1,z_2) \in \mathbb{C}^2\colon |z_1| < 1,\ |z_2| < 1\}$ and the generic norm is given by $p(z_1,z_2) = (1-|z_1|^2)^{\mu_1}(1-|z_2|^2)^{\mu_2}$. By the straightforward computations of $F(t)$, the virtual Bergman kernel for (Ω, p) can be easily obtained:

$$L^{(\mu)}(z,w;r) = \frac{1}{\pi^2}\frac{1}{\prod_{l=1}^{2}(1-z_l\overline{w}_l)^2}\sum_{k=0}^{\infty}\prod_{l=1}^{2}(k\mu_l+1)\left(\frac{r}{\prod_{l=1}^{2}(1-z_l\overline{w}_l)^{\mu_l}}\right)^k.$$

Note that

$$\begin{aligned}\prod_{l=1}^{2}(k\mu_l+1) &= (k\mu_1+1)(k\mu_2+1)\\ &= \mu_1\mu_2(k+1)_2 + (\mu_1+\mu_2-3\mu_1\mu_2)(k+1) + (1-\mu_1)(1-\mu_2).\end{aligned}$$

Thus we have

$$F(t) = \frac{2\mu_1\mu_2}{(1-t)^3} + \frac{\mu_1+\mu_2-3\mu_1\mu_2}{(1-t)^2} + \frac{(1-\mu_1)(1-\mu_2)}{1-t}.$$

If we differentiate the virtual Bergman kernel $L_0^{(\mu)}(z,w;r)$ with respect to r, then the Bergman kernel $\hat{K}_1((z,\zeta),(w,\eta))$ for the domain $\hat{\Omega}_1$ is

$$\begin{aligned}\hat{K}_1\big((z,\zeta),(w,\eta)\big) &= \frac{6\mu_1\mu_2\prod_{l=1}^{2}(1-z_l\overline{w}_l)^{3\mu_l-2}}{\pi^2\rho^4}\\ &\quad+ \frac{2(\mu_1+\mu_2-3\mu_1\mu_2)\prod_{l=1}^{2}(1-z_l\overline{w}_l)^{2\mu_l-2}}{\pi^2\rho^3}\\ &\quad+ \frac{(1-\mu_1)(1-\mu_2)\prod_{l=1}^{2}(1-z_l\overline{w}_l)^{\mu_l-2}}{\pi^2\rho^2},\end{aligned}$$

where $\rho = \prod_{l=1}^{2}(1 - z_l\overline{w}_l)^{\mu_l} - \zeta\bar{\eta}$. In particular, if $\mu_1 = \mu_2 = 1$, then the Bergman kernel for the domain

$$\left\{(z_1, z_2, \zeta) \in \mathbb{C}^3:\ |\zeta|^2 < \left(1 - |z_1|^2\right)\left(1 - |z_2|^2\right)\right\} \tag{2.8}$$

is

$$\frac{6(1 - z_1\overline{w}_1)(1 - z_2\overline{w}_2)}{\pi^2\rho^4} - \frac{2}{\pi^2\rho^3} = \frac{4(1 - z_1\overline{w}_1)(1 - z_2\overline{w}_2) + 2\zeta\bar{\eta}}{\pi^2\{(1 - z_1\overline{w}_1)(1 - z_2\overline{w}_2) - \zeta\bar{\eta}\}^4}.$$

Furthermore, this kernel function is zero-free. Suppose that the above function is zero at (z_1, z_2, ζ), (w_1, w_2, η). Then we have

$$\begin{aligned}
|\zeta\bar{\eta}|^2 &= 4|1 - z_1\overline{w}_1|^2|1 - z_2\overline{w}_2|^2 \\
&\geqslant 4\left(1 - |z_1|^2\right)\left(1 - |w_1|^2\right)\left(1 - |z_2|^2\right)\left(1 - |w_2|^2\right) \\
&> 4|\zeta|^2|\eta|^2,
\end{aligned}$$

which is a contradiction. Thus the domain (2.8) is a Lu Qi-Keng domain.

3. Lu Qi-Keng problem

The explicit formula of the Bergman kernel function for the domain D enables us to investigate whether the Bergman kernel has zeros in $D \times D$ or not. We will call this kind of problem a *Lu Qi-Keng problem*. The motivation of this problem comes from the Riemann mapping theorem. If $n \geqslant 2$, then there is no analogue of Riemann mapping theorem in $\mathbb{C}^n$. Thus the following natural question arises: Are there canonical representatives of biholomorphic equivalence classes of domains? In higher dimensions, Bergman himself [1] introduced a representative domain to which a given domain may be mapped by representative coordinates. Let $K(z, w)$ be the Bergman kernel for a bounded domain D in $\mathbb{C}^n$, and define

$$T(z, w) = (g_{ij}) = \left(\frac{\partial^2 \log K(z, w)}{\partial z_i \partial \bar{z}_j}\right).$$

Then its converse is $T^{-1}(z, w) = (g_{ji}^{-1})$. Hence the local representative coordinate $f(z) = (f_1, \ldots, f_n)$ based at the point ζ is given by

$$f_i(z) = \sum_{j=1}^{n} g_{ji}^{-1} \frac{\partial}{\partial \overline{w}_j} \log \frac{K(z, w)}{K(w, w)}\bigg|_{w=\zeta}$$

for $i = 1, \ldots, n$. For the details, see [1,32]. This coordinates take ζ to 0 and the complex Jacobian matrix at ζ is the identity.

In 1966, Lu Qi-Keng [20] observed the following phenomenon: It is necessary that the Bergman kernel $K(z, w)$ has no zeros in order to define the Bergman representative coordinates.

Definition 3.1. If the Bergman kernel for a bounded domain does not have zeros, then the domain will be called a *Lu Qi-Keng* domain.

From (2.3), it follows that all bounded symmetric domains are Lu Qi-Keng domains. For non-symmetric domains, many Lu Qi-Keng and non-Lu Qi-Keng domains have been found [3,24,32]. Recently, Demmad-Abdessameud [6] solved this problem in $\hat{\Omega}_1$ when Ω is just a symmetric domain and the complex dimension of Ω is less than or equal to 4 ($q = 1$).

Since each symmetric domain of Ω_l is a Lu Qi-Keng domain in Theorem 2.5, the zero set of the Bergman kernel of this Hartogs domain depends only on the function $F^{(m)}(t)$. Recall that

$$F^{(m)}(t) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_q=0}^{n_q} \prod_{l=1}^{q} c(\mu_l, j_l) \sum_{j=\max\{j_1,\dots,j_q\}}^{j_1+\cdots+j_q} d_j^{j_1,\dots,j_q} \frac{(j+m)!}{(1-t)^{j+m+1}}.$$

If $((z,\zeta),(w,\eta)) \in \hat{\Omega}_m \times \hat{\Omega}_m$, then by the Cauchy–Schwartz inequality, we have

$$\frac{|\langle \zeta, \eta\rangle|}{|\prod_{l=1}^{q} N_{\Omega_l}(z_l, w_l)^{\mu_l}|} < 1.$$

Since the holomorphic function $t \mapsto 1/(1-t)$ maps the unit disk, $\{t \in \mathbb{C}\colon |t| < 1\}$ onto the set $\{t \in \mathbb{C}\colon \operatorname{Re} t > 1/2\}$, we have the following consequence of Theorem 2.5. Let $G_m(t) := (1-t)^{m+1} F^{(m)}(t)/m!$ and

$$\begin{aligned} H_m(\zeta) &:= G_m\left(1 - \frac{1}{\zeta + 1/2}\right) \\ &= \sum_{j_1=0}^{n_1} \cdots \sum_{j_q=0}^{n_q} \prod_{l=1}^{q} c(\mu_l, j_l) \sum_{j=\max\{j_1,\dots,j_q\}}^{j_1+\cdots+j_q} d_j^{j_1,\dots,j_q} (m+1)_j \left(\zeta + \frac{1}{2}\right)^j. \end{aligned} \tag{3.9}$$

Lemma 3.2. *For any positive integer m, the domain $\hat{\Omega}_m$ is a Lu Qi-Keng domain if and only if all zeros of the polynomial $H_m(\zeta)$ lie in the closed left half-plane $\{\zeta \in \mathbb{C}\colon \operatorname{Re}\zeta \leqslant 0\}$.*

The Routh–Hurwitz criterion [15] is the most efficient method for determining whether the polynomial $H_m(\zeta)$ has zeros in the open left half-plane. Let $f(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n$ with real coefficients and $a_0 > 0$ and define Δ_j^n for $j = 1, \dots, n$ by

$$\Delta_j^n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2j-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2j-2} \\ 0 & a_1 & a_3 & \cdots & a_{2j-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_j \end{vmatrix},$$

where $a_j = 0$ if $j < 0$ or $j > n$.

Lemma 3.3 *(Routh–Hurwitz/Liénard–Chipart). (See [15].) All zeros of given polynomial $f(\zeta)$ lie in the open left half-plane $\{\zeta \in \mathbb{C}\colon \operatorname{Re}\zeta < 0\}$ if and only if*

$$\Delta_1^n > 0, \quad \ldots, \quad \Delta_n^n > 0.$$

This condition is also equivalent to any one of the following four forms:

(i) $a_n > 0, a_{n-2} > 0, a_{n-4} > 0, \ldots; \Delta_1^n > 0, \Delta_3^n > 0, \ldots,$
(ii) $a_n > 0, a_{n-2} > 0, a_{n-4} > 0, \ldots; \Delta_2^n > 0, \Delta_4^n > 0, \ldots,$
(iii) $a_n > 0; a_{n-1} > 0, a_{n-3} > 0, \ldots; \Delta_1^n > 0, \Delta_3^n > 0, \ldots,$
(iv) $a_n > 0; a_{n-1} > 0, a_{n-3} > 0, \ldots; \Delta_2^n > 0, \Delta_4^n > 0, \ldots.$

Remark 3.4. In Lemma 3.2, we need a criterion for closed left half-plane. From Lemma 3.3, we might guess that all zeros of given polynomial $f(\zeta)$ lie in the closed left half-plane if and only if

$$\Delta_1^n \geqslant 0, \quad \ldots, \quad \Delta_n^n \geqslant 0. \tag{3.10}$$

But, (3.10) does not imply that all zeros of $f(\zeta)$ lie in the closed left half-plane. For example, consider

$$f(\zeta) = (\zeta + a)\big(\zeta^2 + p\big) = \zeta^3 + a\zeta^2 + p\zeta + ap.$$

Then $\Delta_1^3 = a$, $\Delta_2^3 = \Delta_3^3 = 0$. If $p < 0$ and $a \geqslant 0$, then $\Delta_j^3 \geqslant 0$ for $j = 1, 2, 3$. But, $f(\zeta)$ has a zero $\sqrt{-p}$ in the open right half-plane.

It follows from Lemmas 3.2 and 3.3 that

Proposition 3.5. *Let Δ_j^n be constants constructed from $H_m(\zeta)$ for $\hat{\Omega}_m$ in* (3.9)*. If $\Delta_1^n > 0$, $\ldots, \Delta_n^n > 0$, then $\hat{\Omega}_m$ is a Lu Qi-Keng domain.*

But the converse is not true. In singular cases (some Δ_j^n vanish), (3.10) cannot be a necessary and sufficient condition that $\hat{\Omega}_m$ is a Lu Qi-Keng domain. In Section 5, the converse of Proposition 3.5 is obtained for the cases when $n = 2$ and $n = 3$.

Now we need to compute the coefficients of $H_m(\zeta)$. At first, define the coefficients A_j for $1 \leqslant j \leqslant n$ by

$$A_j = \sum_{|\alpha|=0}^{j} \prod_{l=1}^{q} c(\mu_l, n_l - \alpha_l) d_{n-j}^{n_1-\alpha_1,\ldots,n_q-\alpha_q} (m+1)_{n-j}, \tag{3.11}$$

and

$$a_j = \sum_{k=0}^{j} A_{j-k} \binom{n-j+k}{k} \left(\frac{1}{2}\right)^k, \tag{3.12}$$

where $\alpha = (\alpha_1, \ldots, \alpha_q)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_q$. With these notations, we can rewrite (3.9) with

$$\begin{aligned} H_m(\zeta) &= A_0\left(\zeta + \frac{1}{2}\right)^n + A_1\left(\zeta + \frac{1}{2}\right)^{n-1} + \cdots + A_{n-1}\left(\zeta + \frac{1}{2}\right) + A_n \\ &= a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n, \end{aligned}$$

where $n = n_1 + \cdots + n_q$.

Next, for the time being, we deal with an irreducible symmetric domain ($q = 1$ in (1.2)). Let D be one of symmetric domains of six types in Section 1 and n its dimension. Then from Definition 2.4 the Hua polynomial $\chi = \chi_D$ of D is decomposed as

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{n} c(\mu, j)(k+1)_j. \tag{3.13}$$

To investigate the more obvious information on the coefficients $c(\mu, j)$ in (3.13), we introduce the following.

Definition 3.6. For a fixed j, define $c_\nu^j(D)$ with $\nu = j, j+1, \ldots$ by the relation

$$c(\mu, j) = \frac{1}{\chi(0)} \sum_{\nu=j}^{n} c_\nu^j(D)\mu^\nu = \frac{\mu^n}{\chi(0)} \sum_{\nu=j}^{n} \frac{c_\nu^j(D)}{\mu^{n-\nu}} = c(\mu, n) \sum_{\nu=j}^{n} \frac{c_\nu^j(D)}{\mu^{n-\nu}}.$$

The above coefficients $c_\nu^j(D)$'s are well defined, since $c(\mu, j)$ is a polynomial in μ of degree n. Since $c(\mu, j)$ depends on the numerical invariants of the domain, $c_\nu^i(D)$ also does. But $c_n^j(D)$ depends only on the dimension of the domain, not the other invariants a, b, r defined in Section 2.

In Section 4, it will turn out that $c_n^{n-j}(D)$ is a polynomial in n of degree $2j$ (for details, see Remark 4.3(ii)). More precisely, for fixed n, we may write this polynomial in a variable t as

$$c_n^{n-j}[t] = \sum_{\alpha=0}^{2j} \tilde{a}_\alpha^j t^\alpha. \tag{3.14}$$

Here the coefficient $\tilde{a}_\alpha^j$ varies only when the dimension of D does (there are several different types of symmetric domains with the same dimension). Note that $c_n^j[n] = c_n^j(D)$ and in general, $c_\beta^j(D) \neq c_n^j[\beta]$. In the next section, for a fixed n, without changing the coefficients $\tilde{a}_\alpha^j$ and varying the variable t, we will show that $c_\beta^j(D)$, $j \leqslant \beta < n$ can be represented in terms of a numerical constant $T_{n-\beta}(D)$ and $c_n^{n-\beta+j}[\beta]$, and again $c_n^{n-\beta+j}[n]$ is, up to constant, the *Stirling number of the second kind* (for the definitions of $T_{n-\beta}(D)$ and the Stirling number of the second kind, see Definition 4.1 and the definition stated before Proposition 4.8).

4. The explicit formulas of coefficients in $H_m(\zeta)$

In this section we develop an algorithm to obtain the explicit form of A_j, which enables us to calculate an explicit form of a_j by (3.12). We introduce the following concepts which are frequently used in the theory of combinatorics.

Definition 4.1. Let p be a nonnegative integer and D one of symmetric domains as in Section 1.

- (i) Let $E_p(x_1,\dots,x_n)$ be the sum of all possible products of p elements chosen from $\{x_1,\dots,x_n\}$ without replacement where order doesn't matter. Here $x_1,\dots,x_n$ may not be distinct. As a convention, let $E_0(x_1,\dots,x_n)=1$.
- (ii) $S_p(j_1;\dots;j_q) := E_p(1,2,\dots,j_1,\dots,1,2,\dots,j_q)$.
- (iii) $T_p(D) := E_p(y_1,\dots,y_n)$, where y_i is an element of $\{k+(j-1)\frac{a}{2}\colon k=1,2,\dots,1+b+(r-j)a,\ j=1,2,\dots,r\}$. Here a, b, r are numerical invariants with respect to D.

The following lemma shows the interplay between $c_n^i[\beta]$ and $c_\nu^j(D)$.

Lemma 4.2. *Let D be one of symmetric domains of six types.*

- (i) $k^{n-l}T_l(D)=\sum_{j=0}^n c_{n-l}^j(D)(k+1)_j$.
- (ii) *For any nonnegative integers α, β with $\alpha\geqslant\beta$, we have*

$$c_{n-\beta}^{n-\alpha}(D)=\begin{cases}-\sum_{p=1}^{\alpha-\beta}S_p(n-\alpha+p)c_{n-\beta}^{n-\alpha+p}(D), & \text{if } \alpha\geqslant\beta+1,\\ T_\alpha(D), & \text{if } \alpha=\beta.\end{cases}$$

- (iii) *For $0\leqslant\beta\leqslant\alpha$, we have $c_{n-\beta}^{n-\alpha}(D)=T_\beta(D)c_n^{n-\alpha+\beta}[n-\beta]$.*

Remark 4.3. (i) If $l=0$, then $k^n=\sum_{j=0}^n c_n^j(D)(k+1)_j$, which means that $c_n^j(D)$ depends only on the dimension n of D. However, if $l\geqslant 1$, then $c_{n-l}^j(D)$ depends on the other numerical invariants of D.

(ii) By this iterative formula, we see that $c_n^{n-i}(D)$ is a polynomial in n of degree $2i$. Moreover, the explicit formula of $S_p(n)=E_p(1,2,\dots,n)$ can be obtained by

$$S_p(n)=\sum_{1\leqslant i_1<\cdots<i_p\leqslant n} i_1\cdots i_p=\sum_{i_p=1}^{n}\sum_{i_{p-1}=1}^{i_p-1}\cdots\sum_{i_1=1}^{i_2-1} i_1\cdots i_p. \tag{4.15}$$

Proof. From the definition of Hua polynomial and (3.13), we have

$$\prod_{j=1}^r\left(k\mu+1+(j-1)\frac{a}{2}\right)_{1+b+(r-j)a}=\sum_{j=0}^n\left(\sum_{\nu=j}^n c_\nu^j(D)\mu^\nu\right)(k+1)_j.$$

If the coefficients of μ^{n-l} and k^{n-l} of the one side of the above equality are compared to the other, then (i) and (ii) are easily seen, respectively. For (iii), we use the induction on β. If $\beta=\alpha$, then $c_{n-\alpha}^{n-\alpha}(D) = T_\alpha(D)$. Assume that it is true for $\beta = \alpha - \gamma$ with $\gamma = 0, 1, \ldots, k-1$. If we use (i) and (ii) successively, by the induction hypothesis, we see that

$$\begin{aligned}
c_{n-\alpha+k}^{n-\alpha}(D) &= -\sum_{p=1}^{k} S_p(n-\alpha+p) c_{n-\alpha+k}^{n-\alpha+p}(D) \\
&= -S_k(n-\alpha+k) c_{n-\alpha+k}^{n-\alpha+k}(D) - \sum_{p=1}^{k-1} S_p(n-\alpha+p) c_{n-\alpha+k}^{n-\alpha+p}(D) \\
&= -S_k(n-\alpha+k) T_{\alpha-k}(D) \\
&\quad - \sum_{p=1}^{k-1} S_p(n-\alpha+p) T_{\alpha-k}(D) c_n^{n-k+p}[n-\alpha+k] \\
&= T_{\alpha-k}(D) \left\{ -\sum_{p=1}^{k} S_p(n-\alpha+p) c_n^{n-k+p}[n-\alpha+k] \right\} \\
&= T_{\alpha-k}(D) \left\{ -\sum_{p=1}^{k} S_p(n-k+p) c_n^{n-k+p}(D) \right\} [n-\alpha+k] \\
&= T_{\alpha-k}(D) c_n^{n-k}[n-\alpha+k].
\end{aligned} \tag{4.16}$$

In (4.16), $\{\cdots\}$ is a polynomial in n. Here by $\{\cdots\}[n-\alpha+k]$, we mean that one evaluates the polynomial $\{\cdots\}$ at $(n-\alpha+k)$. $\square$

We transform A_j and a_j into another form, which enables one to easily calculate them by successive iteration. Let $\Omega = \Omega_1 \times \cdots \times \Omega_q$, where Ω_j is an irreducible symmetric domain defined in Section 1. First of all, we calculate this when $q = 1$. Recall

$$\prod_{l=1}^{q} (k+1)_{j_l} = \sum_{j=\max\{j_1,\ldots,j_q\}}^{j_1+\cdots+j_q} d_j^{j_1,\ldots,j_q} (k+1)_j. \tag{4.17}$$

Note that

$$d_j^{j_1,\ldots,j_q} = 1 \quad \text{for } j = j_1 + \cdots + j_q \tag{4.18}$$

and moreover, if $q = 1$, then

$$d_j^i = \delta_j^i, \tag{4.19}$$

where δ_j^i is the Kronecker delta. Hence, by (3.11), the definition of $c_\nu^j(\Omega)$, and Lemma 4.2(iii), we have

$$\begin{aligned}A_j &= \sum_{\alpha=0}^{j} c(\mu, n-\alpha) d_{n-j}^{n-\alpha}(m+1)_{n-j} = c(\mu, n-j)(m+1)_{n-j} \\ &= c(\mu, n)(m+1)_{n-j} \sum_{\beta=0}^{j} \frac{c_{n-\beta}^{n-j}(\Omega)}{\mu^\beta} \\ &= c(\mu, n)(m+1)_{n-j} \sum_{\beta=0}^{j} \frac{T_\beta(\Omega)}{\mu^\beta} c_n^{n-j+\beta}[n-\beta].\end{aligned}$$

If we rewrite the last summation, by (3.12), we have

$$\begin{aligned}a_j &= \sum_{k=0}^{j} A_{j-k} \binom{n-j+k}{k} \left(\frac{1}{2}\right)^k \\ &= c(\mu, n) \sum_{k=0}^{j} (m+1)_{n-j+k} \sum_{\beta=0}^{j-k} \frac{T_\beta(\Omega)}{\mu^\beta} c_n^{n-j+k+\beta}[n-\beta] \binom{n-j+k}{k} \left(\frac{1}{2}\right)^k \\ &= c(\mu, n)(m+1)_{n-j} \\ &\quad \times \sum_{\beta=0}^{j} \frac{T_\beta(\Omega)}{\mu^\beta} \sum_{k=0}^{j-\beta} (m+n-j+1)_k \binom{n-j+k}{k} \left(\frac{1}{2}\right)^k c_n^{n-j+k+\beta}[n-\beta].\end{aligned}$$

For the systematic calculation, define $B(t, s)$ by

$$B(t, s) = \sum_{k=0}^{s} (m+t-s+1)_k \binom{t-s+k}{k} \left(\frac{1}{2}\right)^k c_n^{n-s+k}[t]. \tag{4.20}$$

Since $c(\mu, n) = \mu^n / \chi(0)$, by (4.20), we have

$$a_j = \frac{\mu^n}{\chi(0)} (m+1)_{n-j} \sum_{\beta=0}^{k} \frac{T_\beta(\Omega)}{\mu^\beta} B(n-\beta, j-\beta). \tag{4.21}$$

Next, we transform a_j and A_j into another simpler form when $\Omega = \Omega_1 \times \cdots \times \Omega_q$. Since we deal with the product of symmetric domains, we have to extend the above procedure to multi-variables. To do this, we need preliminary lemmas that are interesting also from the viewpoint of combinatorics. Both sides of (4.17) are polynomials in k. If we compare the coefficients of $k^{j_1+\cdots+j_q-l}$ of the left-hand side in (4.17) with the right-hand side, we obtain

Lemma 4.4. *For* $1 \leqslant l \leqslant j_1 + \cdots + j_q$, *we have*

$$d_{j_1+\cdots+j_q-l}^{j_1,\ldots,j_q} = S_l(j_1; \ldots; j_q) - \sum_{p=1}^{l} d_{j_1+\cdots+j_q-l+p}^{j_1,\ldots,j_q} S_p(j_1 + \cdots + j_q - l + p).$$

On the other hand, when $q=1$, we note that by Lemma 4.2(ii), if $\beta=0$, then

$$\sum_{p=0}^{\alpha} S_{\alpha-p}(n-p)c_n^{n-p}(\Omega)=\sum_{\alpha=0}^{l} S_{l-\alpha}(n-\alpha)c_n^{n-\alpha}(\Omega)=0.$$

Now we prove a multivariate version of the above identity. For the simplification of notation, we write $c_{n_l}^i=c_{n_l}^i(\Omega_l)$, $1\leqslant l\leqslant q$ and $c_n^i=c_n^i(D)$, where D is arbitrary symmetric domain of dimension $n=n_1+\cdots+n_q$. Note that these are well defined and make no confusion, since these are dependent only on the dimension.

Lemma 4.5. *For each j with $1\leqslant j\leqslant n$, we have*

$$\sum_{|\alpha|=0}^{j} S_{j-|\alpha|}(n_1-\alpha_1;\ldots;n_q-\alpha_q)c_{n_1}^{n_1-\alpha_1}\cdots c_{n_q}^{n_q-\alpha_q}=0.$$

Proof. Consider the decomposition

$$\prod_{l=1}^{q}\frac{\chi_l(k\mu_l)}{\chi_l(0)}=\prod_{l=1}^{q}\left(\sum_{j_l=0}^{n_l}c(\mu_l,j_l)(k+1)_{j_l}\right).$$

If we compare the coefficients of k^{n-j} of one side in the above identity with those of the other side, we get

$$\begin{aligned}
&\frac{1}{\prod_{l=1}^{q}\chi_l(0)}\sum_{|\beta|=n-j}\mu_1^{\beta_1}\cdots\mu_q^{\beta_q}T_{n_1-\beta_1}(\Omega_1)\cdots T_{n_q-\beta_q}(\Omega_q)\\
&\quad=\sum_{|\alpha|\leqslant j}\prod_{l=1}^{q}c(\mu_l,n_l-\alpha_l)S_{j-|\alpha|}(n_1-\alpha_1;\ldots;n_q-\alpha_q)\\
&\quad=\sum_{|\alpha|\leqslant j}\frac{1}{\prod_{l=1}^{q}\chi_l(0)}\prod_{l=1}^{q}\left(\sum_{\nu=n_l-\alpha_l}^{n_l}c_\nu^{n_l-\alpha_l}\mu_i^\nu\right)S_{j-|\alpha|}(n_1-\alpha_1;\ldots;n_q-\alpha_q).
\end{aligned}$$

Again, comparing the coefficients of $\prod_{l=1}^{q}\mu_l^{n_l}$ with the other side, we obtain

$$\sum_{|\alpha|\leqslant j}\left(\prod_{l=1}^{q}c_{n_l}^{n_l-\alpha_l}\right)S_{j-|\alpha|}(n_1-\alpha_1;\ldots;n_q-\alpha_q)=0. \qquad \square$$

Using the above two lemmas, we prove the following relation which plays a crucial role in computing the explicit forms of A_j and a_j.

Lemma 4.6. *For each j with $0 \leqslant j \leqslant n$, we have*

$$\sum_{|\alpha|=0}^{j} d_{n-j}^{n_1-\alpha_1,\dots,n_q-\alpha_q} c_{n_1}^{n_1-\alpha_1} \cdots c_{n_q}^{n_q-\alpha_q} = c_n^{n-j}. \tag{4.22}$$

Proof. By (4.18) and Lemma 4.2(i), we see that $d_n^{n_1,\dots,n_q} = 1$, $c_{n_l}^{n_l} = 1$, and $c_n^n = 1$. Hence if $j = 0$, then (4.22) is clear. Now assume that $j \geqslant 1$. We will show this by induction on j. First, using Lemmas 4.4 and 4.5, we can rewrite the left-hand side of (4.22) as

$$\begin{aligned}
&\sum_{|\alpha|=0}^{j} d_{n-j}^{n_1-\alpha_1,\dots,n_q-\alpha_q} c_{n_1}^{n_1-\alpha_1} \cdots c_{n_q}^{n_q-\alpha_q} \\
&\quad = \sum_{|\alpha|=0}^{j} \Biggl(S_{j-|\alpha|}(n_1-\alpha_1; \dots; n_q-\alpha_q) \\
&\qquad - \sum_{p=1}^{j-|\alpha|} d_{n-j+p}^{n_1-\alpha_1,\dots,n_q-\alpha_q} S_p(n-j+p) \Biggr) c_{n_1}^{n_1-\alpha_1} \cdots c_{n_q}^{n_q-\alpha_q} \\
&\quad = -\sum_{|\alpha|=0}^{j} \sum_{p=1}^{j-|\alpha|} d_{n-j+p}^{n_1-\alpha_1,\dots,n_q-\alpha_q} S_p(n-j+p) c_{n_1}^{n_1-\alpha_1} \cdots c_{n_q}^{n_q-\alpha_q}.
\end{aligned}$$

For the time being, we denote the summation in the last equality of the above formula by $F(j)$. It is enough to show that $F(j) = c_n^{n-j}$. If $j = 1$, by Lemma 4.2(ii), then

$$F(1) = -d_n^{n_1,\dots,n_q} S_1(n) c_{n_1}^{n_1} \cdots c_{n_q}^{n_q} = -S_1(n) = c_n^{n-1}.$$

Assume that $F(\nu) = c_n^{n-\nu}$ for $\nu \leqslant j$. Then, by Lemma 4.2(ii) and the induction hypothesis, we have

$$\begin{aligned}
F(j+1) &= -\sum_{|\alpha|=0}^{j+1} \sum_{p=1}^{j+1-|\alpha|} d_{n-j-1+p}^{n_1-\alpha_1,\dots,n_q-\alpha_q} S_p(n-j-1+p) c_{n_1}^{n_1-\alpha_1} \cdots c_{n_q}^{n_q-\alpha_q} \\
&= -\sum_{p=1}^{j+1} S_p(n-j-1+p) \sum_{|\alpha|=0}^{j+1-p} d_{n-j-1+p}^{n_1-\alpha_1,\dots,n_q-\alpha_q} c_{n_1}^{n_1-\alpha_1} \cdots c_{n_q}^{n_q-\alpha_q} \\
&= -\sum_{p=1}^{j+1} S_p(n-j-1+p) c_n^{n-j-1+p} \\
&= c_n^{n-j-1}. \qquad \square
\end{aligned}$$

Now we want to formulate a multivariate version of (4.21). By (3.11), Definition 3.6, and Lemma 4.2(ii), we have

$$\begin{aligned}
A_j &= P(m+1)_{n-j} \sum_{|\alpha|=0}^{j} d_{n-j}^{n_1-\alpha_1,\dots,n_q-\alpha_q} \sum_{|\beta|=0}^{j} \prod_{l=1}^{q} \frac{c_{n_l-\beta_l}^{n_l-\alpha_l}(\Omega_l)}{\mu_l^{\beta_l}} \\
&= P(m+1)_{n-j} \sum_{|\beta|=0}^{j} \frac{1}{\prod_{l=1}^{q} \mu_l^{\beta_l}} \sum_{|\alpha|=0}^{j} d_{n-j}^{n_1-\alpha_1,\dots,n_q-\alpha_q} \prod_{l=1}^{q} c_{n_l-\beta_l}^{n_l-\alpha_l}(\Omega_l) \\
&= P(m+1)_{n-j} \sum_{|\beta|=0}^{j} \prod_{l=1}^{q} \frac{T_{\beta_l}(\Omega_l)}{\mu_l^{\beta_l}} \sum_{|\alpha|=0}^{j} d_{n-j}^{n_1-\alpha_1,\dots,n_q-\alpha_q} \prod_{l=1}^{q} c_{n_l}^{n_l-\alpha_l+\beta_l}[n_l-\beta_l],
\end{aligned}$$

where $P = \prod_{l=1}^{q} c(\mu_l, n_l) = \prod_{l=1}^{q} (\mu_l^{n_l}/\chi_l(0))$. Again, using (4.22), we can simplify the summation term of multi-variables in the last line of the above formula with a single term of one variable. More precisely, by (4.22),

$$\begin{aligned}
&\sum_{|\alpha|=0}^{j} d_{n-j}^{n_1-\alpha_1,\dots,n_q-\alpha_q} \prod_{l=1}^{q} c_{n_l}^{n_l-\alpha_l+\beta_l}[n_l-\beta_l] \\
&\quad= \left(\sum_{|\alpha|=0}^{j} d_{n-(j-|\beta|)}^{n_1+\beta_1-\alpha_1,\dots,n_q+\beta_q-\alpha_q} \prod_{l=1}^{q} c_{n_l}^{n_l-\alpha_l+\beta_l} \right)[n_1-\beta_1,\dots,n_q-\beta_q] \\
&\quad= \left(\sum_{|\alpha|=0}^{j-|\beta|} d_{n-(j-|\beta|)}^{n_1-\alpha_1,\dots,n_q-\alpha_q} \prod_{l=1}^{q} c_{n_l}^{n_l-\alpha_l} \right)[n_1-\beta_1,\dots,n_q-\beta_q] \\
&\quad= c_n^{n-j+|\beta|}\big[n-|\beta|\big].
\end{aligned}$$

Here, by $(\cdots)[n_1-\beta_1,\dots,n_q-\beta_q]$, we mean that one evaluates the polynomial $(\cdots)$ in multi-variable $(n_1,\dots,n_q)$ at $(n_1-\beta_1,\dots,n_q-\beta_q)$. The last equality can be justified, since $c_n^{n-j+|\beta|}$ is a polynomial in $n = n_1+\cdots+n_q$. This observation forces one to write $c_n^{n-j+|\beta|}[t_1,\dots,t_q] = c_n^{n-j+|\beta|}[|t|]$. Thus we have

$$A_j = P(m+1)_{n-j} \sum_{|\beta|=0}^{j} \prod_{l=1}^{q} \frac{T_{\beta_l}(\Omega_l)}{\mu_l^{\beta_l}} c_n^{n-j+|\beta|}\big[n-|\beta|\big].$$

Theorem 4.7. *The polynomial of* (3.9)*,*

$$H_m(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n$$

has the following form

$$a_j = P(m+1)_{n-j} \left(\sum_{|\beta|=0}^{j} \prod_{l=1}^{q} \frac{T_{\beta_l}(\Omega_l)}{\mu_l^{\beta_l}} \right) B\big(n-|\beta|, j-|\beta|\big),$$

where

$$P = \prod_{l=1}^{q} \frac{\mu_l^{n_l}}{\chi_l(0)}, \qquad B(t,s) = \sum_{k=0}^{s} (m+t-s+1)_k \binom{t-s+k}{k} \left(\frac{1}{2}\right)^k c_n^{n-s+k}[t].$$

Proof. Recall

$$a_j = \sum_{k=0}^{j} A_{j-k} \binom{n-j+k}{k} \left(\frac{1}{2}\right)^k.$$

Plugging A_j in the formula of a_j and using the definition of $B(t,s)$, one obtains the conclusion. □

If one tries to calculate a_j using Theorem 4.7, one has to know how to obtain c_n^{n-l}, $l = 0, 1, \dots, n$. This can be done through the *Stirling number of the second kind*. For $k \geqslant l$, the number of partitions of $\{1, 2, \dots, k\}$ into l blocks is denoted $S(k,l)$ and called the *Stirling number of the second kind*. Then we have the following relation between c_n^j and $S(k,l)$.

Proposition 4.8. *For $l = 0, 1, \dots, n$, we have*

$$c_n^{n-l} = (-1)^l S(n+1, n+1-l). \tag{4.23}$$

For the proof of this proposition, we need a lemma. We omit the proof of the following lemma, which is well known in the theory of combinatorics (see Section 2.1 in [21]).

Lemma 4.9. *The Stirling number of the second kind has the following properties.*

(i) *For any positive integer p,*

$$x^p = \sum_{r=1}^{p} S(p,r)(x-r+1)_r.$$

(ii) *For any r with $0 \leqslant r \leqslant p$, we have*

$$\begin{aligned} S(p,r) &= \frac{1}{r!} \sum_{j=1}^{r} (-1)^{r+j} \binom{r}{j} j^p \\ &= \frac{1}{r!} \sum_{s_1+\dots+s_r=p} \frac{p!}{s_1! \cdots s_r!} =: \frac{1}{r!} \sum_{s_1+\dots+s_r=p} \binom{p}{s_1, \dots, s_r}. \end{aligned}$$

Proof of Proposition 4.8. In Lemma 4.2(i), if $l = 0$, then we have

$$k^n = \sum_{j=0}^{n} c_n^j (k+1)_j. \tag{4.24}$$

By Lemma 4.9(i), we have

$$k^{n+1} = \sum_{j=1}^{n+1} S(n+1, j)(k-j+1)_j. \tag{4.25}$$

Since $(k-j+1)_j = (k-j+1)_{j-1} \cdot k$ for $j \geqslant 1$, we can rewrite (4.25) as

$$k^n = \sum_{j=0}^{n} S(n+1, j+1)(k-j)_j.$$

It follows that

$$(-k)^n = \sum_{j=0}^{n} S(n+1, j+1)(-k-j)_j = \sum_{j=0}^{n} S(n+1, j+1)(-1)^j (k+1)_j,$$

so that

$$k^n = \sum_{j=0}^{n} (-1)^{n-j} S(n+1, j+1)(k+1)_j. \tag{4.26}$$

Then we get (4.23) from (4.24) and (4.26). □

Combining Lemma 3.3 and Theorem 4.7, theoretically, we can find the conditions for $(\mu_1, \ldots, \mu_q)$ that $\hat{\Omega}_m$ is a Lu Qi-Keng domain. In the next section, we will give several examples which show how our method works efficiently to tell whether a given $\hat{\Omega}_m$ is a Lu Qi-Keng domain or not.

5. Examples

Recall that the Cartan–Hartogs domain $\hat{\Omega}_m = \{(z, \zeta) \in \Omega \times \mathbb{C}^m \colon \|\zeta\|^2 < p(z)\}$ is defined by

$$\Omega = \Omega_1 \times \cdots \times \Omega_q, \qquad p(z_1, \ldots, z_q) = \prod_{l=1}^{q} N_{\Omega_l}(z_l, z_l)^{\mu_l}, \quad \mu_l > 0.$$

Here Ω_j's are irreducible symmetric domains as in Section 1. Let n be the dimension of Ω. In this section, we obtain the sufficient and necessary condition that $\hat{\Omega}_m$ is a Lu Qi-Keng domain when $n \leqslant 3$. For higher dimensional cases, we only investigate the asymptotic condition, that is, for sufficiently large μ_j, $j = 1, \ldots, q$, that $\hat{\Omega}_m$ is a Lu Qi-Keng domain. Finally, using some polynomials in m, we present a necessary condition for $\hat{\Omega}_m$ to be a Lu Qi-Keng domain.

To accomplish these, first of all, we need a lot of numerical data on $S(k, l)$, c_n^j, and $B(n, j)$ in turn. Using Lemma 4.9(ii), we have

$$\begin{aligned}
S(k,k) &= 1,\\
S(k,k-1) &= \frac{1}{2}k(k-1),\\
S(k,k-2) &= \frac{1}{24}k(k-1)(k-2)(3k-5),\\
S(k,k-3) &= \frac{1}{48}k(k-1)(k-2)^2(k-3)^2,\\
S(k,k-4) &= \frac{1}{5760}k(k-1)(k-2)(k-3)(k-4)\bigl(15k^3-150k^2+485k-502\bigr),\\
S(k,k-5) &= \frac{1}{11\,520}k(k-1)(k-2)(k-3)(k-4)^2(k-5)^2\bigl(3k^2-23k+38\bigr).
\end{aligned}$$

For example, to compute $S(k,k-3)$, we use Lemma 4.9(ii):

$$\begin{aligned}
S(k,k-3) &= \frac{1}{(k-3)!}\sum_{j_1+\cdots+j_{k-3}=k}\binom{k}{j_1,\ldots,j_{k-3}}\\
&= \frac{1}{(k-3)!}\left\{\binom{k}{1,1,\ldots,4}\binom{k-3}{1}+2\binom{k}{1,1,\ldots,1,2,3}\binom{k-3}{2}\right.\\
&\quad\left.+\binom{k}{1,1,\ldots,1,2,2,2}\binom{k-3}{3}\right\}\\
&= \frac{1}{48}k(k-1)(k-2)^2(k-3)^2.
\end{aligned}$$

Again, using $c_n^{n-l}=(-1)^l S(n+1,n-l+1)$, one directly gets

$$\begin{aligned}
c_n^n &= 1,\\
c_n^{n-1} &= -\frac{1}{2}n(n+1),\\
c_n^{n-2} &= \frac{1}{24}n(n-1)(n+1)(3n-2),\\
c_n^{n-3} &= -\frac{1}{48}n(n+1)(n-2)^2(n-1)^2,\\
c_n^{n-4} &= \frac{1}{5760}(n+1)n(n-1)(n-2)(n-3)\bigl(15n^3-105n^2+230n-152\bigr),\\
c_n^{n-5} &= -\frac{1}{11\,520}(n+1)n(n-1)(n-2)(n-3)^2(n-4)^2\bigl(3n^2-17n+18\bigr).
\end{aligned}$$

The above formulas and (4.20) directly give

$$\begin{aligned}
B(n,0) &= 1,\\
B(n,1) &= \frac{1}{2}n(m-1),
\end{aligned}$$

$$\begin{aligned}
B(n,2) &= \frac{1}{24}n(n-1)\bigl(3m^2-9m+4-2n\bigr),\\
B(n,3) &= \frac{1}{48}n(n-1)(n-2)(m-1)\bigl(m^2-5m+4-2n\bigr),\\
B(n,4) &= \frac{1}{5760}n(n-1)(n-2)(n-3)\bigl(15m^4-150m^3+465m^2\\
&\quad -570m+180nm+20n^2-132n-60m^2n+208\bigr),\\
B(n,5) &= \frac{1}{11\,520}n(n-1)(n-2)(n-3)(n-4)(m-1)\bigl(3m^4-42m^3\\
&\quad +193m^2-362m+20n^2-20m^2n+100nm-132n+208\bigr).
\end{aligned} \tag{5.27}$$

Combining (5.27) and Lemma 3.3, we have the following characterization of Lu Qi-Keng domains.

At first, a_1 is always positive for any cases.

Lemma 5.1. *For any positive integers m and n, it holds that $a_1 > 0$.*

Proof. By Theorem 4.7, we have

$$a_1 = p(m+1)_{n-1}\left\{B_1(n) + \sum_{l=1}^{q} \frac{T_1(\Omega_l)}{\mu_l}\right\} > 0,$$

since $B_1(n) = \frac{1}{2}n(m-1) \geqslant 0$ for $m \geqslant 1$ and $T_1(\Omega_l) > 0$. □

5.1. Case $n = 2$

Let $H_m(\zeta) = a_0\zeta^2 + a_1\zeta + a_2$ in (3.9). By Proposition 3.5 and Lemma 3.3, if $a_2 > 0$ and $a_1 > 0$, then $\hat{\Omega}_m$ is a Lu Qi-Keng domain. Note that a_1 is always positive by Lemma 5.1. If $a_2 < 0$ (and a_1 is always positive), then $H_m(\zeta) = 0$ has a positive real root, so $\hat{\Omega}_m$ is not a Lu Qi-Keng domain. If $a_2 = 0$ (and a_1 is always positive), then zeros of $H_m(\zeta)$ are 0 and $-a_1/a_0 < 0$. Therefore, $\hat{\Omega}_m$ is a Lu Qi-Keng domain if and only if $a_2 \geqslant 0$.

Theorem 5.2 *($n = 2$). Let m be a positive integer.*

(i) $\{(z,\zeta) \in \mathbb{C}^2 \times \mathbb{C}^m\colon \|\zeta\|^2 < (1-\|z\|^2)^{\mu}\}$ *is a Lu Qi-Keng domain if and only if*

$$8 + 6(m-1)\mu + m(m-3)\mu^2 \geqslant 0.$$

(ii) $\{(z_1,z_2,\zeta) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^m\colon \|\zeta\|^2 < (1-|z_1|^2)^{\mu_1}(1-|z_2|^2)^{\mu_2}\}$ *is a Lu Qi-Keng domain if and only if*

$$4 + 2(m-1)(\mu_1+\mu_2) + m(m-3)\mu_1\mu_2 \geqslant 0.$$

5.2. Case $n = 3$

Let $H_m(\zeta) = a_0\zeta^3 + a_1\zeta^2 + a_2\zeta + a_3$ in (3.9). By Proposition 3.5 and Lemma 3.3, if $a_3 > 0$, $a_1 > 0$, $\Delta_2^3 > 0$, then $\hat{\Omega}_m$ is a Lu Qi-Keng domain. Note also that $a_1 > 0$ by Lemma 5.1. Assume that

$$a_3 \geqslant 0 \quad \text{implies} \quad \Delta_2^3 > 0, \tag{5.28}$$

which will be proved for each case in Theorem 5.3. If $a_3 < 0$, then similarly as in the case when $n = 2$, $\hat{\Omega}_m$ is not a Lu Qi-Keng domain. If $a_3 = 0$, then by Lemma 5.1 and (5.28), $H_m(\zeta) = \zeta(a_0\zeta^2 + a_1\zeta + a_2)$ has two zeros in an open left half-plane except for the origin. Therefore, $\hat{\Omega}_m$ is a Lu Qi-Keng domain if and only if $a_3 \geqslant 0$.

Theorem 5.3 *($n = 3$). Let m be a positive integer.*

(i) $\{(z, \zeta) \in \mathbb{C}^3 \times \mathbb{C}^m\colon \|\zeta\|^2 < (1 - \|z\|^2)^{\mu}\}$ *is a Lu Qi-Keng domain if and only if*

$$48 + 44(m-1)\mu + 12m(m-3)\mu^2 + (m-1)\big(m^2 - 5m - 2\big)\mu^3 \geqslant 0.$$

(ii) $\{(z_1, z_2, \zeta) \in \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^m\colon \|\zeta\|^2 < (1 - |z_1|^2)^{\mu_1}(1 - \|z_2\|^2)^{\mu_2}\}$ *is a Lu Qi-Keng domain if and only if*

$$\begin{aligned} &16 + 4(m-1)(2\mu_1 + 3\mu_2) \\ &\quad + 2m(m-3)\big(\mu_2^2 + 3\mu_1\mu_2\big) + (m-1)\big(m^2 - 5m - 2\big)\mu_1\mu_2^2 \geqslant 0. \end{aligned}$$

(iii) $\{(z_1, z_2, z_3, \zeta) \in \mathbb{C}^3 \times \mathbb{C}^m\colon \|\zeta\|^2 < (1 - |z_1|^2)^{\mu_1}(1 - |z_2|^2)^{\mu_2}(1 - |z_3|^2)^{\mu_3}\}$ *is a Lu Qi-Keng domain if and only if*

$$1 + \frac{1}{2}(m-1)r_1 + \frac{1}{4}m(m-3)r_2 + \frac{1}{8}(m-1)\big(m^2 - 5m - 2\big)r_3 \geqslant 0,$$

where

$$r_1 = \mu_1 + \mu_2 + \mu_3, \qquad r_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \qquad r_3 = \mu_1\mu_2\mu_3.$$

Proof. Here we prove (iii) only. (i) and (ii) can be proved similarly. In this case, $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3$, where each Ω_l is the unit disk $\{z \in \mathbb{C}\colon |z| < 1\}$. In fact, this is the Type I for $m = n = 1$ defined in Section 1. Note that the numerical invariants of this type are $a = 2$, $b = 0$, $r = 1$, and $g = 2$, respectively. Hence by Definition 4.1, we have

$$T_p(\Omega_l) = E_p(1) = \begin{cases} 1, & \text{if } p = 0 \text{ or } 1; \\ 0, & \text{if } p \geqslant 2. \end{cases} \tag{5.29}$$

By Theorem 4.7, $H_m(\zeta) = a_0\zeta^3 + a_1\zeta^2 + a_2\zeta + a_3$, where

$$a_j = (m+1)_{3-j} \sum_{|\beta|=0}^{j} T_{\beta_l}(\Omega_l)\mu_l^{1-\beta_l} B\big(3 - |\beta|, j - |\beta|\big). \tag{5.30}$$

Combining (5.27), (5.29), and (5.30), we have

$$\begin{aligned}
a_0 &= (m+1)_3 r_3,\\
a_1 &= \frac{1}{2}(m+1)_2\bigl\{3(m-1)r_3 + 2r_2\bigr\},\\
a_2 &= \frac{1}{4}(m+1)\bigl\{4r_1 + 4(m-1)r_2 + \bigl(3m^2 - 9m - 2\bigr)r_3\bigr\},\\
a_3 &= 1 + \frac{1}{2}(m-1)r_1 + \frac{1}{4}m(m-3)r_2 + \frac{1}{8}(m-1)\bigl(m^2 - 5m - 2\bigr)r_3,
\end{aligned}$$

where

$$r_1 = \mu_1 + \mu_2 + \mu_3, \qquad r_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \qquad r_3 = \mu_1\mu_2\mu_3.$$

By direct calculations, we obtain

$$\begin{aligned}
\Delta_2^3 = (m+1)_2\bigl\{&2mr_3 + m(m-1)\bigl(m^2 - 2m - 2\bigr)r_3^2 + (m-1)(m+1)r_2^2\\
&+ m(m-1)r_1r_3 + (m+1)(r_1r_2 - 3r_3) + \bigl(2m^3 - 3m^2 - 2m + 1\bigr)r_2r_3\bigr\}.
\end{aligned}$$

Now we will prove (5.28). Since $\Delta_2^3 > 0$ for $m \geqslant 3$, it is enough to prove (5.28) for $m = 1, 2$. If $m = 1$, then we have

$$a_3 = 1 - \frac{1}{2}r_2 \quad \text{and} \quad \Delta_2^3 = 12(r_1r_2 - 2r_3 - r_2r_3).$$

If $a_3 = 0$, then $r_2 = 2$, so that $\mu_3 = \frac{2-\mu_1\mu_2}{\mu_1\mu_2}$. It follows that

$$\Delta_2^3 = 24(r_1 - 2r_3) = \frac{24}{\mu_1 + \mu_2}\bigl\{(\mu_1 - \mu_2)^2 + 2(\mu_1\mu_2)^2 - \mu_1\mu_2 + 2\bigr\} > 0.$$

For some small $0 < \mu_1, \mu_2 < 1$, $a_3(\mu_1, \mu_2) = 0$ always implies $\Delta_2^3(\mu_1, \mu_2) > 0$. For example, if $r_2 = 1$, then $a_3 = 1/2 > 0$ and so it is easily checked that $\Delta_2^3 = 12(r_1 - 3r_3) > 0$ in a similar way. Thus (5.28) is true for $m = 1$.

Next, if $m = 2$, then

$$\begin{aligned}
a_3 &= 1 + \frac{1}{2}r_1 - \frac{1}{2}r_2 - r_3,\\
\Delta_2^3 &= 12\bigl(3r_1r_2 + r_2r_3 + 2r_3r_1 - 5r_3 + 3r_2^2 - 4r_3^2\bigr).
\end{aligned}$$

Assuming $a_3 = 0$, so that $r_1 = r_2 + 2r_3 - 2$, we have $\Delta_2^3 = 36(r_2 - 1)(2r_2 + 3r_3)$. Since $r_2 \geqslant 3\sqrt[3]{r_3^2}$, we have $r_2^{3/2} \geqslant 3\sqrt{3}r_3 > \frac{3}{2}\sqrt{3}(2 - r_2)$ and $2r_2^{3/2} + 3\sqrt{3}r_2 > 6\sqrt{3}$. Suppose that $r_2 \leqslant 1$, then $2{r_2}^{3/2} + 3\sqrt{3}r_2 \leqslant 2 + 3\sqrt{3} < 6\sqrt{3}$, which leads to a contradiction. It follows that $r_2 > 1$, so that $\Delta_2^3 > 0$. Similarly as the case when $m = 1$, this shows that (5.28) holds also for $m = 2$. $\square$

5.3. Case $n \geqslant 4$

Next, we consider the case that $n = 4, 5$, where n is the dimension of Ω. Even though we have some condition that $\hat{\Omega}_m$ is a Lu Qi-Keng domain, practically, it is not useful, since that condition involves too many terms. To overcome this complexity, we assume that the exponent μ_j, $j = 1, \ldots, q$, is sufficiently large. We define the leading terms of a_j and Δ_j^k by

$$\tilde{a}_j = P(m+1)_{n-j} B(n, j)$$

and

$$\tilde{\Delta}_j^k = \begin{vmatrix} \tilde{a}_1 & \tilde{a}_3 & \tilde{a}_5 & \cdots & \tilde{a}_{2j-1} \\ \tilde{a}_0 & \tilde{a}_2 & \tilde{a}_4 & \cdots & \tilde{a}_{2j-2} \\ 0 & \tilde{a}_1 & \tilde{a}_3 & \cdots & \tilde{a}_{2j-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{a}_j \end{vmatrix},$$

where $\tilde{a}_j = 0$ if $j < 0$ or $j > k$ and $P = \prod_{l=1}^{q}(\mu_l^{n_l}/\chi_l(0))$. The other terms of a_j, not $\tilde{a}_j$ are negligible, that is, if μ_j is sufficiently large and the fiber dimension m is less than certain number, say m_0, then $a_j = \tilde{a}_j + \epsilon(\mu)$. Here we can make $\epsilon(\mu)$ as small as possible by choosing μ_j sufficiently large. Thus we have

Proposition 5.4. *Let $n = 4$ or 5 and m_0 given. If $\mu_1, \ldots, \mu_q$ are sufficiently large and $m \leqslant m_0$, then we have*

(i) *$\tilde{a}_j > 0$ if and only if $a_j > 0$, for each $j = 1, \ldots, n$,*
(ii) *$\tilde{\Delta}_j^n > 0$ if and only if $\Delta_j^n > 0$, for each $j = 1, \ldots, n$.*

Remark 5.5. Proposition 5.4 holds for any fixed n if we choose μ_j sufficiently large. Here we restricted the dimension, since we have numerical data of $B(n, j)$'s only for $n \leqslant 5$ in the preceding part.

Example 5.6. Let m_0 be a given large number. Choose $\mu_1, \ldots, \mu_q$ sufficiently large so that Proposition 5.4 holds.

(1) Let $n = 4$. By Lemma 3.3(i) and Proposition 5.4, $\hat{\Omega}_m$ is a Lu Qi-Keng domain if

$$\tilde{a}_1 > 0, \qquad \tilde{a}_4 > 0, \qquad \tilde{a}_2 > 0, \qquad \tilde{\Delta}_3^4 > 0.$$

For the simplification of notation, we omit the coefficient P in the definitions $\tilde{a}_j$ and $\tilde{\Delta}_j^n$. In fact, this does not change anything, since we need only the positiveness of $\tilde{a}_j$ and $\tilde{\Delta}_j^n$. Moreover, without the coefficient P, we abuse the notations of $\tilde{a}_j$ and $\tilde{\Delta}_j^n$. By (5.27), we have

$$\tilde{a}_0 = (m+1)_4,$$
$$\tilde{a}_2 = \frac{1}{2}(m+1)(m+2)\left(3m^2 - 9m - 4\right),$$

$$\tilde{a}_4 = \frac{1}{16} m\big(m^3 - 10m^2 + 15m + 10\big),$$
$$\tilde{\Delta}_3^4 = m(m+2)(m+3)(m-1)^2(m+1)^2\big(m^5 - 5m^4 - 4m^3 + 20m^2 + 15m + 5\big).$$

It follows that $\tilde{a}_4, \tilde{a}_2, \tilde{a}_1, \tilde{\Delta}_3^4 > 0$ for $m \geqslant 8$. In fact, $\tilde{a}_4 > 0$ implies $\tilde{a}_1 > 0$, $\tilde{a}_2 > 0$, $\tilde{\Delta}_3^4 > 0$. So, for sufficiently large $\mu_1, \dots, \mu_q$ and any $m \geqslant 8$, $\hat{\Omega}_m$ is a Lu Qi-Keng domain.

(2) Let $n = 5$. Similarly as $n = 4$, by Lemma 3.3(ii) and Proposition 5.4, the condition for $\hat{\Omega}_m$ to be a Lu Qi-Keng domain is characterized by

$$\tilde{a}_1 > 0, \qquad \tilde{a}_3 > 0, \qquad \tilde{a}_5 > 0, \qquad \tilde{\Delta}_2^5 > 0, \qquad \tilde{\Delta}_4^5 > 0.$$

Then we have

$$\tilde{a}_1 = \frac{5}{2}(m+1)(m+2)(m+3)(m+4)(m-1),$$
$$\tilde{a}_3 = \frac{5}{4}(m+1)^2(m+2)(m-1)(m-6),$$
$$\tilde{a}_5 = \frac{1}{32}(m-1)\big(m^4 - 14m^3 + 31m^2 + 46m + 16\big)$$

and

$$\tilde{\Delta}_2^5 = 5m(m-1)(m+4)(m+3)\big(m^2 - 6\big)(m+2)^2(m+1)^2,$$
$$\tilde{\Delta}_4^5 = m^2(m+4)(m+3)\big(m^{10} - 11m^9 + 14m^8 + 134m^7 - 201m^6 - 523m^5 + 236m^4 + 936m^3 + 741m^2 + 351m + 72\big)(-1+m)^2(m+2)^2(m+1)^2.$$

Note that $\tilde{a}_5 > 0$, that is, $m \geqslant 11$ implies the positiveness of $\tilde{a}_1$, $\tilde{a}_3$, $\tilde{\Delta}_2^5$, $\tilde{\Delta}_4^5$. So, for sufficiently large $\mu_1, \dots, \mu_q$ and any $m \geqslant 11$, $\hat{\Omega}_m$ is a Lu Qi-Keng domain.

Finally we give a necessary condition for $\hat{\Omega}_m$ to be a Lu Qi-Keng domain. By Routh–Hurwitz criterion (Lemma 3.3), if $\hat{\Omega}_m$ is a Lu Qi-Keng domain, then $a_n > 0$. Note that

$$a_n = \prod_{l=1}^{q} \frac{\mu_l^{n_l}}{\chi_l(0)} \left(\sum_{|\beta|=0}^{n} \prod_{l=1}^{q} \frac{T_{\beta_l}(\Omega_l)}{\mu_l^{\beta_l}} \right) B\big(n - |\beta|, n - |\beta|\big).$$

Now we investigate the formula of $B(t,t)$ more closely. By Proposition 4.8, we have

$$\begin{aligned} B(t,t) &= \sum_{k=0}^{t} (m+1)_k \left(\frac{1}{2}\right)^k c_n^k[t] \\ &= \sum_{k=0}^{t} (m+1)_k \left(\frac{1}{2}\right)^k (-1)^{t-k} S(t+1, k+1) \\ &= (-1)^t \sum_{k=0}^{t} (m+1)_k \left(-\frac{1}{2}\right)^k S(t+1, k+1). \end{aligned}$$

Denote $B(t)[m] := B(t,t)$ and for any $l \leqslant t$, define $p_{l,k}(m)$ by

$$B(t)[m] = (-1)^t \sum_{k=l}^{t} (m+1)_{k-l} \left(-\frac{1}{2}\right)^k p_{l,k}(m) S(t+1-l, k+1-l).$$

Since $B(t)[m]$ is a polynomial in m, we see that $p_{l,k}(m)$ is uniquely defined. Actually, $p_{l,k}(m)$ is a polynomial in m of degree l as we will see.

Lemma 5.7. *For $k \geqslant m \geqslant 2$, we have $S(k+1,m) = S(k,m-1) + mS(k,m)$.*

Proof. See Theorem 2.1.20 in [21]. $\square$

It follows from the above lemma that

$$\begin{aligned}
\frac{B(t)[m]}{(-1)^t} &= S(t+1,1) + (m+1)_t \left(-\frac{1}{2}\right)^t S(t+1,t+1) \\
&\quad + \sum_{k=1}^{t-1} (m+1)_k \left(-\frac{1}{2}\right)^k \bigl\{S(t,k) + (k+1)S(t,k+1)\bigr\} \\
&= S(t,1) + (m+1)_t \left(-\frac{1}{2}\right)^t S(t,t) \\
&\quad + \sum_{k=1}^{t-1} (m+1)_k \left(-\frac{1}{2}\right)^k S(t,k) + \sum_{k=2}^{t} (m+1)_{k-1} \left(-\frac{1}{2}\right)^{k-1} kS(t,k) \\
&= \sum_{k=1}^{t} \left\{ (m+1)_k \left(-\frac{1}{2}\right)^k + (m+1)_{k-1} \left(-\frac{1}{2}\right)^{k-1} k \right\} S(t,k) \\
&= \sum_{k=1}^{t} (m+1)_{k-1} \left(-\frac{1}{2}\right)^k \bigl\{(m+k) - 2k\bigr\} S(t,k).
\end{aligned}$$

Thus $p_{0,k}(m) = 1$ and $p_{1,k}(m) = (m+k) - 2k$. In this way, we obtain the recurrence relation for $p_{l,k}(m)$:

$$\begin{aligned}
p_{0,k}(m) &= 1, \\
p_{l+1,k}(m) &= (m+k-l)p_{l,k}(m) - 2(k-l)p_{l,k-1}(m).
\end{aligned} \tag{5.31}$$

Therefore if $l = t$, then we have

$$B(t)[m] = \left(\frac{1}{2}\right)^t p_{t,t}(m).$$

Theorem 5.8. *If $\hat{\Omega}_m$ is Lu Qi-Keng domain for all $(\mu_1, \dots, \mu_q)$, then $p_{t,t}(m) > 0$ for all $t = 0, 1, \dots, n$.*

Example 5.9. The recurrence relation (5.31) gives

$$\begin{aligned}
p_{1,1}(m) &= m-1,\\
p_{2,2}(m) &= m(m-3),\\
p_{3,3}(m) &= (m-1)\big(m^2-5m-2\big),\\
p_{4,4}(m) &= m\big(m^3-10m^2+15m+10\big),\\
p_{5,5}(m) &= (m-1)\big(m^4-14m^3+31m^2+46m+16\big),\\
p_{6,6}(m) &= m\big(m^5-21m^4+105m^3-35m^2-210m-112\big).
\end{aligned}$$

If we combine Example 5.6 and Theorem 5.8, it seems that if the dimension of the base domain Ω is greater than or equal to 4, then

$$p_{t,t}(m) > 0 \quad \text{for } m \geqslant 8+3(t-4) \text{ and } t \geqslant 4$$

and the positiveness of $p_{t,t}(m)$ is also sufficient condition for a Cartan–Hartogs domain to be a Lu Qi-Keng. Also, this insight has been confirmed as we have checked the positiveness of $p_{t,t}(m)$ for many large t with the help of a computer program (Maple or Mathematica).

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References

[1] S. Bergman, The Kernel Function and Conformal Mapping, revised ed., Math. Surveys, vol. V, American Mathematical Society, Providence, RI, 1970.
[2] H.P. Boas, The Lu Qi-Keng conjecture fails generically, Proc. Amer. Math. Soc. 124 (7) (1996) 2021–2027.
[3] H.P. Boas, S. Fu, E.J. Straube, The Bergman kernel function: explicit formulas and zeroes, Proc. Amer. Math. Soc. 127 (3) (1999) 805–811.
[4] J.P. D'Angelo, A note on the Bergman kernel, Duke Math. J. 45 (2) (1978) 259–265.
[5] J.P. D'Angelo, An explicit computation of the Bergman kernel function, J. Geom. Anal. 4 (1) (1994) 23–34.
[6] F.Z. Demmad-Abdessameud, Polynômes de Hua, noyau de Bergman des domaines de Cartan–Hartogs et problème de Lu Qikeng (Hua polynomial, Bergman kernel of Cartan–Hartogs domains and the Lu Qi-Keng problem), Rend. Semin. Mat. Univ. Politec. Torino 67 (1) (2009) 55–89 (in French, French summary).
[7] A. Edigarian, W. Zwonek, Geometry of the symmetrized polydisc, Arch. Math. (Basel) 84 (4) (2005) 364–374.
[8] M. Engliš, Zeroes of the Bergman kernel of Hartogs domains, Comment. Math. Univ. Carolin. 41 (1) (2000) 199–202 (English summary).
[9] M. Engliš, A Forelli–Rudin construction and asymptotics of weighted Bergman kernels, J. Funct. Anal. 177 (2) (2000) 257–281.
[10] J. Faraut, S. Kaneyuki, A. Korányi, Q.-K. Lu, G. Roos, Jordan triple system, in: Analysis and Geometry on Complex Homogeneous Domains, in: Progr. Math., vol. 185, Birkhäuser Boston Inc., Boston, MA, 2000.
[11] J. Faraut, S. Kaneyuki, A. Korányi, Q.-K. Lu, G. Roos, A. Koranyi, Function spaces and reproducing kernels on bounded symmetric domains, J. Funct. Anal. 88 (1) (1990) 64–89.
[12] G. Francsics, N. Hanges, The Bergman kernel of complex ovals and multivariable hypergeometric functions, J. Funct. Anal. 142 (2) (1996) 494–510.
[13] K. Fujita, Bergman transformation for analytic functionals on some balls, in: Microlocal Analysis and Complex Fourier Analysis, World Sci. Publ., River Edge, NJ, 2002, pp. 81–98.

[14] K. Fujita, Bergman kernel for the two-dimensional balls, Complex Var. Theory Appl. 49 (3) (2004) 215–225.
[15] F.R. Gantmacher, The Theory of Matrices, vol. 1, Translated from the Russian by K.A. Hirsch, Reprint of the 1959 translation, AMS Chelsea Publishing, Providence, RI, 1998.
[16] S. Gong, X.A. Zheng, The Bergman kernel function of some Reinhardt domains, Trans. Amer. Math. Soc. 348 (5) (1996) 1771–1803.
[17] L.K. Hua, Inequalities involving determinants, Acta Math. Sinica 5 (1955) 463–470 (in Chinese); Translated into English: Transl. Amer. Math. Soc. Der. II 32 (1963) 265–272.
[18] L.K. Hua, Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains, Translated from the Russian by Leo Ebner and Adam Korányi, American Mathematical Society, Providence, RI, 1963.
[19] E. Ligocka, Forelli–Rudin constructions and weighted Bergman projections, Studia Math. 94 (1989) 257–272.
[20] Q.-K. Lu, On Kähler manifolds with constant curvature, Chinese Math.–Acta 8 (1966) 283–298.
[21] R. Merris, Combinatorics, second ed., Wiley–Interscience (John Wiley & Sons), New York, 2003.
[22] G.D. Mostow, Y.T. Siu, A compact Kähler surface of negative curvature not covered by the ball, Ann. of Math. (2) 112 (2) (1980) 321–360.
[23] N. Nikolov, W. Zwonek, The Bergman kernel of the symmetrized polydisc in higher dimensions has zeros, Arch. Math. (Basel) 87 (5) (2006) 412–416.
[24] K. Oeljeklaus, P. Pflug, E.H. Youssfi, The Bergman kernel of the minimal ball and applications, Ann. Inst. Fourier (Grenoble) 47 (3) (1997) 915–928.
[25] J.-D. Park, New formulas of the Bergman kernels for complex ellipsoids in $\mathbb{C}^2$, Proc. Amer. Math. Soc. 136 (12) (2008) 4211–4221.
[26] P. Pflug, E.H. Youssfi, The Lu Qi-Keng conjecture fails for strongly convex algebraic domains, Arch. Math. (Basel) 71 (3) (1998) 240–245.
[27] G. Roos, Weighted Bergman kernels and virtual Bergman kernels, Sci. China Ser. A 48 (suppl.) (2005) 225–237.
[28] N. Suita, A. Yamada, On the Lu Qi-keng conjecture, Proc. Amer. Math. Soc. 59 (2) (1976) 222–224.
[29] A. Wang, L. Zhang, J. Bai, W. Zhang, J.-D. Park, Zeros of Bergman kernels on some Hartogs domains, Sci. China Ser. A 52 (12) (2009) 2730–2742.
[30] W. Yin, Two problems on Cartan domains, J. China Univ. Sci. Tech. 16 (2) (1986) 130–146.
[31] W. Yin, The Bergman kernels on Cartan–Hartogs domains, Chinese Sci. Bull. 44 (21) (1999) 1947–1951.
[32] W. Yin, Lu Qi-Keng conjecture and Hua domain, preprint, arXiv:math/0605428v2, 2006.
[33] W. Yin, K. Lu, G. Roos, New classes of domains with explicit Bergman kernel, Sci. China Ser. A 47 (3) (2004) 352–371.
[34] E.H. Youssfi, Proper holomorphic liftings and new formulas for the Bergman and Szegő kernels, Studia Math. 152 (2) (2002) 161–186.
[35] L. Zhang, W. Yin, Lu Qi-Keng's problem on some complex ellipsoids, J. Math. Anal. Appl. 357 (2) (2009) 364–370.