Fuzzy Inf. Eng. (2015) 7: 49–59

HOSTED BY ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Information and Engineering

http://www.elsevier.com/locate/fiae

Fuzzy Information and Engineering
An International Journal

ORIGINAL ARTICLE

Center of Intersection Graph of Fuzzy Submodules of Modules

CrossMark

Kukil Kalpa Rajkhowa · Helen K. Saikia

Received: 17 May 2014/ Revised: 5 January 2015
Accepted: 15 February 2015/

Abstract In this paper, we discuss the graphical aspects of fuzzy algebra. Let M be an $R - module$ with zero element θ and $F(M)$ be collection of all fuzzy submodules of M. The intersection graph of $F(M)$ is an undirected graph G with vertex set as $F(M)$ and two vertices α and β are adjacent if and only if $\alpha \cap \beta \neq \chi_\theta$, where χ_θ is the characteristic function on θ. We find the girth of G and also study some properties of center of $G - \chi_\theta$.

Keywords Fuzzy set · Fuzzy submodule · Intersection graph · Center

1. Introduction

This paper is motivated by the profound concept, zero divisor graph. In 1988, Istvan Beck [6], introduced the motivating insight zero divisor graph of commutative ring. This prominent introduction has been well expanded in various dimensions in the field of graphical aspects of algebraic structures. The significance of graphical aspects of algebraic structures is observed in comaximal graph of commutative ring by Sharma and Bhatwadekar [21], total graph of commutative ring by Anderson and Badawi [3], intersection graph of ideals of a ring by Chakrabarty et al. [7]

Kukil Kalpa Rajkhowa (✉)
Department of Mathematics, Girijananda Chowdhury Institute of Management and Technology, Guwahati-781017, India
email: kukilrajkhowa@yahoo.com
Helen K. Saikia
Gauhati University, Guwahati-781014, India
Peer review under responsibility of Fuzzy Information and Engineering Branch of the Operations Research Society of China.

http://dx.doi.org/10.1016/j.fiae.2015.03.004

etc. These types of correspondences sense the same of graph and fuzzy set. In 1975, Rosenfeld [19] interpreted the concept of fuzzy graph which has been influencing the researchers gradually. The idea of fuzzy graph automatically touches its relation with fuzzy algebra as in the same of graph and algebra. But before going to this generalized correlation, an intermediate insight arises, i.e., the correspondence between graph and fuzzy algebra. Having taken this intention, we proceed to the correlation of graph and fuzzy algebra. We study characteristics of center of intersection graph of fuzzy submodules of a module and establish its existence by establishing some results relating with corresponding crisp concepts. This intersection graph of fuzzy submodules of a module is an infinite graph.

Now we state some definitions from [4-5, 8-15] and [17], which are needed in the sequel.

Throughout this discussion M is a left module over a ring R, θ is the zero element of M. A fuzzy subset α is a mapping from a set X into [0,1]. For $x \in X$, $\alpha(x)$ is called membership value of x. The collection of all fuzzy subsets of X is denoted by $[0,1]^X$. Let $\alpha, \beta \in [0,1]^X$. Then α is contained in β, if $\alpha(x) \leq \beta(x), \forall x \in X$ and is denoted by $\alpha \subseteq \beta$. The intersection and union of α and β are denoted by $\alpha \cup \beta$ and $\alpha \cap \beta$, are defined as $(\alpha \cup \beta)(x) = \alpha(x) \vee \beta(x)$ and $(\alpha \cap \beta)(x) = \alpha(x) \wedge \beta(x)$, for all $x \in X$, respectively. If $t \in (0,1]$, then the set $\alpha_t = \{x \in X \mid \alpha(x) \geq t\}$ is called level subset of α. Also the set $\alpha^* = \{x \in X \mid \alpha(x) > 0\}$ is called the support of α. The sum of α and β is defined as

$$(\alpha + \beta)(x) = \vee\{\alpha(y) \wedge \beta(z) \mid y + z = x; y, z \in X\}$$

for $x \in X$. Similarly, for $\alpha_i \in [0,1]^X, i \in I$, where I is an index set, $\sum_{i \in I} \alpha_i$ is defined as

$$\sum_{i \in I} \alpha_i(x) = \vee\{\wedge_{i \in I} \alpha_i(x_i) \mid \sum_{i \in I} x_i = x; x_i \in X, \forall i\}$$

for $x \in X$. Now we give the definition of fuzzy point from [16], which is an important one for this discussion. A fuzzy subset α of X of the form

$$\alpha(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. We write $x_t \in \alpha$ if and only if $x \in \alpha_t$. To avoid confusion of fuzzy point with level subset we use the notation $(x)_t$ for fuzzy point in place of x_t. Let α be a member of $[0,1]^R$. Then α is said to be an F-ideal of R if $\alpha(x - y) \geq \alpha(x) \wedge \alpha(y)$ and $\alpha(xy) \geq \alpha(x) \vee (y)$, for x, y in R. Let α, β be two F-ideals of R. We define $\alpha\beta$, which is also an F-ideal of R, as

$$(\alpha\beta)(x) = \vee\{\alpha(y) \wedge \beta(z) \mid yz = x\}$$

for x in R. An F-ideal γ of R is called prime if γ is non-constant and $\alpha\beta \subseteq \gamma$ implies $\alpha \subseteq \gamma$ or $\beta \subseteq \gamma$, for F-ideals α, β of R. Again if α is a member of $[0,1]^M$, then α is said to be an F-submodule of M if $\alpha(\theta) = 1$, $\alpha(x+y) \geq \alpha(x) \wedge \alpha(y)$ and $\alpha(rx) \geq \alpha(x)$, for x, y in M and r in R. The collection of all F-submodules of M is denoted by $F(M)$. Let $\alpha, \beta \in F(M)$. Then α is an F-submodule of β if α is contained in β and we write $\alpha \leq \beta$. In this discussion, the notation $F(\alpha)$ is used for the collection of all F-submodules of α.

An undirected graph G consists of a set $V(G)$ of vertices or points and a collection $E(G)$ of unordered pairs of vertices called edges or lines. If u and v are two vertices of a graph and if unordered pair u, v is an edge denoted by e, we say that e is an edge between u and v or u and v are adjacent. In this case, the vertices u and v are said to be incident on e and e is incident to both u and v. Two or more edges that join the same pair of distinct vertices are called parallel edges. An edge represented by an unordered pair in which the two elements are not distinct is known as a loop. A simple graph is a graph with no parallel edges and loops. In our discussion, all graphs are simple. The degree of a vertex v in a graph is the number of edges incident on that vertex. It is denoted by $deg(v)$. If $deg(v)$ is 0, then v is said to be an isolated vertex of G. The graph $H = (W, F)$ is a subgraph of the graph $G = (V, E)$ if W is a subset of V and F is a subset of E. If $H = (W, F)$ is a subgraph of the graph $G = (V, E)$ such that an edge exists in F between two vertices in W if and only if an edge exists in E between those two vertices, the subgraph H is said to be induced by the set W, which is maximal subgraph of G with respect to the set. A walk in a graph is an alternating sequence of vertices and edges $v_0 x_1 v_1 \cdots x_n v_n$ in which each edge x_i is $v_{i-1} v_i$. A walk is said to be closed if its beginning and ending vertices are same, otherwise open. The length of a walk is n, the number of occurrences of edge in it. A path is a walk in which all vertices are distinct. A closed walk is said to be a circuit or cycle, if all, other than the beginning and ending vertices are distinct. For vertices x and y of G, we define $d(x, y)$ to be the length of any shortest path from x to y. G is said to be connected, if there exists a path between every pair of vertices of it, otherwise it is said to be disconnected. The girth, $girth(G)$ of G is the length of a shortest cycle (if any). The eccentricity $e(v)$ of a vertex v in G is the distance from v to the vertex farthest from v in G. A vertex with minimum eccentricity in G is called a center of G.

Now, we are going to remember the definition of intersection graph from Harary [11]. Let $F = \{S_1, S_2, \cdots, S_p\}$ be a nonempty family of distinct nonempty subsets of a non empty set S such that $S_1 \cup S_2 \cup \cdots = S$. Then the intersection graph of F is denoted by $\Omega(F)$. The vertex set of $\Omega(F)$ is F with two distinct vertices S_i and S_j are adjacent if $S_i \cap S_j \neq \emptyset$. Whenever there is a family F of subsets of S for which a graph G and $\Omega(F)$ are isomorphic, G is an intersection graph on S. An interesting result on intersection graph is-every graph is an intersection graph. Chakrabarty et al. [7] in 2009 introduced intersection graph of ideals of a ring. According to them, the intersection graph of ideals of a ring is an undirected graph with vertex set as the collection of nontrivial ideals of the ring such that any two vertices are adjacent if their intersection is not zero. They studied some enjoyable characteristics of intersection graph of ideals of a ring. This concept is also extended to module. The intersection graph of submodules of a module was discussed by Akbari et al. in [1].

In our discussion, we first define intersection graph of submodules of a module in slightly different way than in [7]. The intersection graph G_M of submodules of M is an undirected graph with vertex set $V(G_M)$ is the collection of all submodules of M and any two distinct $A, B \in V(G_M)$, A and B are adjacent if and only if $A \cap B \neq 0$. In the same sense, the intersection graph $G_{1_M}(= G)$ of $F(M)$ is an undirected graph with $V(G) = F(M)$ and any two distinct $\alpha, \beta \in F(M)$, α and β are adjacent if and only if $\alpha \cap \beta \neq \chi_\theta$, and we write $\alpha\beta$ or α *adj* β. If α and β are not adjacent, we write α *nadj* β. Henceforth, G is the intersection graph of $F(M)$. We consider the symbol G_{1_M} for the intersection graph of $F(M)$ as any fuzzy submodule of M is contained in 1_M, where $1_M(x) = 1$ for all $x \in M$. We say α is an F-submodule of 1_M rather than α is an F-submodule of M. In the same sense, the intersection graph of $F(\alpha)$ is denoted by G_α and so is for crisp concept.

We observe that G contains all the F-submodules of 1_M as vertices rather than the collection of nontrivial ideals of rings, defined by Chakrabarty et al. [7]. This utilization provides the prospect in the interdisciplinary study for graphical aspects for essential fuzzy submodules of modules. Clearly, the submodule χ_θ is an isolated vertex of G. Thus, G is a disconnected graph. But, the induced subgraph $G - \chi_\theta$, which does not contain that the vertex χ_θ, of G is a connected graph, as 1_M is adjacent to every vertex of $G - \chi_\theta$. This observation leads that the eccentricity of 1_M, $e(1_M) = 1$. So, connectedness of $G - \chi_\theta$ concludes 1_M is a center of it. From this onward, the notation $G - \chi_\theta$ stands for the induced subgraph of G which does not contain the vertex χ_θ.

Any undefined terminology is available in [4-5, 8-15] and [17].

2. Center of Intersection Graph of $F(1_M)$

We start this section with the following examples. Consider two fuzzy subsets α and β of 1_R where $R = \{0, 1, 2, \cdots, 11\}$ under addition and multiplication modulo 12 as follows

$$\alpha(x) = \begin{cases} 1, & \text{if } x = 0, \\ t, & \text{if } x \in \{4, 8\}, \\ 0, & \text{if } x \notin \{0, 4, 8\}, \end{cases} \qquad \beta(x) = \begin{cases} 1, & \text{if } x = 0, \\ t, & \text{if } x \in \{2, 4, 6, 8, 10\}, \text{ where } t > 0. \\ 0, & \text{if } x \notin \{0, 2, 4, 6, 8, 10\}. \end{cases}$$

Then $\beta \in V(G_{1_R} - \chi_\theta)$ and $\alpha \in V(G_\beta - \chi_\theta)$. Again consider the vertex $\gamma \in V(G_\beta - \chi_\theta)$ as follows

$$\gamma(x) = \begin{cases} 1, & \text{if } x = 0, \\ t, & \text{if } x = 6, \\ 0, & \text{if } x \notin \{0, 6\}. \end{cases}$$

We observe that α is not a center in $G_\beta - \chi_\theta$ as γ *nadj* α. It can be easily noticed that β is a center in $G_{1_R} - \chi_\theta$ but not α.

The minimum length of all cycles in G is the extremum one. This is established in the immediate succeeding theorem. In [1], an exact result is observed for the concept of crisp set theory. We, in our next establishment, follow the same way.

Theorem 2.1 *If G contains a cycle, then girth$(G) = 3$.*

Proof Toward a contradiction, assume that $girth(G) \geq 4$. Then every pair distinct non-trivial fuzzy submodules of 1_M with non-zero intersection should be comparable. If not, G will contain cycle of length 3, a contradiction. Suppose $\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$ is a path in G, as $girth(G) \geq 4$. Since any two fuzzy submodules are comparable and chain of non-trivial fuzzy submodules of length 2 induces a cycle of length 3 in G, the only possible cases are $\alpha_1 \subseteq \alpha_2, \alpha_3 \subseteq \alpha_2, \alpha_3 \subseteq \alpha_4$ or $\alpha_2 \subseteq \alpha_1, \alpha_2 \subseteq \alpha_3$. From the first case, we get $\alpha_3 \subseteq \alpha_2 \cap \alpha_4$ and hence $\alpha_2 \cap \alpha_4 \neq \chi_\theta$. Thus $\alpha_2 - \alpha_3 - \alpha_4 - \alpha_2$ is a triangle, a contradiction. Similarly, by the second case, $\alpha_2 \subseteq \alpha_1 \cap \alpha_3$, from which we obtain the triangle $\alpha_1 - \alpha_2 - \alpha_3 - \alpha_1$ as $\alpha_1 \cap \alpha_3 \neq \chi_\theta$. Again a contradiction toward the assumption. Therefore $girth(G) = 3$ and this completes the proof.

In [18], Saikia and Kalita introduced the concept of essentiality in fuzzy set theory. The following results show the graphical aspects of their discussion. It is well observed that the center plays the role of essentiality in the corresponding graphical character. We first urge in the correlation of fuzzy set and crisp set, which is clearly noticed in Theorems 2.2, 2.3, 2.5 and 2.6.

Theorem 2.2 *A is a center in $G_M - 0$ if and only if χ_A is a center in $G - \chi_\theta$.*

Theorem 2.3 *If A and B are two submodules of M, then A is a center in $G_B - 0$ if and only if χ_A is a center in $G_{\chi_B} - \chi_\theta$.*

Theorem 2.4 *If $M = R$ is prime (ring) then every vertex of $G - \chi_\theta$ is center.*

Proof If possibly suppose that α is not a center in $G - \chi_\theta$, then there exists a vertex β in $G - \chi_\theta$ with β *nadj* α, then $\alpha\beta \subseteq \chi_\theta$ as $\alpha\beta \subseteq \alpha \cap \beta$ by Theorem 3.1.32 [5]. Also θ is prime since M is prime . This implies that χ_θ is a prime F-ideal of 1_M. But then we get $\alpha = \chi_\theta$ or $\beta = \chi_\theta$, a contradiction. This contradiction implies that α is a center in $G - \chi_\theta$. Hence the theorem is obtained.

Theorem 2.5 *α is a center in $G - \chi_\theta$ if and only if α^* is a center in $G_M - \theta$ for $\alpha \in V(G - \chi_\theta)$.*

Proof First we assume α is a center in $G - \chi_\theta$. Let $A \in V(G_M - \theta)$. Then $\chi_A \in V(G - \chi_\theta)$ and so α *adj* χ_A since α is a center in $G - \chi_\theta$. Then we get $(\alpha \cap \chi_A)(x) > 0$, for some $x \neq \theta$. From this we have $\alpha(x) > 0, \chi_A(x) > 0$ and thus $\alpha^* \cap A \neq \theta$. Hence α^* is a center in $G_M - \theta$. Conversely, suppose that α^* is a center in $G_M - \theta$. Consider a vertex β in $G - \chi_\theta$, then the support $\beta^* \in V(G_M - \theta)$. We show that $\alpha \cap \beta \neq \chi_\theta$. Now α^* is a center in $G_M - \theta$ and so α^* *adj* β^*, i.e., $\alpha^* \cap \beta^* \neq \theta$. This means that $\alpha(x) > 0$ and $\beta(x) > 0$ for some $x \neq \theta$. Thus $\alpha \cap \beta \neq \chi_\theta$ and so the proof is complete.

Theorem 2.6 *α is a center in $G_\beta - \chi_\theta$ if and only if α^* is a center in $G_{\beta^*} - \theta$ for $\alpha, \beta \in V(G - \chi_\theta)$.*

Proof First suppose that α is a center in $G_\beta - \chi_\theta$, then α^* is a non-zero submodule of M, as α is non-zero and also $\alpha^* \subseteq \beta^*$. Let A be a vertex in $G_{\beta^*} - \theta$. We define a fuzzy submodule γ as $\gamma(x) = \beta(x)$, if $x \in A$ and $\gamma(x) = 0$, if $x \notin A$. It is clear that $\gamma^* = A$. Now, $\gamma(x) = \beta(x) > 0$ for some non-zero $x \in A$, as A is non-zero. This implies that γ is non-zero and also $\gamma \subseteq \beta$. But α is a center in $G_\beta - \chi_\theta$ therefore α *adj* γ, i.e., $\alpha \cap \gamma \neq \chi_\theta$. This gives for some non-zero y, we have $\alpha(y) > 0$ and $\gamma(y) > 0$. Thus

$\alpha^* \cap A \neq \theta$, i.e., α^* *adj* A. Hence α^* is a center in $G_{\beta^*} - \theta$. Conversely, let α^* be a center in $G_{\beta^*} - \theta$. Consider a vertex γ in $G_\beta - \chi_\theta$. It is easy to check that γ^* is a vertex in $G_{\beta^*} - \theta$. Since α^* is a center in $G_{\beta^*} - \theta$ and so α^* *adj* γ^*. Using this, it is obtained that α *adj* γ. This concludes that α is a center in $G_\beta - \chi_\theta$. The proof is complete.

Now, we consider the vertices α_n $(n = 2, 3, 4, \cdots)$ in $G - \chi_\theta$, where $M = \mathbb{Z}$ is module over $\mathbb{Z}$, as follows

$$\alpha_n(x) = \begin{cases} 1, & \text{if } x = 0, \\ t, & \text{if } x \in n\mathbb{Z} - \theta, \\ 0, & \text{if } x \notin n\mathbb{Z}, \end{cases}$$

where $t > 0$.

It is easy to check that $\alpha_n^* = n\mathbb{Z}$ is a center in $G_\mathbb{Z} - \theta$. Therefore α_n is a center in $G - \chi_\theta$. But we see that $(\cap n\mathbb{Z})^*$ is not a center in $G_\mathbb{Z} - \theta$. Thus $\cap n\mathbb{Z}$ is not a center in $G - \chi_\theta$.

Theorem 2.7

(a) *If $\alpha \leq \beta \leq 1_M$, then α is a center in $G - \chi_\theta$ if and only if α and β are centers in $G_\beta - \chi_\theta$ and $G - \chi_\theta$, respectively.*

(b) *Let $\beta, \beta' \leq 1_M$. If α and α' are centers in $G_\beta - \chi_\theta$ and $G_{\beta'} - \chi_\theta$, respectively; then so is $\alpha \cap \alpha'$ in $G_{\beta \cap \beta'} - \chi_\theta$.*

(c) *Let $f : M \to N$ be a module homomorphism. If α is center in $G_{1_N} - \theta$, then so is $f^{-1}(\alpha)$ in $G_{1_M} - \theta$.*

Proof (a) First, we assume that α is a center in $G - \chi_\theta$, consider a vertex γ of $G_\beta - \chi_\theta$, then γ is also a vertex of $G - \chi_\theta$. From the assume condition, we have $e(\alpha) = 1$ in $G - \chi_\theta$. Using this, we get $d(\alpha, \gamma) = 1$, and so $e(\alpha) = 1$ in $G_\beta - \chi_\theta$. Hence α is a center in $G_\beta - \chi_\theta$. Again, if we consider a vertex δ of $G - \chi_\theta$, then α *adj* δ. This immediately implies that β *adj* δ, and so $e(\beta) = 1$ in $G - \chi_\theta$. Thus β is also a center in $G - \chi_\theta$. Conversely, we suppose that α and β are centers in $G_\beta - \chi_\theta$ and $G - \chi_\theta$, respectively. Now, for a vertex η of $G - \chi_\theta$, we have β *adj* η, as β is a center in $G - \chi_\theta$. Then, we notice that α *adj* η, since α is a center in $G_\beta - \chi_\theta$. From this, it is observed that $e(\alpha) = 1$ in $G - \chi_\theta$. Thus, α is a center in $G - \chi_\theta$. This completes the proof.

(b) Assume that α and α' are centers in $G_\beta - \chi_\theta$ and $G_{\beta'} - \chi_\theta$, respectively. By Theorem 2.6, we have α^* and α'^* are centers in $G_{\beta^*} - \theta$ and $G_{\beta'^*} - \theta$, respectively. Also using Theorem 1.1 (b) [10], $\alpha^* \cap \alpha'^*$ is a center in $G_{\beta^* \cap \beta'^*} - \theta$. Again by Theorem 2.6, it follows that $\alpha \cap \alpha'$ is a center in $G_{\beta \cap \beta'} - \chi_\theta$. The proof is complete.

(c) It can be easily verified that $f^{-1}(\alpha^*) = (f^{-1}(\alpha))^*$. Suppose that α is a center in $G_{1_N} - \chi_\theta$. It follows from that Theorem 2.6 that α^* is a center in $G_N - \theta$. By Theorem 1.1 (c) [10], $f^{-1}(\alpha^*)$ is a center in $G_M - \theta$. Thus again by Theorem 2.6 we get $f^{-1}(\alpha)$ is a center in $G_{1_M} - \theta$. Hence the result is obtained.

The following example asserts that the converse of the preceding theorem is not true. Consider the ring $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ under addition and multiplication modulo 8 and the mapping $f : R \longrightarrow R$ is defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \text{ and } x \text{ is even}, \\ 4, & \text{if } x \text{ is odd}. \end{cases}$$

It is easy to check that f is a module homomorphism. Consider the vertex α in $G_{1_R} - \chi_\theta$ with $\alpha(0) = 1, \alpha(4) = s, \alpha(2) = \alpha(6) = t, \alpha(x) = 0$; otherwise, where $s, (\geq) t \in (0, 1]$. Clearly, $\alpha^* = \{0, 2, 4, 6\}$ is a center in $G_R - 0$. Thus by Theorem 2.6, α is a center in $G_{1_R} - \chi_0$. But $f(\alpha)$ is not a center in $G_{1_R} - \chi_0$.

For the continuation of our discussion, we give the definition of disconnected set of vertices of G.

Definition 2.1 *A set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of vertices of G is said to be a disconnected set of vertices if α_i nadj $(\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_n)$ for every $i \in \{1, 2, \cdots, n\}$.*

Theorem 2.8 *If $\{\alpha_i\}_{i=1}^n$ is a disconnected set of vertices of G, and α_i is a center in $G_{\beta_i} - \chi_\theta$ for each $i \in \{1, 2, \cdots, n\}$, then $\{\beta_i\}_{i=1}^n$ is also a disconnected set and $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is a center in $G_{\beta_1+\beta_2+\cdots+\beta_n} - \chi_\theta$.*

Proof First we show that the theorem is true for $n = 2$. We consider a disconnected set $\{\alpha_1, \alpha_2\}$ of vertices of G such that α_1 and α_2 are centers in $G_{\beta_1} - \chi_\theta$ and $G_{\beta_2} - \chi_\theta$, respectively. Using Theorem 2.6, α_1^* and α_2^* are centers in $G_{\beta_1^*} - \theta$ and $G_{\beta_2^*} - \theta$, respectively. Again by Theorem 2.3 (b) [10], we have $\alpha_1^* \cap \alpha_2^*$ is a center in $G_{\beta_1^* \cap \beta_2^*} - \theta$. By the given condition, $\alpha_1 \cap \alpha_2 = \chi_\theta$ which concludes that $\alpha_1^* \cap \alpha_2^* = \theta$. Thus $\beta_1^* \cap \beta_2^* = \theta$ and from this we see that $\{\beta_1, \beta_2\}$ is also a disconnected set of vertices.

Now take the projection maps $\pi : \beta_1^* + \beta_2^* \rightarrow \beta_1^*$ and $\eta : \beta_1^* + \beta_2^* \rightarrow \beta_2^*$. With the help of Theorem 2.3 (c) [10], it can be easily seen that $\pi^{-1}(\alpha_1^*)$ $(= \alpha_1^* + \beta_2^*)$ and $\eta^{-1}(\alpha_2^*)$ $(= \alpha_2^* + \beta_1^*)$ are centers in $G_{\beta_1^*+\beta_2^*} - \theta$. Observe that α_1^* *nadj* β_2^* and α_2^* *nadj* β_1^*. Again using Theorem 2.3 (b) [10], we get $\alpha_1^* + \alpha_2^*$ is a center in $G_{\beta_1^*+\beta_2^*} - \theta$. Hence by Theorem 3.3.7 [17], $(\alpha_1 + \alpha_2)^*$ is a center in $G_{(\beta_1+\beta_2)^*} - \theta$. Observe that α_1 *nadj* α_2 and β_1 *nadj* β_2. Next assume that the theorem is true for $n-1$, then $\{\beta_1, \beta_2, \cdots, \beta_{n-1}\}$ is a disconnected set of vertices and $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$ is a center in $G_{\beta_1+\beta_2+\cdots+\beta_{n-1}} - \chi_\theta$. Now by the above case it is clear that $(\beta_1 + \beta_2 + \cdots + \beta_{n-1})$ *nadj* β_n and $(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) + \alpha_n$ is a center in $G_{(\beta_1+\beta_2+\cdots+\beta_{n-1})+\beta_n} - \chi_\theta$. Hence $\{\beta_i\}_{i=1}^n$ is also a disconnected set and $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is a center in $G_{\beta_1+\beta_2+\cdots+\beta_n} - \chi_\theta$. The proof is complete.

Now we define maximal non-adjacent (MNA) vertex of G.

Definition 2.2 *Let $\alpha \in V(G)$. Then $\beta \in V(G)$ is said to be the MNA vertex if β is maximal with respect to the property α nadj β.*

We observe the role of MNA vertex in our discussion. It is noticed that the complemented fuzzy submodule takes necessary role in the concept of essential fuzzy submodules of modules in [18]. We first give the definition of complemented intersection graph to study its graphical aspects.

Definition 2.3 *Let $\alpha \in V(G)$. Then a vertex $\beta \in V(G)$ is said to be a complement of α if α nadj β and $\alpha + \beta = 1_M$. G is said to be a complemented graph if every vertex of G has a complement.*

χ_θ and 1_M are the trivial complements of G.

Theorem 2.9 *For any $\alpha \in V(G)$, there is a non-adjacent vertex γ to α such that $\alpha + \gamma$ is a center in $G - \chi_\theta$.*

Proof We consider that α is a non-zero F-submodule of 1_M. Let $\mathfrak{F} = \{\beta \leq 1_M \mid \alpha \cap \beta = \chi_\theta\}$. Clearly, $\mathfrak{F} \neq \phi$. By Zorn's lemma $\mathfrak{F}$ has a maximal element γ (MNA vertex), say, with respect to $\alpha \cap \gamma = \chi_\theta$. Thus we obtain a non-adjacent vertex γ to α. Now, we show that $\alpha + \gamma$ is a center in $G - \chi_\theta$. Suppose $\alpha + \gamma$ is not a center in $G - \chi_\theta$, this means $e(\alpha + \gamma) > 1$, then, there is a non-zero $\delta \in V(G)$ such that *$(\alpha + \gamma)$ nadj δ*, and this gives *α nadj $(\gamma + \delta)$*. But, maximality of γ with respect to $\alpha \cap \gamma = \chi_\theta$ implies that $\gamma + \delta = \gamma$. Therefore, we get $\delta = \delta \cap (\alpha + \gamma) = \chi_\theta$, which is absurd. Hence the theorem is obtained.

In the same way, the following theorem can be obtained.

Theorem 2.10 *For any $\alpha \in V(G_\beta)$, there is a non-adjacent vertex γ to α such that $\alpha + \gamma$ is a center in $G_\beta - \chi_\theta$.*

Theorem 2.11 *If G is a complemented graph, then so is G_α for any F-submodule α of 1_M.*

Proof Let $\beta \in V(G_\alpha)$. Then $\beta' \cap \alpha$ is a complement of β in G_α, where β' is a complement of β in G. The proof is complete.

Remark 1 [2] A fuzzy submodule α of 1_M is called maximal if α is a maximal element in the set of all non-constant fuzzy submodules of 1_M, with respect to the set inclusion.

Remark 2 [18] Let α, β, γ be three fuzzy submodules of 1_M. If $\beta \subseteq \gamma$, then $\gamma \cap (\alpha + \beta) = (\gamma \cap \alpha) + \beta$.

Theorem 2.12 *Let α be a vertex in G. If G_α is a complemented graph, then there is a non-trivial maximal F-submodule of α which is not a center in $G_\alpha - \chi_\theta$.*

Proof It is sufficient to show that for $t \in (0, 1]$ and for each $(x)_t (\neq \chi_\theta) \notin \alpha$, we have a maximal fuzzy submodule μ of α such that $(x)_t \notin \mu$. We consider $(x)_t \notin \alpha$. Let $\mathfrak{F} = \{\beta \mid \beta \in F(\alpha), (x)_t \notin \beta\}$. Then $\mathfrak{F} \neq \phi$. By Zorn's lemma, $\mathfrak{F}$ has a maximal element μ (say) with $(x)_t \notin \mu$. We show that μ is a maximal F-submodule of α. Let $\mu_t \subseteq \gamma_t \subseteq \alpha_t$. Since $\gamma_t \subseteq \alpha_t$, so $\gamma \subseteq \alpha$. As $F(\alpha)$ is complemented, therefore there exists $\gamma' \in F(\alpha)$ with $\alpha = \gamma + \gamma'$ and γ *nadj* γ'. Now

$$
\begin{aligned}
\gamma \cap (\mu + \gamma') &= \mu + (\gamma \cap \gamma') \\
&= \mu + \chi_\theta \\
&= \mu.
\end{aligned}
$$

Thus $(x)_t \notin \mu$ implies either $(x)_t \notin \gamma$ or $(x)_t \notin (\mu + \gamma')$. If $(x)_t \notin \gamma$, then $\gamma = \mu$, as μ is maximal with $(x)_t \notin \mu$. So $\gamma_t = \mu_t$. Also if $(x)_t \notin (\mu + \gamma')$, then $\mu + \gamma' = \mu$. This gives $\mu_t + \gamma'_t = \mu_t$. Therefore $\alpha = \gamma + \gamma'$ gives $\alpha_t = \gamma_t$. Thus μ is maximal with $(x)_t \notin \mu$. From this, we get that there exists a maximal F-submodule μ of α with $(x)_t \notin \mu$ if $(x)_t(\neq \chi_\theta) \in \alpha$. We observe that $\bigcap\{\mu \mid \mu$ is a maximal submodule of $\alpha\} = \chi_\theta$. Hence the theorem is obtained.

The uniqueness character of center in intersection graph follows from the immediate succeeding theorem. Before this, we state the definition of simple F-submodule from [18].

Definition 2.4 *An F-submodule ν of M is said to be simple F-submodule if $\mu \subseteq \nu$, where $\mu \in F(M)$ implies either $\mu = \chi_\theta$ or $\mu = \nu$.*

Theorem 2.13 *If 1_M is the sum of simple F-submodules of 1_M, then 1_M is the only center in $G - \chi_\theta$.*

Proof If possible, we assume that α is a center in $G - \{\chi_\theta\}$ which is distinct from 1_M. But 1_M is the sum of simple F-submodules of 1_M, let $\{\alpha_i\}_{i \in I}$ be the collection of all simple F-submodules of 1_M. Then $1_M = \sum_i \alpha_i$. Since $e(M) = 1$, therefore α *adj* α_i for every i. As for every i, α_i is a simple F-submodule of 1_M, thus $\alpha \cap \alpha_i \leq \alpha_i$ gives $\alpha \cap \alpha_i = \alpha_i$. That means α contains all simple F-submodules of 1_M. From this $1_M \leq \alpha$, which is absurd. The proof is complete.

Theorem 2.14 *If 1_M is the only center in $G - \chi_\theta$, then the intersection of maximal proper F-submodules of 1_M is χ_θ.*

Proof Let $\alpha \leq 1_M$. Then by Theorem 2.9, there is a non-adjacent vertex β with $\alpha + \beta$ is a center in G. From the given condition $\alpha + \beta = 1_M$. This means that β is a complement of α. Thus G is a complemented graph. Now, following the same way of Theorem 2.12, we get the result.

Theorem 2.15 *If 1_M is the only center in $G - \chi_\theta$, then 1_M is the sum of all F-submodules of 1_M.*

Proof Since 1_M is the only center in $G - \chi_\theta$, therefore, from proof of Theorem 2.8, G is a complemented graph. Let $\{\alpha_i\}_{i \in I}$ be the collection of all simple F-submodules of 1_M. Then $\sum_i \alpha_i \leq 1_M$ gives that there is a vertex β such that $\sum_i \alpha_i$ *nadj* β and $\sum_i \alpha_i + \beta = 1_M$. Then, we have G_β is complemented. Therefore, using Theorem 2.8, we can show that 1_M is the sum of all F-submodules of 1_M. This completes the proof.

The next definition is nothing but the graphical aspect of closed submodule of module of crisp concept.

Definition 2.5 *A vertex α of G is said to be an isolated center if α is a center only in $G_\alpha - \chi_\theta$, i.e., if α is a center in $G_\beta - \chi_\theta$, then $\alpha = \beta$.*

Theorem 2.16 *If α nadj β with $\gamma = \alpha + \beta$, then α is an isolated center in G_γ.*

Proof Towards a contradiction assume that α is not an isolated center in G_γ. So we have a vertex α' in $G_\gamma - \chi_\theta$ with α is a center in $G_{\alpha'} - \chi_\theta$. Then it can be easily obtained that $(\alpha' \cap \beta)$ *nadj* α, which gives α' *nadj* β. Now $\gamma = \alpha + \beta \subseteq \alpha' + \beta \subseteq \gamma$ and so $\alpha' + \beta = \gamma(= \alpha + \beta)$. Observe that α_t *nadj* β_t for any $t > 0$. Also it is easy to see that $(\alpha + \beta)_t = \alpha_t + \beta_t$. Our claim is $\alpha = \alpha'$, and for this we assume $(x)_t \in \alpha'$, $t \neq 0$, i.e., $\alpha'(x) \geq t$. Now, if $x + y = z + w$, where $y, w \in \beta_t, z \in \alpha_t$, then we notice that $x - z = \{\theta\}$. Thus $x = z \in \alpha_t$, so we have $(x)_t \in \alpha$. Next if $\alpha'(x) = p(\neq 0)$, then $x_p \in \alpha'$ and this gives $\alpha(x) = p$. Also it is well known that $(\alpha'^{*})^c \subseteq (\alpha^{*})^c$, as $\alpha \subseteq \alpha'$. Observe that $(\alpha^{*})^c = \{x \in M \mid \alpha(x) = 0\}$. If $\alpha'(y) = 0$, then we obtain $y \in (\alpha^{*})^c$. Therefore $\alpha(y) = 0$ and hence $\alpha = \alpha'$. Thus the proof is complete.

Theorem 2.17 *If β is an MNA vertex for α in G, then β is an isolated center in G.*

Proof Suppose β is a center in $G_{\beta'} - \{\chi_\theta\}$, where $\beta' \in V(G)$. Since β is an MNA vertex for α in G, therefore α *nadj* β'. This implies that $(\alpha \cap \beta')$ *nadj* β. As β is a center, so α *nadj* β'. But β is an MNA vertex gives $\beta' = \beta$. The proof is complete.

3. Conclusion

In this paper, we observe a new direction that is the graphical aspects of fuzzy algebra. The center plays a vital role in our discussion. In fact, the concept of essential submodule plays this role in graphical characterizations.

Acknowledgments

Authors would like to thank the referees for their helpful comments.

References

[1] S. Akbari, H.A. Tavallaee, S.K. Ghezelahmad, Intersection graph of submodules of a module, Journal of Algebra and Its Applications 10(5) (2011) 1-8.

[2] R. Ameri, R. Mahjoob, Spectrum of prime L-submodules, Fuzzy Sets and Systems 159 (2008) 1107-1115.

[3] D.F. Anderson, A. Badawi, The total graph of a commutative ring, Journal of Algebra 320 (2008) 2706-2719.

[4] M.F. Atiyah, I.G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, London, 1969.

[5] R. Balkrishnan, K. Ranganathan, A Text Book of Graph Theory, Springer-verlag New York, Inc. Reprint, 2008.

[6] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988) 208-226.

[7] I. Chakrabarty, S. Ghosh, T.K. Mukherjee, M.K. Sen, Intersection graphs of ideals of rings, Discrete Math. 309 (17) (2009) 5381-5392.

[8] A.W. Chatters, C.R. Hazarnavis, Rings with Chain Condition, Pitman Advanced Publishing Program, 1980.

[9] N. Deo, Graph Theory with Applications to Engineering and Computer Science, Prentice-Hall of India Private Limited, New Delhi, 2003.

[10] K.R. Goodearl, Ring Theory, Marcel Dekker, 1976.

[11] F. Harary, Graph Theory, Addison-Wesley Publishing Company, Inc., Reading, Mass. 1969.

[12] J.A. Huckaba, Commutative Rings with Zero-divisors, Marcel-Dekker, New York, Basel, 1988.

[13] J. Lambeck, Lectures on Rings and Modules, Blaisdell Publishing Company, Waltham, Toronto, London, 1966.

[14] I. Kaplansky, Commutative Rings, Revised Edition, University of Chicago Press, Chicago, 1974.

[15] R. Kumar, Fuzzy Algebra, University of Delhi Publication Division, 1993.

[16] P.M. Pu, L.M. Liu, Fuzzy topology I: Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571-599.

[17] J.N. Mordeson, D.S. Malik, Fuzzy Commutative Algebra, World Scientific Publishing, Singapore, 1998.

[18] K.K. Rajkhowa, H.K. Saikia, On fuzzy socle of modules, Jour. Pure Math. 25 (2008) 51-56.

[19] A. Rosenfeld, Fuzzy graphs, in: L.A. Zadeh, K.S. Fu, K. Tanaka, M. Shimura (Eds.), Fuzzy Sets and Their Applications to Cognitive and Decision Processes, New York, Academic Press, 1975, pp.77-95.

[20] H.K. Saikia, M.C. Kalita, On fuzzy essentials submodules, The Journal of Fuzzy Mathematics 17(1) (2009) 109-117.

[21] P.K. Sharma, S.M. Bhatwadekar, A note on graphical representation of rings, Journal of Algebra 176 (1995) 124-127.