Cardinal functions of Pixley–Roy hyperspaces

Masami Sakai 1

Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan

Abstract
Let \( F[X] \) be the Pixley–Roy hyperspace of a regular space \( X \), and let \( F_n[X] = \{ F \in F[X] : |F| \leq n \} \). For tightness \( t \) and supertightness \( st \), we show the following equalities:

1. \( t(F[X]) = \sup \{ st(X^n) : n \in \mathbb{N} \} \),
2. \( \sup \{ t(F_n[X]) : n \in \mathbb{N} \} = \sup \{ t(X^n) : n \in \mathbb{N} \} \).

The first equality answers a question posed in Sakai (1983) [18]. The inequality \( \sup \{ t(X^n) : n \in \mathbb{N} \} \leq \sup \{ st(X^n) : n \in \mathbb{N} \} \) is strict, indeed there is a space \( Z \) such that \( \sup \{ t(X^n) : n \in \mathbb{N} \} < \sup \{ st(X^n) : n \in \mathbb{N} \} \). The discrete countable chain condition and weak Lindelöf property of \( F[X] \) are also investigated.

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1. Introduction

All spaces are assumed to be regular. The symbol \( \mathbb{N} \) is the set of all positive integers. Unexplained notions and terminology are the same as in [7].

For a space \( X \), let \( F[X] \) be the space of all nonempty finite subsets of \( X \) with the Pixley–Roy topology [14]: for \( A \in F[X] \) and an open set \( U \subseteq X \), let

\[
[A, U] = \{ B \in F[X] : A \subseteq B \subseteq U \};
\]

the family \([A, U] : A \in F[X], U \text{ open in } X\) is a base for the Pixley–Roy topology. It is known that for a \( T_1 \)-space \( X \), \( F[X] \) is always zero-dimensional, completely regular and every subspace of \( F[X] \) is metacompact: see van Douwen [6]. For each \( n \in \mathbb{N} \), we put \( F_n[X] = \{ F \in F[X] : |F| \leq n \} \). Each \( F_n[X] \) is closed in \( F[X] \), and each \( F_n[X] \setminus F_{n-1}[X] \) is a discrete space.

The following facts are used in the next section.

Lemma 1.1. ([15, Proposition 1.2]) Let \( Y \) be a subspace of a space \( X \). Then \( F[Y] \) is homeomorphic to the closed subspace \( \{ A \in F[X] : A \subseteq Y \} \) of \( F[X] \).

Lemma 1.2. ([11, Theorem 2.8]) For spaces \( X_1, \ldots, X_k \), \( F[X_1] \times \cdots \times F[X_k] \) can be embedded as a closed subspace of \( F[X_1 \times \cdots \times X_k] \).

E-mail address: sakaim01@kanagawa-u.ac.jp.

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2. The tightness of $\mathcal{F}[X]$

**Definition 2.1.** For a space $X$ and a point $x \in X$, let $t(x, X)$ be the smallest cardinal number $\kappa \geq \omega$ with the property that if $A \subset X$ and $x \in A \setminus \overline{A}$, then there is a subset $B \subset A$ such that $x \in B$ and $|B| \leq \kappa$. The cardinal number $t(X) = \sup\{t(x, X) : x \in X\}$ is called the tightness of $X$.

Concerning cardinal functions of Pixley–Roy hyperspaces, the following question was posed in [18, Question 2], where $\Psi(X)$ (resp., $\psi_\Delta(X)$) is the closed pseudocharacter (resp., the diagonal degree) of a space $X$.

**Question 2.2.** Determine exactly $t, \Psi$ and $\psi_\Delta$ on $\mathcal{F}[X]$ in terms of those on $X$.

Answering this question, Tanaka [20] gave the equalities $\Psi(\mathcal{F}[X]) = \psi_\Delta(\mathcal{F}[X]) = \psi(X)$. In this section, we answer the case of $t(\mathcal{F}[X])$.

For a space $X$ and a point $x \in X$, a family $\mathcal{P}$ of nonempty subsets of $X$ is said to be a $\pi$-network at $x$ if every neighborhood of $x$ contains some member of $\mathcal{P}$.

**Definition 2.3.** ([13]) For a space $X$ and a point $x \in X$, let $st(x, X)$ be the smallest cardinal number $\kappa \geq \omega$ with the property that if $\mathcal{P}$ is a $\pi$-network at $x$ consisting of finite subsets of $X^n$, then there is a subfamily $\mathcal{Q} \subset \mathcal{P}$ such that $\mathcal{Q}$ is a $\pi$-network at $x$ and $|\mathcal{Q}| \leq \kappa$. The cardinal number $st(x, X) = \sup\{st(x, X) : x \in X\}$ is called the supertightness of $X$.

The supertightness of a space $X$ was denoted by $p(X)$ in [13]. Obviously $t(X) \leq st(X)$ holds. There is a supercompact Fréchet–Urysohn space $Z$ with $st(Z) = 2^\omega$ [13, Example 2.6].

**Theorem 2.4.** For a space $X$, the equality $t(\mathcal{F}[X]) = \sup\{st(X^n) : n \in \mathbb{N}\}$ holds.

**Proof.** Assume $t(\mathcal{F}[X]) = \kappa$, and fix an $n \in \mathbb{N}$ and a point $x = (x_1, \ldots, x_n) \in X^n$. We show $st(x, X^n) \leq \kappa$. Let $\mathcal{P}$ be a $\pi$-network at $x$ consisting of finite subsets of $X^n$. We take an open neighborhood $U_1$ of $x_1$ such that $U_i = U_j$ if $x_i = x_j$, and $U_1 \cap U_j = \emptyset$ if $x_1 \neq x_j$. Let $A = \{x_1, \ldots, x_n\}$ and $U = U_1 \cup \cdots \cup U_n$. Let

$$\mathcal{D} = \{F \in [A, U] : \text{there is a member } P \in \mathcal{P} \text{ with } P \subset (U_1 \times \cdots \times U_n) \cap F^n\}.$$

We observe $A \in \mathcal{D}$. Take any basic open neighborhood $[A, V]$ of $A$. Since $\mathcal{P}$ is an open neighborhood of $x$, there is a member $P \in \mathcal{P}$ with $P \subset (U_1 \times \cdots \times U_n) \cap (V \cap F^n) \neq \emptyset$. Let

$$F = A \cup p_1(P) \cup \cdots \cup p_n(P),$$

where $p_i$ is the projection of $X^n$ to the $i$-th coordinate. Obviously $F \in [A, V] \cap [A, U]$. Since $F^n$ contains $P$, $P \subset (U_1 \times \cdots \times U_n) \cap F^n$, thus $F \in [A, V] \cap \mathcal{D}$. Since $t(\mathcal{F}[X]) = \kappa$, there is a subfamily $\{F_\alpha : \alpha < \kappa\} \subset \mathcal{D}$ such that $A \in \{F_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, take a member $P_\alpha \in \mathcal{P}$ such that $P_\alpha \subset (U_1 \times \cdots \times U_n) \cap (F_\alpha \cap F^n)$. We observe that $\{P_\alpha : \alpha < \kappa\}$ is a $\pi$-network at $x$. Let $W_1 \times \cdots \times W_n$ be an open neighborhood of $x$, where $W_i$ is an open neighborhood of $x_i$ such that $W_i \subset U_i$, and $W_i = W_j$ if $x_i = x_j$. Take some $\alpha < \kappa$ with $F_\alpha \in [A, W_1 \cup \cdots \cup W_n]$. Then we have

$$P_\alpha \subset (U_1 \times \cdots \times U_n) \cap (F_\alpha \cap F^n) \subset (U_1 \times \cdots \times U_n) \cap (W_1 \cup \cdots \cup W_n)^n = W_1 \times \cdots \times W_n.$$

Thus $st(x, X^n) \leq \kappa$.

Conversely assume $\sup\{st(X^n) : n \in \mathbb{N}\} = \kappa$. We show $t(\mathcal{F}[X]) \leq \kappa$. Let $A = \{x_1, \ldots, x_n\} \in \mathcal{F}[X]$ and assume $A \in \overline{A} \setminus \overline{A}$ for $A \subset \mathcal{F}[X]$. Take an open neighborhood $U_1$ of $x_1$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Since $[A, U_1 \cup \cdots \cup U_n]$ is an open neighborhood of $A$, we may assume $A \subset [A, U_1 \cup \cdots \cup U_n]$. Let

$$\mathcal{P} = \{(U_1 \cap B) \times \cdots \times (U_n \cap B) : B \in \mathcal{A}\}.$$

Obviously each member of $\mathcal{P}$ is nonempty and finite. We observe that $\mathcal{P}$ is a $\pi$-network at the point $x = (x_1, \ldots, x_n) \in X^n$. Let $W_1 \times \cdots \times W_n$ be an open neighborhood of $x$, where $W_i \subset U_i (1 \leq i \leq n)$. Take a point $B \in [A, W_1 \cup \cdots \cup W_n] \cap \mathcal{A}$. Then

$$(U_1 \cap B) \times \cdots \times (U_n \cap B) = (W_1 \cap B) \times \cdots \times (W_n \cap B) \subset W_1 \times \cdots \times W_n.$$

Thus $\mathcal{P}$ is a $\pi$-network at $x$. By $st(x, X^n) \leq \kappa$, there is a subfamily $\mathcal{B}_\alpha : \alpha < \kappa \subset \mathcal{A}$ such that $\{(U_1 \cap B_\alpha) \times \cdots \times (U_n \cap B_\alpha) : \alpha < \kappa\}$ is a $\pi$-network at $x$. We observe $A \in \{B_\alpha : \alpha < \kappa\}$. Take a basic open neighborhood $[A, V]$ of $A$. Since $(U_1 \cap V) \times \cdots \times (U_n \cap V)$ is an open neighborhood of $x$, there is some $\alpha < \kappa$ such that

$$(U_1 \cap B_\alpha) \times \cdots \times (U_n \cap B_\alpha) \subset (U_1 \cap V) \times \cdots \times (U_n \cap V).$$

Since $B_\alpha$ is contained in $U_1 \cup \cdots \cup U_n$ (remember $\mathcal{A} \subset [A, U_1 \cup \cdots \cup U_n]$),

$$B_\alpha = (U_1 \cap B_\alpha) \cup \cdots \cup (U_n \cap B_\alpha) \subset (U_1 \cap V) \cup \cdots \cup (U_n \cap V) \subset V.$$

Hence $B_\alpha \subset [A, V]$. Thus we have $t(A, \mathcal{F}[X]) \leq \kappa$. □
Lemma 2.5. If a family \( \{X_n: n \in \mathbb{N}\} \) of spaces has the property that \( st(X_1 \times \cdots \times X_n) \leq \kappa \) for all \( n \in \mathbb{N} \), then \( st(\prod_{n \in \mathbb{N}} X_n) \leq \kappa \).

Proof. Let \( x = (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n \) and \( \mathcal{P} \) be a \( \pi \)-network at \( x \) of finite subsets of \( \prod_{n \in \mathbb{N}} X_n \). For each \( n \in \mathbb{N} \), let \( \mathcal{Q}_n = \{p_n(P): P \in \mathcal{P}\} \), where \( p_n: \prod_{n \in \mathbb{N}} X_n \to X_1 \times \cdots \times X_n \) be the projection. Then \( \mathcal{Q}_n \) is a \( \pi \)-network at \( (x_1, \ldots, x_n) \). For each \( n \in \mathbb{N} \), take a subfamily \( \mathcal{P}_n \subset \mathcal{P} \) such that \( |\mathcal{P}_n| \leq \kappa \) and \( \{p_n(P): P \in \mathcal{P}_n\} \) is a \( \pi \)-network at \( (x_1, \ldots, x_n) \). Let \( \mathcal{P}' = \bigcup \{\mathcal{P}_n: n \in \mathbb{N}\} \), then \( |\mathcal{P}'| \leq \kappa \) and \( \mathcal{P}' \) is a \( \pi \)-network at \( x \). \( \square \)

By Theorem 2.4 and the preceding lemma, we have the following.

Corollary 2.6. For a space \( X \), the equality \( t(\mathcal{F}[X]) = st(\mathcal{X}^X) \) holds.

For a space \( X \), let \( \mathcal{F}^1[X] = \mathcal{F}[X] \), and let \( \mathcal{F}^n[X] = \mathcal{F}[\mathcal{F}^{n-1}[X]] \) for \( n \geq 2 \). The \( n \)-times power of \( \mathcal{F}[X] \) is denoted by \( \mathcal{F}^n[X] \).

Proposition 2.7. The following statements hold:

1. \( t(\mathcal{F}[X]) = st(\mathcal{F}[X]) \).
2. \( t(\mathcal{F}[X]) = t(\mathcal{F}^2[X]) \) for all \( n \in \mathbb{N} \).
3. \( t(\mathcal{F}[X]) = t(\mathcal{F}^n[X]) \) for all \( n \in \mathbb{N} \).
4. \( t(\mathcal{F}[X]) = t(\mathcal{F}^n[X]) \) for all \( n \in \mathbb{N} \).

Proof. (1): Assume \( t(\mathcal{F}[X]) = \kappa \). Let \( A \in \mathcal{F}[X] \) and let \( \mathcal{P} \) be a \( \pi \)-network at \( A \) consisting of finite subsets of \( \mathcal{F}[X] \). Without loss of generality, we may assume \( A \subset [A, X] \) for all \( A \in \mathcal{P} \) (i.e., every member of \( \bigcup \{A: A \in \mathcal{P}\} \) contains \( A \)). For each \( A \in \mathcal{P} \), let \( F(A) = \bigcup A \). Note that for a basic open neighborhood \( [A, U] \in \mathcal{F} \), \( F(A) \in [A, U] \) if and only if \( A \subset [A, U] \). We observe \( A \in [F(A): A \in \mathcal{P}] \). Take any basic open neighborhood \( [A, U] \in \mathcal{F} \). Then \( A \subset [A, U] \) for some \( A \in \mathcal{P} \), hence \( F(A) \in [A, U] \). By \( t(\mathcal{F}[X]) = \kappa \), there is a subfamily \( \mathcal{Q} \subset \mathcal{P} \) such that \( |\mathcal{Q}| \leq \kappa \) and \( A \in [F(A): A \in \mathcal{Q}] \). This implies \( \mathcal{Q} \) is a \( \pi \)-network at \( A \). Thus we have \( st(\mathcal{F}[X]) \leq \kappa \).

(2): Fix an \( n \in \mathbb{N} \). Using Lemma 1.1, we immediately have \( t(\mathcal{F}[X]) \leq t(\mathcal{F}^n[X]) \). Conversely let \( t(\mathcal{F}[X]) = \kappa \). Then, by Theorem 2.4, the supertightness of every finite power of \( X \) is less than or equal to \( \kappa \), so is the supertightness of every finite power of \( X^n \). By Theorem 2.4, we have \( t(\mathcal{F}^n[X]) \leq \kappa \).

(3): This follows from the previous statement (2) and Lemma 1.2.

(4): First we show the equality \( t(\mathcal{F}[X]) = t(\mathcal{F}^2[X]) \). By Theorem 2.4,

\[
\text{Then obviously } \sup \{st(\mathcal{F}^n[X]): n \in \mathbb{N}\} \geq \sup \{st(\mathcal{F}^n[X]): n \in \mathbb{N}\} \geq t(\mathcal{F}[X]). \]

Thus we have \( t(\mathcal{F}^2[X]) \geq t(\mathcal{F}[X]) \). On the other hand, using Lemma 1.2 and the statements (1), (2) in this proposition, we have \( \sup \{st(\mathcal{F}^n[X]): n \in \mathbb{N}\} \leq \sup \{st(\mathcal{F}^n[X]): n \in \mathbb{N}\} \leq t(\mathcal{F}^2[X]) \). Thus \( t(\mathcal{F}^2[X]) \leq t(\mathcal{F}[X]) \). Inductively we have \( t(\mathcal{F}^n[X]) \leq t(\mathcal{F}^{n+1}[X]) \).

We denote by \( hd(X) \) (resp., \( hl(X) \)) the hereditary density (resp., hereditary Lindelöf degree) of a space \( X \).

Proposition 2.8. For a space \( X \), \( t(\mathcal{F}[X]) \leq \sup(\text{hd}(X^n)): n \in \mathbb{N} \) holds.

Proof. Let \( \kappa = \sup(\text{hd}(X^n)): n \in \mathbb{N} \). First we show \( st(X) \leq \kappa \). Let \( x \in X \) and \( \mathcal{P} \) be a \( \pi \)-network at \( x \) of finite subsets of \( X \). For each \( n \in \mathbb{N} \), let \( \mathcal{P}_n = \{P \in \mathcal{P}: |P| = n\} \). For each \( P \in \mathcal{P}_n \), put \( P = (x_1(P), \ldots, x_n(P)) \) and \( x(P) = (x_1(P), \ldots, x_n(P)) \in X^n \). By \( \text{hd}(X^n) \leq \kappa \), we can take a subfamily \( \mathcal{Q}_n \subset \mathcal{P}_n \) such that \( |\mathcal{Q}_n| \leq \kappa \) and \( x(Q) = (x_1(P), \ldots, x_n(P)) \). On the other hand, using Lemma 1.2 and the statements (1), (2) in this proposition, we have \( \sup(\text{hd}(X^n)): n \in \mathbb{N} \) \leq \sup(\text{hd}(X)): n \in \mathbb{N} \) \leq t(\mathcal{F}^2[X]) \). Thus \( t(\mathcal{F}^2[X]) \leq t(\mathcal{F}[X]) \). In the equality \( t(\mathcal{F}^2[X]) = t(\mathcal{F}[X]) \), replacing \( X \) by \( \mathcal{F}[X] \), we have \( t(\mathcal{F}^2[X]) = t(\mathcal{F}^2[X]) \). Inductively we have \( t(\mathcal{F}^n[X]) = t(\mathcal{F}^n[X]) \).

We denote by \( hd(X) \) (resp., \( hl(X) \)) the hereditary density (resp., hereditary Lindelöf degree) of a space \( X \).

Remark 2.9. For a Tychonoff space \( X \), we denote by \( C_p(X) \) the space of all real-valued continuous functions on \( X \) with the topology of pointwise convergence. Let \( l(X) \) be the Lindelöf degree of a space \( X \). In [17, Theorem 2.1], the inequality \( \text{sup}(\text{hd}(X^n)): n \in \mathbb{N} \) \leq l(C_p(X)) \) was proved for a Tychonoff space \( X \). Moreover, Zenor gave the equality \( hl(C_p(X)) = \text{sup}(\text{hd}(X^n)): n \in \mathbb{N} \) in [23, Theorem 4*]. Hence, for a Tychonoff space \( X \), we have

\[
t(\mathcal{F}[X]) \leq l(C_p(X)) \leq hl(C_p(X)) = \text{sup}(\text{hd}(X^n)): n \in \mathbb{N} \).
\]

Now we show the second equality.
Lemma 2.10. Let \( k \in \mathbb{N} \) and assume \( t(X^k) \leq \lambda \). If \( x \in X \) and \( \mathcal{P} \) is a \( \pi \)-network at \( x \) such that \( |P| = k \) for all \( P \in \mathcal{P} \), then there is a subfamily \( \mathcal{P}^* \subset \mathcal{P} \) such that \( |\mathcal{P}^*| \leq \lambda \) and \( \mathcal{P}' \) is a \( \pi \)-network at \( x \).

Proof. For each \( P \in \mathcal{P} \), let \( P = \{x_1(P), \ldots, x_k(P)\} \) and \( x(P) = (x_1(P), \ldots, x_k(P)) \). Then the point \( (x, \ldots, x) \in X^k \) is in the closure of \( \{x(P) : P \in \mathcal{P}\} \subset X^k \). Using \( t(X^k) \leq \lambda \), we have a subfamily \( \mathcal{P}^* \subset \mathcal{P} \) such that \( |\mathcal{P}^*| \leq \lambda \) and the point \((x, \ldots, x) \in X^k\) is in the closure of \( \{x(P) : P \in \mathcal{P}^*\} \). Obviously \( \mathcal{P}' \) is a \( \pi \)-network at \( x \).

Lemma 2.11. Let \( m, k \in \mathbb{N} \) and assume \( t(X^{mk}) \leq \lambda \). If \( x \in X^m \) and \( \mathcal{P} \) is a \( \pi \)-network at \( x \) in \( X^m \) such that \( |P| = k \) for all \( P \in \mathcal{P} \), then there is a subfamily \( \mathcal{P}' \subset \mathcal{P} \) such that \( |\mathcal{P}'| \leq \lambda \) and \( \mathcal{P}' \) is a \( \pi \)-network at \( x \).

Proof. In Lemma 2.10, replace \( X \) by \( X^m \).

Theorem 2.12. For a space \( X \), the equality \( \sup(t(F_n[X])) : n \in \mathbb{N} \) = \( \sup(t(X^n)) : n \in \mathbb{N} \) holds.

Proof. Assume \( \sup(t(F_n[X])) : n \in \mathbb{N} \leq \lambda \). We show \( t(X^n) \leq \lambda \) for all \( n \in \mathbb{N} \). Fix an \( n \in \mathbb{N} \). Let \( x = (x_1, \ldots, x_n) \in X^n \), and \( x \in Y \) \( \setminus \{Y\} \). Take an open neighborhood \( U_1 \) of \( x \) such that \( U_1 = U_1 \) if \( x_1 = x_2 \), and \( U_1 \cap U_2 = \emptyset \) if \( x_1 \neq x_2 \). We may assume \( Y \subset U_1 \times \cdots \times U_n \). Let \( A = \{x_1, \ldots, x_n\} \). For each \( y = (y_1, \ldots, y_n) \in Y \), we put \( F(y) = A \cup \{y_1, \ldots, y_n\} \in F_{2n}[X] \). Let \( A = \{y : y \in Y\} \). We observe \( A \notin \lambda \). Obviously \( A \notin \mathcal{A} \), because of \( x \notin Y \). Take a basic open neighborhood \([A, V] \) of \( A \). Since \( x \in V^n \), there is a \( y = (y_1, \ldots, y_n) \in Y \). Using \( t(F_{2n}[X]) \leq \lambda \), we have a subset \( \lambda \subset Y \) such that \( |\lambda| \leq \lambda \) and \( A = \{F(y) : y \in \lambda\} \). We observe \( x \in \lambda \). Take a basic open neighborhood \( W_1 \times \cdots \times W_m \) of \( x \), where \( W_1 \subset U_1 \), and \( W_i = W_j \) if \( x_i = x_j \). By \( \mathcal{A} \), there is a \( y \in Y \) such that \( F(y) \in [A, W_1 \cup \cdots \cup W_n] \). Then \( y_1, \ldots, y_n \in W_1 \cup \cdots \cup W_n \), and \( y_i \in U_i \cap (W_1 \cup \cdots \cup W_n) \). Thus \( y \in W_1 \times \cdots \times W_n \), consequently \( t(X^n) \leq \lambda \).

Conversely assume \( \sup(t(X^n)) : n \in \mathbb{N} = \sup(t(F_n[X])) : n \in \mathbb{N} \leq \lambda \). Since \( F_1[X] \) is discrete, obviously \( t(F_1[X]) \leq \lambda \). Fix any \( n > 1 \), and assume \( t(F_{n-1}[X]) \leq \lambda \). Let \( A \in F_n[X] \). \( A \subset F_n[X] \) and \( A \in \mathcal{A} \). Since every point in \( F_n[X] \setminus F_{n-1}[X] \) is isolated in \( F_n[X] \), there is a \( 1 \leq m \leq n \) such that \( A \not\in F_n[X] \setminus F_{m-1}[X] \). If \( A \not\in \mathcal{A} \setminus (F_{m-1}[X] \setminus F_{m-1}[X]) \), then by \( t(F_{m-1}[X]) \leq \lambda \) there is nothing to prove. Assume \( A \not\in \mathcal{A} \setminus (F_{m-1}[X] \setminus F_{m-1}[X]) \). Let \( A = \{a_1, \ldots, a_m\} \) and take an open neighborhood \( U(a_i) \) of \( a_i \) such that \( U(a_i) \cup U(a_j) = \emptyset \) if \( i \neq j \). Considering the basic open neighborhood \( [A, U_1 \cup \cdots \cup U_m] \) of \( A \), we may assume \( A \subset [A, U_1 \cup \cdots \cup U_m] \). Let \( A = \{a_1, \ldots, a_m\} \) and take an open neighborhood \( U(a_i) \) of \( a_i \) such that \( U(a_i) \cup U(a_j) = \emptyset \) if \( i \neq j \). Let \( P = \{a_i : 1 \leq i \leq k \} \subset X^m \) and \( \mathcal{P} = \{P : F \in \mathcal{A}\} \). Note that \( |P| = k \) for all \( P \in \mathcal{A} \). We observe that \( \mathcal{P} \) is a \( \pi \)-network at \( (a_1, \ldots, a_m) \). Take a basic open neighborhood \( V_1 \times \cdots \times V_m \) of \( (a_1, \ldots, a_m) \), where \( V_i \subset U_i \) for all \( i \). By \( \mathcal{A} \), there is an \( \mathcal{F} \in [V_1 \cup \cdots \cup V_m] \cap \mathcal{A} \). Then for each \( 1 \leq i \leq k \), \( x_i \in U(a_i) \). Thus \( x_i \in U_i \cap (V_1 \cup \cdots \cup V_m) \). Hence \( P = \{x_i : 1 \leq i \leq k \} \subset W \) and \( \mathcal{P} = \{P : F \in \mathcal{A}\} \). This implies \( \mathcal{F} \in [A, W] \), consequently \( t(F_n[X]) \leq \lambda \).

Lemma 2.13. ([7, p. 227]) If a family \( X_0 : n \in \mathbb{N} \) of spaces has the property that \( t(X_1 \times \cdots \times X_0) \leq \kappa \) for all \( n \in \mathbb{N} \), then \( t(\prod_{n \in \mathbb{N}} X_0) \leq \kappa \).

By Theorem 2.12 and the preceding lemma, we have the following.

Corollary 2.14. For a space \( X \), the equality \( \sup(t(F_n[X])) : n \in \mathbb{N} = t(X^n) \) holds.

Let \( Z \) be the supercompact space in [13, Example 2.6]. This space satisfies \( t(Z) = \omega \) (indeed, Fréchet–Urysohn) and \( st(Z) = 2^\omega \). Since \( Z \) is compact, \( t(Z^n) = \omega \) for all \( n \in \mathbb{N} \) [7, 3.12.8(f)]. Therefore \( \sup(t(F_n[Z])) : n \in \mathbb{N} = \omega \), but \( t(F[Z]) = 2^\omega \). Let \( S_\kappa \) be the quotient space obtained by identifying all limit points of \( \kappa \) many convergent sequences. It is well known that \( t(S_\kappa \times S_\omega) \) is uncountable. Hence we can see that \( t(F_3[S_\omega]) \) is uncountable.

3. DCCC and CCC of Pixley–Roy hyperspaces

Definition 3.1. A space \( X \) satisfies the discrete countable chain condition (shortly, DCCC) [22] if every discrete family of nonempty open subsets of \( X \) is countable. A space \( X \) satisfies the countable chain condition (shortly, CCC) if every pairwise disjoint family of nonempty open subsets of \( X \) is countable.
In this section, we investigate some properties concerning DCCC and CCC of Pixley–Roy hyperspaces. For convenience of the readers, first of all we give a diagram of the notions appeared in this section, where $hl$ (resp., $hd$) is hereditary Lindelöf degree (resp., hereditary density).

$$
\mathcal{F}[X] : \sigma \text{-centered} \rightarrow \text{precaliber } \omega_1 \rightarrow \text{CCC} \rightarrow \text{weakly Lindelöf} \rightarrow \text{DCCC}
$$

$X : \text{cosmic} \rightarrow (C) \rightarrow (C^*) \rightarrow (\text{WS}_f) \rightarrow ? \rightarrow ? \rightarrow ?$

$$
l(h(X^\omega)) = hd(X^\omega) = \omega \quad \text{WS} \rightarrow h(X) = hd(X) = \omega
$$

**Definition 3.2.** A space $(X, \tau)$ is $\sigma$-centered if $\tau \setminus \{\emptyset\}$ is the union of countably many centered subfamilies. A space $X$ has precaliber $\omega_1$ if for every family $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of nonempty open subsets of $X$, there is an uncountable subset $I \subset \omega_1$ such that the family $\{U_\alpha : \alpha \in I\}$ is centered. A space $X$ is weakly Lindelöf if every open cover $\mathcal{U}$ has a countable subfamily $V \subset \mathcal{U}$ such that $\bigcup V$ is dense in $X$.

For an arbitrary space, each implication of "$\sigma$-centered $\rightarrow \cdots \rightarrow \text{DCCC}$" holds obviously, or follows from a simple observation. Note that regularity is needed to show "weakly Lindelöf $\rightarrow \text{DCCC}$".

**Definition 3.3.** A space is cosmic if it has a countable network. A space $X$ satisfies condition $(C)$ [9, Definition 2] if for every subspace $Y \subset X$ of cardinality $\omega_1$, every open family $\mathcal{U}$ in $Y$ of cardinality $\omega_1$ has a countable network (i.e., there is a countable family $N'$ of subsets of $Y$ such that every member of $\mathcal{U}$ is the union of certain members of $N'$).

Obviously a cosmic space satisfies condition $(C)$. The following two theorems are due to van Douwen, Hajnal and Juhász respectively.

**Theorem 3.4.** ([6, Lemma 2.2, Theorem 3.3(b)]) A space $X$ is cosmic if and only if $\mathcal{F}[X]$ is $\sigma$-centered.

**Theorem 3.5.** ([9, Theorems 1, 2]) The following hold:

1. If a space $X$ satisfies condition $(C)$, then $\mathcal{F}[X]$ satisfies CCC.
2. Under $\text{MA}_{\omega_1}$, $\mathcal{F}[X]$ satisfies CCC if and only if $X$ satisfies condition $(C)$.
3. Under $\text{CH}$, there is a space $X$ such that $\mathcal{F}[X]$ satisfies CCC, but $X$ does not satisfy condition $(C)$.

We introduce condition $(C')$ and recall a weakly separated subset.

**Definition 3.6.** A space $X$ satisfies condition $(C')$ if for every subset $\{x_\alpha : \alpha < \omega_1\} \subset X$ and a family $\{U_\alpha : \alpha < \omega_1\}$ of open subsets of $X$ with $x_\alpha \in U_\alpha$, there is an uncountable subset $I \subset \omega_1$ such that $\{x_\alpha : \alpha \in I\} \subset \bigcap\{U_\alpha : \alpha \in I\}$. A subset $Y$ of $X$ is weakly separated [21] if for each point $y \in Y$, one can assign an open neighborhood $U_y$ of $y$ such that for distinct $y, y' \in Y$, $y \not\in U_{y'}$ or $y' \not\in U_y$ holds. If a space $X$ has no uncountable weakly separated subset, then we say that $X$ satisfies (WS). If no finite power of a space $X$ has an uncountable weakly separated subset, then we say that $X$ satisfies (WS$_f$).

Condition $(C)$ obviously implies $(C')$. We can easily see that $(C')$ implies (WS), and that $(C')$ is closed under finite powers. Hence $(C')$ implies (WS$_f$). It is known that, if a space $X$ satisfies (WS$_f$), then $\mathcal{F}[X]$ satisfies CCC [12, Theorem]. Moreover, we can easily see that, if $\mathcal{F}[X]$ satisfies CCC, then $X$ satisfies (WS).

**Lemma 3.7.** ([23, Theorems 3, 3*]) If a family $\{X_n : n \in \mathbb{N}\}$ of spaces has the property that $X_1 \times \cdots \times X_n$ is hereditarily Lindelöf (resp., hereditarily separable) for all $n \in \mathbb{N}$, then $\prod_{n \in \mathbb{N}} X_n$ is also hereditarily Lindelöf (resp., hereditarily separable).

A weakly separated subset is a common generalization of a left-separated subset and a right-separated subset. Therefore, if a space satisfies (WS), then it is hereditarily Lindelöf and hereditarily separable. In particular, if $\mathcal{F}[X]$ satisfies CCC, then $X$ is hereditarily Lindelöf and hereditarily separable. If a space $X$ satisfies (WS$_f$), then every finite power of $X$ is hereditarily Lindelöf and hereditarily separable, hence $X^\omega$ is hereditarily Lindelöf and hereditarily separable by Lemma 3.7.

A cover $C$ of a set $X$ is said to be an $\omega$-cover [8] if every finite subset of $X$ is contained in some member of $C$.

**Lemma 3.8.** ([8, Proposition]) Every finite power of a space $X$ is Lindelöf if and only if every open $\omega$-cover of $X$ has a countable $\omega$-subcover.

**Lemma 3.9.** Let $\mathcal{U}$ be an open family of a space $X$. Let $V(\mathcal{U}) = \{F \in \mathcal{F}[X] : F \subset U$ for some $U \in \mathcal{U}\}$, then it is open-and-closed in $\mathcal{F}[X]$. 


Proof. If \( F \in V(\mathcal{U}) \), then \( F \subseteq U \) for some \( U \in \mathcal{U} \). Obviously \( \{ F, U \} \subseteq V(\mathcal{U}) \), thus \( V(\mathcal{U}) \) is open in \( \mathcal{F}(X) \). On the other hand, if \( F \in \mathcal{F}(X) \setminus V(\mathcal{U}) \), then \( \{ F, X \} \cap V(\mathcal{U}) = \emptyset \), thus \( V(\mathcal{U}) \) is closed in \( \mathcal{F}(X) \). \( \square \)

**Theorem 3.10.** If \( \mathcal{F}(X) \) satisfies DCCC, then the following hold:

1. \( X \) is hereditarily Lindelöf, in particular \( |X| \leq 2^\omega \).
2. For finitely many open subsets \( U_1, \ldots, U_n \) of \( X \), \( U_1 \times \cdots \times U_n \) is Lindelöf.

**Proof.** First of all we show that every finite power of \( X \) is Lindelöf. By Lemma 3.8, we have only to show that every open \( \omega \)-cover of \( X \) has a countable \( \omega \)-subcover. Let \( U = \{ U_\alpha : \alpha < \kappa \} \) be an open \( \omega \)-cover of \( X \). For each \( \alpha \in \kappa \), let

\[
V_\alpha = V(\{ U_\alpha \}) \setminus V(\{ U_\beta : \beta < \alpha \}).
\]

By Lemma 3.9, each \( V_\alpha \) is open-and-closed in \( \mathcal{F}(X) \). The family \( \{ V_\alpha : \alpha < \kappa \} \) is a cover of \( \mathcal{F}(X) \). Indeed, let \( F \in \mathcal{F}(X) \) and put \( \gamma = \min\{ \alpha < \kappa : F \subseteq U_\alpha \} \), then \( F \subseteq V_\gamma \). Moreover, \( V_\alpha \cap V_\beta = \emptyset \) if \( \alpha < \beta < \kappa \). Indeed, \( F \in V_\alpha \) implies \( F \subseteq U_\alpha \), and \( F \in V_\beta \) implies \( F \setminus U_\beta \neq \emptyset \), this is a contradiction. Since \( \{ V_\alpha : \alpha < \kappa \} \) is a pairwise disjoint cover consisting of open-and-closed subsets of \( \mathcal{F}(X) \), by DCCC of \( \mathcal{F}(X) \) the set \( \Gamma = \{ \alpha < \kappa : V_\alpha \neq \emptyset \} \) must be countable. Let \( \Gamma = \{ \alpha_n : n \in \omega \} \). We observe that \( \{ U_{\alpha_n} : n \in \omega \} \) is an \( \omega \)-cover of \( X \). Let \( F \in \mathcal{F}(X) \), then there is an \( n \in \omega \) with \( F \subseteq V_{\alpha_n} \). This obviously implies \( F \subseteq U_{\alpha_n} \).

To show that \( X \) is hereditarily Lindelöf, it suffices that every open subset of \( X \) is Lindelöf. Let \( U \) be an open subset of \( X \). It is easy to see that \( \mathcal{F}(U) \) is homeomorphic to the open-and-closed subset \( V(U) \) in \( \mathcal{F}(X) \). Hence \( \mathcal{F}(U) \) also satisfies DCCC, by the argument in the preceding paragraph, \( U \) is Lindelöf. The fact \( |X| \leq 2^\omega \) is well known for hereditarily Lindelöf spaces: see [10, Remark, p. 13].

Let \( U_1, \ldots, U_n \) be open subsets of \( X \). Since \( X \) is hereditarily Lindelöf, every open subset of \( X \) is an \( F_\sigma \)-set. Hence \( U_1 \times \cdots \times U_n \) is an \( F_\sigma \)-subset of the Lindelöf space \( X^n \), therefore it is Lindelöf. \( \square \)

**Example 3.11.** Let \( X \) be the two arrows space [7, 3.10.C]. This space \( X \) is compact, first-countable, hereditarily Lindelöf and hereditarily separable. Hence \( X \) satisfies (1) and (2) in Theorem 3.10. But we show that \( \mathcal{F}(X) \) does not satisfy DCCC. For convenience of the readers, we recall the two arrows space. Let \( X = (\{0,1\} \times \{0,1\}) \setminus \{(0,0), (1,1)\} \), consider the order \( \prec \) on \( X \) defined as follows: \((x,i) \prec (y,j)\) if \( x < y \), or \( x = y \) and \( i < j \). The two arrows space is the space \( X \) with the order topology induced by \( \prec \). In the sequel, to avoid confusion, a point \((r,i) \in X \) is denoted by \( \langle r,i \rangle \), and \((a,b)\) stands for an open interval. For each \( r \in (0,1/4) \), let

\[
U_r = \{(r,1), (1-r,1)\} \cup \{p(i) : p \in (r,1/4) \cup (1-r,1), i = 0,1\}.
\]

Each \( U_r \) is open in \( X \). Let \( \mathcal{O}_r = \{(r,1), (1-r,1)\}, U_r \) for \( r \in (0,1/4) \). Obviously \( \mathcal{O}_r \cap \mathcal{O}_{r'} = \emptyset \) if \( r \neq r' \). Assume that \( \{ \mathcal{O}_r : r \in (0,1/4) \} \) is first-countable, there are \( r_0 \in (0,1/4) \) and \( A_n \in \mathcal{O}_{r_n} \) \((n \in \omega)\) such that \( A \subseteq A_0 \) and \( A_n \to A \) in \( \mathcal{F}(X) \). By \( \langle r_0,1 \rangle, (1-r_0,1) \in A_n \), there are distinct two points \( x, y \in A \) and an infinite subset \( J \subseteq \mathcal{C}(A) \) such that \( \langle r_0,1 \rangle \to x \), \( (1-r_0,1) \to y \) \((n \in J)\). For simplicity, we may assume \( J = \omega \). Then \( x, y \in A \subseteq A_0 \subseteq U_{rn} \) for all \( n \in \omega \). Assume \( x = (p,1) \) for some \( p \in (0,1/4) \). By the condition \( \langle r_0,1 \rangle \to x \), \( (1-r_0,1) \to y \) \((n \in J)\). This is a contradiction. So let \( x = (0,p) \) for some \( p \in (0,1/4) \). Then \( r_n < p \) for all but finitely many \( n \in \omega \) and \( y = (1-p,1) \) holds. This means \( y = (1-p,1) \in U_{rn} \) for only finitely many \( n \in \omega \). This is also a contradiction. We conclude that \( \{ \mathcal{O}_r \} : r \in (0,1/4) \) is a discrete family in \( \mathcal{F}(X) \).

Daniels [5, Theorem 1A] noted that, if \( \mathcal{F}(X) \) is weakly Lindelöf, then every finite power of \( X \) is Lindelöf. The statement (2) in Theorem 3.10 is an improvement of Daniels’ result.

A space \( X \) is said to be semi-stratifiable [4] if for each open set \( U \subseteq X \), one can assign a sequence \( \{ U_n : n \in \omega \} \) of closed subsets of \( X \) such that \( \{ U_n : n \in \omega \} = U \). Moreover since a semi-stratifiable Lindelöf space is hereditarily separable [4, Theorem 2.8], \( X^n \) is hereditarily separable. Our conclusion follows from Lemma 3.7. \( \square \)

Concerning weak Lindelöfness of \( \mathcal{F}(X) \), we note the following.

**Proposition 3.13.** If \( \mathcal{F}(X) \) is weakly Lindelöf, then every closed subset of \( X \) is separable. If \( t(X) = \omega \) holds additionally, then \( X \) is hereditarily separable.
Proof. Let $Y$ be a closed subset of $X$. Consider the open cover
\[ \left\{ \left[ \left\{ y \right\} \times X \right] : y \in Y \right\} \cup \left\{ \left\{ F, X \setminus Y \right\} : F \in \mathcal{F}[X], F \cap Y = \emptyset \right\} \]
of $\mathcal{F}[X]$. Take countable subsets $\{y_n : n \in \omega\} \subset Y$ and $\{F_n : n \in \omega\} \subset \mathcal{F}[X]$ with $F_n \cap Y = \emptyset$ ($n \in \omega$) such that the union of the family $\{\left[ \left\{ y_n \right\} \times X \right] : n \in \omega\} \cup \{\left[ F_n, X \setminus Y \right] : n \in \omega\}$ is dense in $\mathcal{F}[X]$. Assume that there is a point $y \in Y \setminus \{y_n : n \in \omega\}$. Take the open set $\left[ \left\{ y \right\} \times G \right]$, where $G = X \setminus \{y_n : n \in \omega\}$. Obviously $\left[ \left\{ y \right\} \times G \right] \cap \{F_n : n \in \omega\} \notin \emptyset$, so $\left[ \left\{ y \right\} \times G \right] \notin \emptyset$ for some $n \in \omega$. This implies $y \in G$, a contradiction. Hence $\{y_n : n \in \omega\}$ is dense in $Y$. Additionally assume $t(X) = \omega$ and let $Y$ be a subset of $X$. Then $\mathcal{F}$ has a countable dense subset $\{y_n : n \in \omega\}$. For each $n \in \omega$, take a countable set $Y_n \subset Y$ with $y_n \in Y_n$. Then $\bigcup \{y_n : n \in \omega\}$ is countable and dense in $Y$. □

The author does not know if there is a non-separable regular space $X$ such that $\mathcal{F}[X]$ satisfies DCCC. But we show that there is such a space among $T_2$-spaces.

Lemma 3.14. Assume $2^\omega > \omega_1$. If $K$ is an uncountable compact metric space, and $A$ is a subset of $X$ such that $|A| = \omega_1$, then the set $B = \{x \in K : |A \cup \{x\}| = \omega_1\}$ for any neighborhood $U$ of $x$ has cardinality $2^\omega$.

Proof. Recall that every uncountable compact metric space has cardinality $2^\omega$. For each $x \in K \setminus (A \cup B)$, take an open neighborhood $U_x$ of $x$ such that $|A \cup U_x| \leq \omega$. Since $K \setminus (A \cup B)$ is Lindelöf, $K \setminus (A \cup B) \subset \bigcup \{U_x : \alpha \in \omega\}$ for a countable subset $\{x_\alpha : \alpha \in \omega\} \subset K \setminus (A \cup B)$. Then
\[ A \setminus \bigcup \{U_{x_\alpha} : \alpha \in \omega\} = \left( A \setminus \bigcup \{U_{x_\alpha} : \alpha \in \omega\} \right) \cup B, \]
and it is an uncountable compact metric space. Hence we have $|B| = 2^\omega$. □

Lemma 3.15. If a space $X$ has cardinality $\omega_1$ and every countable subset of $X$ is closed in $X$, then $\mathcal{F}[X]$ does not satisfy DCCC.

Proof. Let $X = \{x_\alpha : \alpha < \omega_1\}$. Since every countable subset is closed, $U_\alpha = \{x_\beta : \beta \geq \alpha\}$ is open in $X$. Let $O = \{\{x_\alpha, U_\alpha\} : \alpha < \omega_1\}$. Obviously $O$ is pairwise disjoint. Let $F = \{x_\alpha, \ldots, x_{\alpha_n}\} \in \mathcal{F}[X]$, where $\alpha_1 < \cdots < \alpha_n$, then $F \in \{\{x_\alpha, U_\alpha\} : \alpha < \omega_1\}$. Thus $O$ is a cover. Therefore $\mathcal{F}[X]$ does not satisfy DCCC. □

Let $I = [0, 1]$ be the closed unit interval, and $\tau$ be the usual topology on $I$. We consider a finer topology $\tau' = \{U \setminus D : U \in \tau, D$ is a countable subset of $I\}$ than $\tau$. Obviously $(I, \tau')$ is a non-separable $T_2$-space which is not regular. In Proposition 3.13, regularity was not used. Therefore $\mathcal{F}([I, \tau'])$ is not weakly Lindelöf.

Theorem 3.16. The following are equivalent:

(1) $2^\omega > \omega_1$ holds,
(2) $\mathcal{F}([I, \tau'])$ satisfies DCCC.

Proof. (1) $\Rightarrow$ (2): Let $O$ be a family of nonempty open subsets of $\mathcal{F}([I, \tau'])$ such that $|O| = \omega_1$. We show that $O$ is not discrete. We may put $O = \{\{F_\alpha, U_\alpha \setminus D_\alpha\} : \alpha < \omega_1\}$, where $F_\alpha \in \mathcal{F}([I, \tau])$, $\alpha < \omega_1$, $D_\alpha$ is a countable subset of $I$ and $F_\alpha \subset U_\alpha \setminus D_\alpha$. Let $\mathcal{B}$ be a countable base for $([I, \tau])$ closing under finite unions. For each $\alpha < \omega_1$, take $A_\alpha, B_\alpha \in \mathcal{B}$ such that $F_\alpha \subset A_\alpha \subset \overline{A_\alpha} \subset B_\alpha \subset U_\alpha$, where the closure is taken in $\tau$. Then there are a countable set $J_1 \subset \omega_1$ and $A, B \in \mathcal{B}$ such that $F_\alpha \subset A \subset \overline{A} \subset B \subset U_\alpha$ for all $\alpha \in J_1$. By $\Delta$-system lemma [7, 2.7.10(c)], there are an uncountable set $J_2 \subset J_1$ and a finite set $R \subset I$ such that $F_\alpha \cap F_\beta = \emptyset$ for distinct $\alpha, \beta \in J_2$. Moreover, there are an uncountable set $J_3 \subset J_2$ and a $k \in \mathbb{N}$ such that $|F_\alpha| = k$ for all $\alpha \in J_3$. From these observations, replacing $U_\alpha(\alpha \in J_3)$ by $B$, we may put $O = \{\{F_\alpha, B \setminus D_\alpha\} : \alpha < \omega_1\}$ and this family satisfies
(i) $|F_\alpha| = k$ for all $\alpha < \omega_1$,
(ii) $\bigcup \{F_\alpha : \alpha < \omega_1\} \subset B$, where the closure is taken in $\tau$,
(iii) $F_\alpha \cap F_\beta = \emptyset$ for distinct $\alpha, \beta < \omega_1$.

Let $|F_\alpha \setminus R| = m$, and put $F_\alpha \setminus R = \{x_\alpha,1, \ldots, x_\alpha,m\}$ and $x_\alpha = (x_\alpha,1, \ldots, x_\alpha,m) \in \mathbb{I}^m$. Let $X = \{x_\alpha : \alpha < \omega_1\}$ and let
\[ Y = \{y \in \mathbb{I}^m \setminus X : |X \cap W| = \omega_1 \text{ for any neighborhood } W \text{ of } y \text{ in } [I, \tau'] \}. \]

By Lemma 3.14, $|Y| = 2^\omega$. Hence we can take a point $y = (y_1, \ldots, y_m) \in Y \setminus \left( \bigcup \{F_\alpha \cup D_\alpha : \alpha < \omega_1\} \right)^m$. Let $F_y = \{y_1, \ldots, y_m\}$. Then obviously $F_y \cap (\bigcup \{F_\alpha \cup D_\alpha : \alpha < \omega_1\}) = \emptyset$, and $F_y \subset B$ because of (ii) above. We see that every neighborhood of $F_y \cup R$ intersects with uncountably many members of $O$. Let $[F_y \cup R, V \setminus D]$ be a basic open neighborhood of $F_y \cup R$, where $V \subset \tau$, $D$ is a countable set in $I$ and $F_y \cup R \subset V \setminus D$. Since $y \in Y \cap (V \times \cdots \times V)$ and $D$ is countable, there is an uncountable
set \( J \subset \omega_1 \) such that \( F_\alpha \setminus R \subset V \) and \( (F_\alpha \setminus R) \cap D = \emptyset \) for all \( \alpha \in J \). Then \( F_\alpha \subset V \setminus D \) and \( F_\alpha \cup R \subset B \setminus D_\alpha \) for all \( \alpha \in J \). This means \( \{ F_\alpha \cup R, V \setminus D \} \cap \{ F_\alpha, B \setminus D_\alpha \} \neq \emptyset \) (\( \alpha \in J \)). Thus \( \mathcal{O} \) is not discrete, so \( \mathcal{F}(L, \tau') \) satisfies DCCC.

(2) \( \rightarrow \) (1): This follows from Lemma 3.15. \( \square \)

**Remark 3.17.** In ZFC, there is a non-separable \( T_1 \)-space \( X \) such that \( \mathcal{F}(X) \) satisfies DCCC. Let \( X \) be a set of cardinality \( \omega_2 \). We give \( X \) the topology \( \tau = \{ \emptyset \} \cup \{ X \setminus D : D \) is a countable set in \( X \} \). DCCC of \( \mathcal{F}(X) \) can be proved by the same argument as in Theorem 3.16(1) \( \rightarrow \) (2).

We give a characterization for \( \mathcal{F}(X) \) to have precaliber \( \omega_1 \).

**Theorem 3.18.** For a space \( X \), the following are equivalent:

1. \( \mathcal{F}(X) \) has precaliber \( \omega_1 \).
2. \( X \) satisfies condition (C').

**Proof.** (1) \( \rightarrow \) (2): Let \( \{ x_\alpha : \alpha < \omega_1 \} \subset X \) and \( \{ U_\alpha : \alpha < \omega_1 \} \) be an open family in \( X \) with \( x_\alpha \in U_\alpha \). Consider the open family \( \{ [x_\alpha, U_\alpha'] : \alpha < \omega_1 \} \) in \( \mathcal{F}(X) \). Then, using the condition (1), we have an uncountable subset \( I \subset \omega_1 \) such that \( \{ [x_\alpha, U_\alpha'] : \alpha \in I \} \) is centered. Fix any \( \alpha \in I \). Then, for every \( \beta \in I \), \( [x_\alpha, U_\alpha'] \cap [x_\beta, U_\beta'] \neq \emptyset \), hence \( x_\alpha \in U_\beta' \). Therefore we have \( \{ x_\alpha : \alpha \in I \} \subset \bigcap U_\alpha' : \alpha \in I \} \).

(2) \( \rightarrow \) (1): Let \( \{ U_\alpha : \alpha \in \omega_1 \} \) be a family of cardinality \( \omega_1 \) consisting of nonempty open subsets of \( \mathcal{F}(X) \). We may assume that every member of \( \{ U_\alpha : \alpha < \omega_1 \} \) satisfies condition (2) to \( x_\alpha, U_\alpha \) and \( \{ U_\alpha : \alpha < \omega_1 \} \), we have an uncountable subset \( I_1 \subset \omega_1 \) such that \( \{ x_\alpha, U_\alpha : \alpha \in I_1 \} \subset \bigcap U_\alpha \). Continuing this operation, inductively we have an uncountable subset \( I_k \subset I_{k-1} \) such that \( \bigcup I_k \). Therefore \( \{ x_\alpha, U_\alpha : \alpha \in I_k \} \) is centered. \( \square \)

**Corollary 3.19.** If \( \mathcal{F}(X) \) has precaliber \( \omega_1 \), then \( X^{\omega} \) is hereditarily Lindelöf and hereditarily separable.

Recall the diagram above. Using Theorem 3.5(2), we have the following.

**Corollary 3.20.** Under MA\( \omega_1 \), if \( \mathcal{F}(X) \) satisfies CCC, then \( X^{\omega} \) is hereditarily Lindelöf and hereditarily separable.

The following questions look interesting.

**Question 3.21.** Let \( X \) be a regular space. If \( \mathcal{F}(X) \) satisfies DCCC, then \( X \) (hereditarily) separable? In particular, if \( L \) is a Souslin line, does \( \mathcal{F}(L) \) satisfy DCCC?

**Question 3.22.** Let \( X \) be a regular space. If \( \mathcal{F}(X) \) satisfies DCCC, then \( \mathcal{F}(X) \) weakly Lindelöf?

**Question 3.23.** If \( \mathcal{F}(X) \) is weakly Lindelöf, then \( X \) is hereditarily separable (equivalently, of countable tightness)?

4. An application of Pixley–Roy hyperspaces

We give an application on DCC with Pixley–Roy hyperspaces. According to [2], a space \( X \) is said to be feebly Lindelöf if every locally finite family of nonempty open subsets of \( X \) is countable, and a space \( X \) is said to be star Lindelöf if for every open cover \( \mathcal{U} \) of \( X \), there is a Lindelöf subspace \( L \subset X \) such that \( st(L, \mathcal{U}) = X \), where \( st(L, \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : U \cap L \neq \emptyset \} \). For a regular space, DCCC and feebly Lindelöf property are equivalent [22, Theorem 2.6]. A star Lindelöf space is feebly Lindelöf [2, Theorem 2.7]. Alas et al. asked whether a \( T_4 \) (normal \( T_1 \)) feebly Lindelöf space is star Lindelöf [2, p. 626]. Answering this question, under \( 2^{\omega} = 2^{\omega_1} \) Song gave a counterexample [19, Example 2.2]. We show that under \( MA + 2^{\omega} > \omega_1 \) (Martin’s axiom plus the negation of the continuum hypothesis) there is a \( T_4 \) DCCC (hence, feebly Lindelöf) metacompact Moore space which is not star Lindelöf.

**Lemma 4.1.** ([11, Theorem 2.3]) For a space \( X \), \( \mathcal{F}(X) \) is the union of countably many closed discrete subspaces if and only if every point of \( X \) is \( G_\delta \).

**Proposition 4.2.** For a space \( X \), \( \mathcal{F}(X) \) is star Lindelöf if and only if \( X \) is countable.

**Proof.** Assume that \( \mathcal{F}(X) \) is star Lindelöf, and consider the open cover \( \mathcal{U} = \{ [x] : x \in X \} \) of \( \mathcal{F}(X) \). Take a Lindelöf subspace \( L \subset \mathcal{F}(X) \) such that \( st(L, \mathcal{U}) = \mathcal{F}(X) \). Since a star Lindelöf space is feebly Lindelöf (DCCC), by Theorem 3.10(1)
X is hereditarily Lindelöf, hence every point of X is Gδ. Therefore, by Lemma 4.1, F[X] is the union of countably many closed discrete subspaces. This implies that L is countable. Let L = {Fk : n ∈ ω}. If x ∈ X, then [x] ∈ st(L, U), so there are a point y ∈ X and a k ∈ ω such that Fk ∈ [{y}, X] and [x] ∈ [{y}, X]. Then obviously x = y, so we have x ∈ Fk. Thus X = ∪{Fk : n ∈ ω}.

The converse is trivial. □

**Lemma 4.3.** ([6, Proposition 2.5]) A space X is first-countable if and only if F[X] is a Moore space.

**Theorem 4.4.** ([Przymusiński and Tall [16]]) Under MA + 2ω > ω1, if X is a subspace of the real line with |X| = ω1, then F[X] is normal.

**Example 4.5.** Assume MA + 2ω > ω1, and let X be a subspace of the real line with |X| = ω1. Then, by Proposition 4.2, Lemma 4.3 and Theorem 4.4, F[X] is a T4 CCC (hence, feebly Lindelöf) metacompact Moore space which is not star Lindelöf.

Song’s counterexample in [19] is neither CCC, metacompact nor a Moore space, because it contains the space ω1 with the order topology as an open-and-closed subspace.

Alas et al. asked also whether a first-countable star Lindelöf space is star countable [2, p. 625], where a space X is said to be star countable if for every open cover U of X, there is a countable set A ⊆ X such that st(A, U) = X. This question was solved in the negative [1, Example 3]. Aiken’s counterexample is not pseudocompact. We comment that there is a pseudocompact counterexample. Bell [3, Example 5.1] showed that if a Tychonoff space X is a first-countable, zero-dimensional, locally compact, meta-Lindelöf, non-compact space in which all nonempty open sets have π-weight 2ω, then X has a first-countable, meta-Lindelöf, non-compact pseudocompactification. Let C be the usual Cantor set in the closed unit interval. Let K be the space C[2 with the topology induced by the lexicographic order on it. Let X be the topological sum of ω many copies of Kω. Then, by Bell’s result above, we have a first-countable, meta-Lindelöf, non-compact pseudocompactification Y of X. Since Y has a dense σ-compact space X, obviously it is star Lindelöf. On the other hand, since a meta-Lindelöf star countable space is Lindelöf, Y is not star countable.

**References**