A note on optimal point distributions in $[0, 1)^s$

Gerhard Larcher, Friedrich Pillichshammer,*1

Institut für Finanzmathematik, Universität Linz, Altenbergerstraße 69, A-4040 Linz, Austria

Received 1 June 2006; received in revised form 10 August 2006

Abstract

In this note we determine the infimum of the $L_2$-, the star- and the extreme discrepancy taken over all 2-element point sets in the $s$-dimensional unit cube. Moreover we give very good bounds on the infimum of the isotropic discrepancy taken over all $(s + 1)$-element point sets in $[0, 1)^s$.

© 2006 Elsevier B.V. All rights reserved.

MSC: 11K38; 52A40

Keywords: $L_2$-, star, extreme- and isotropic discrepancy

1. Introduction

In [6] the authors investigate where to place a point $x$ in the $s$-dimensional unit cube $[0, 1)^s$ in order to minimize the $L_2$-discrepancy $L(\mathcal{P})$ and to minimize the star discrepancy $D^*(\mathcal{P})$ of the point set $\mathcal{P}$ consisting of the single point $x$. They used their results for a quick testing of programs for the calculation of the discrepancy of a point set.

In this note we extend these investigations and we will give optimal values for $L(\mathcal{P})$, $D^*(\mathcal{P})$ and also for the extreme discrepancy $D(\mathcal{P})$ for point sets $\mathcal{P}$ consisting of two points $x$ and $y$ in $[0, 1)^s$.

We remind the definitions of these classical distribution measures, see also [1,3,5].

For a point set $\mathcal{P}$ of $N$ points in $[0, 1)^s$ and a subset $Q$ of $[0, 1)^s$ let $A_N(Q)$ denote the number of points of $\mathcal{P}$ in $Q$ and by $\lambda(Q)$ we denote the $s$-dimensional volume of $Q$. Then the $L_2$-discrepancy of $\mathcal{P}$ is given by

$$L(\mathcal{P}) := \left( \frac{1}{N^2} \int_{[0,1]^s} \left( A_N \left( \prod_{i=1}^s [0, u_i) \right) - N \prod_{i=1}^s u_i \right)^2 \, du_1 \ldots du_s \right)^{1/2}$$

the star discrepancy of $\mathcal{P}$ is given by

$$D^*(\mathcal{P}) = \sup_B \left| \frac{A_N(B)}{N} - \lambda(B) \right|,$$
where the supremum is taken over all boxes $B$ in $[0, 1)^s$ with sides parallel to the axes and one corner in the origin, and the extreme discrepancy of $\mathcal{P}$ is given by

$$D(\mathcal{P}) = \sup_B \left| \frac{A_N(B)}{N} - \lambda(B) \right|,$$

where the supremum is taken over all boxes $B$ in $[0, 1)^s$ with sides parallel to the axes.

A further important distribution measure which we consider here is the isotropic discrepancy of $\mathcal{P}$ given by

$$J(\mathcal{P}) = \sup_C \left| \frac{A_N(C)}{N} - \lambda(C) \right|,$$

where the supremum is taken over all convex subsets $C$ of $[0, 1)^s$.

We will determine explicitly

$$L_2^{(s)} := \inf_{\mathcal{P}} L(\mathcal{P}), \quad D_2^{(s)} := \inf_{\mathcal{P}} D^*(\mathcal{P}) \quad \text{and} \quad D_2^{(s)} := \inf_{\mathcal{P}} D(\mathcal{P}),$$

where the infimum is taken over all point sets $\mathcal{P} = \{x, y\}$ of two points in $[0, 1)^s$.

We note here that the exact values for the minimal star discrepancy of $N \in \{1, 2, \ldots, 6\}$ points in the two-dimensional unit square have been given in Ref. [7].

The determination of $\inf_{\mathcal{P}} J(\mathcal{P})$ for 2-point sets is trivial, since $J(\mathcal{P}) = 1$ for every point set $\mathcal{P}$ in $[0, 1)^s$ with $|\mathcal{P}| \leq s$. This can be seen easily by considering an $(s - 1)$-dimensional hyperplane containing $\mathcal{P}$. So the first non-trivial object to study is

$$J_{s+1}^{(s)} := \inf_{\mathcal{P}} J(\mathcal{P}),$$

where the infimum is taken over all point sets $\mathcal{P}$ of $s + 1$ points in $[0, 1)^s$. We will determine $J_{s+1}^{(2)}$ explicitly, and we will give good bounds for $J_{s+1}^{(s)}$ in general. It certainly will be very hard to determine $J_{s+1}^{(s)}$ for $s \geq 3$ explicitly.

2. Results and proofs

First we consider the problem of minimizing the $L_2$-discrepancy of a 2-element point set.

**Theorem 1.** We have

$$L_2^{(s)} = \left( \frac{1}{3^s} - \frac{(1 - \zeta_s)^{s+1}}{8\zeta_s} - 3 \frac{(1 - \eta_s)^{s+1}}{8\eta_s} \right)^{1/2},$$

where $\zeta_s$ is the unique positive real solution of $x(1 + x)^{s-1} - 2^{s-3} = 0$ and where $\eta_s$ is the unique positive real solution of $y(1 + y)^{s-1} - 3 \cdot 2^{s-3} = 0$.

**Proof.** As in [6, Proof of Theorem 4] we use the well-known formula for the $L_2$-discrepancy of a point set (see, for example, [4]) which states that for $\mathcal{P} = \{x_0, \ldots, x_{N-1}\}$ in $[0, 1)^s$ the squared $L_2$-discrepancy is given by

$$L^2(\mathcal{P}) = \frac{1}{3^s} - \frac{2^{1-s}}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{s} (1 - x_{n,i}^2) + \frac{1}{N^2} \sum_{n,m=0}^{N-1} \prod_{i=1}^{s} (1 - \max\{x_{n,i}, x_{m,i}\}),$$

where $x_{n,i}$ is the $i$th component of the point $x_n$. If $\mathcal{P}$ consists only of two points $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$, this formula reduces to

$$L^2(\mathcal{P}) = \frac{1}{3^s} - \frac{1}{2^s} \left( \sum_{i=1}^{s} (1 - x_i^2) + \sum_{i=1}^{s} (1 - y_i^2) \right) + \frac{1}{4} \left( \sum_{i=1}^{s} (1 - x_i) + 2 \sum_{i=1}^{s} (1 - \max\{x_i, y_i\}) + \sum_{i=1}^{s} (1 - y_i) \right).$$
Now we minimize the function
\[
    f(x, y) := -\frac{1}{2s} \left( \prod_{i=1}^{s} (1 - x_i^2) + \prod_{i=1}^{s} (1 - y_i^2) \right) + \frac{1}{4} \left( \prod_{i=1}^{s} (1 - x_i) + 2 \prod_{i=1}^{s} (1 - \max[x_i, y_i]) + \prod_{i=1}^{s} (1 - y_i) \right).
\]

We proceed in an analogous manner as the authors did in [6]. First one assumes that the infimum is reached if the points \( x \) and \( y \) are in the interior of the unit cube. In this case all partial derivatives \( \partial f(x, y) / \partial x_i \) and \( \partial f(x, y) / \partial y_i \) need to be zero for \( 1 \leq i \leq s \). We first show, that we always can assume \( x_i \leq y_i \) for all \( 1 \leq i \leq s \).

Assume that the points \( x \) and \( y \) for which the minimum is reached have \( \rho \in \{0, 1, \ldots, s\} \) equal components, w.l.o.g. \( x_1 = y_1, \ldots, x_\rho = y_\rho, x_\rho+1 \neq y_\rho+1, \ldots, x_s \neq y_s \).

Especially for all \( \rho + 1 \leq j \leq s \) we must have
\[
    \frac{\partial f(x, y)}{\partial x_j} = \frac{2x_j}{2s} \prod_{i \neq j} (1 - x_i^2) - \frac{1}{4} \prod_{i=1}^{\rho} (1 - x_i) \left( \prod_{i=\rho+1}^{s} (1 - x_i) + 2 \cdot 1_{(y_j, 1)}(x_j) \prod_{i=\rho+1}^{s} (1 - \max[x_i, y_i]) \right) = 0
\]
and
\[
    \frac{\partial f(x, y)}{\partial y_j} = \frac{2y_j}{2s} \prod_{i \neq j} (1 - y_i^2) - \frac{1}{4} \prod_{i=1}^{\rho} (1 - y_i) \left( \prod_{i=\rho+1}^{s} (1 - y_i) + 2 \cdot 1_{(x_j, 1)}(y_j) \prod_{i=\rho+1}^{s} (1 - \max[x_i, y_i]) \right) = 0.
\]

Hence, for \( \rho + 1 \leq j < k \leq s \) we obtain
\[
    x_j = \frac{2^{s-3}}{\prod_{i=1}^{\rho}(1 + x_i)} \left[ \frac{1}{\prod_{i \neq j}^{s}(1 + y_i)} + 2 \cdot 1_{(y_j, 1)}(x_j) \prod_{i=\rho+1}^{s} (1 - \max[x_i, y_i]) \right]
\]
and (note that \( x_i = y_i \) for \( i \in \{1, \ldots, \rho\} \))
\[
    y_j = \frac{2^{s-3}}{\prod_{i=1}^{\rho}(1 + y_i)} \left[ \frac{1}{\prod_{i \neq j}^{s}(1 + y_i)} + 2 \cdot 1_{(x_j, 1)}(y_j) \prod_{i=\rho+1}^{s} (1 - \max[x_i, y_i]) \right].
\]

Assume now that there are \( j, k \in \{\rho + 1, \ldots, s\} \) such that \( x_j < y_j \) and \( x_k < y_k \). Then we have
\[
    x_j = \frac{2^{s-3}}{\prod_{i=1}^{\rho}(1 + x_i) \prod_{i \neq j}^{s}(1 + x_i)} \quad \text{and} \quad x_k = \frac{2^{s-3}}{\prod_{i=1}^{\rho}(1 + x_i) \prod_{i \neq k}^{s}(1 + x_i)}
\]
and it follows that
\[
    x_j (1 + x_k) = \frac{2^{s-3}}{\prod_{i \neq [j, k]} (1 + x_i)} = x_k (1 + x_j).
\]
This implies \( x_j = x_k \).
Further we have

\[
y_j = \frac{2^{s-3}}{\prod_{i=1}^{\rho} (1 + x_i)} \left[ \frac{1}{\prod_{i=\rho+1}^{s} (1 + y_i)} + 2 \cdot \frac{1 - \max\{x_i, y_i\}}{1 - y_i^2} \right]
\]

and

\[
y_k = \frac{2^{s-3}}{\prod_{i=1}^{\rho} (1 + x_i)} \left[ \frac{1}{\prod_{i=\rho+1}^{s} (1 + y_i)} + 2 \cdot \frac{1 - \max\{x_i, y_i\}}{1 - y_i^2} \right].
\]

Then

\[
y_j(1 + y_k) = \frac{2^{s-3}}{\prod_{i=1}^{\rho} (1 + x_i)} \left[ \frac{1}{\prod_{i=\rho+1}^{s} (1 + y_i)} + 2 \cdot \frac{1 - \max\{x_i, y_i\}}{1 - y_i^2} \right]
\]

where

\[
y_k(1 + y_j).
\]

Again we obtain \( y_j = y_k \).

In the same way we obtain \( x_j = x_k \) and \( y_j = y_k \) if \( x_j > y_j \) and \( x_k > y_k \). Therefore \((x_{\rho+1}, \ldots, x_s)\) has at most two different components \( x \) and \( \overline{x} \) and also \((y_{\rho+1}, \ldots, y_s)\) has at most two different components \( y \) and \( \overline{y} \).

Let \( x < y \) and \( \overline{x} > \overline{y} \) and let \( k \in \{0, \ldots, s - \rho\} \) be the number of components of \((x_{\rho+1}, \ldots, x_s)\) equal to \( x \) (this is of course also the number of components of \((y_{\rho+1}, \ldots, y_s)\) equal to \( y \)). Then

\[
f(x, y) = -\frac{1}{2^s} \prod_{i=1}^{\rho} (1 - x_i^2)((1 - x_i^2)^k(1 - x_i^2)^{s - \rho - k} + (1 - y_i^2)^k(1 - y_i^2)^{s - \rho - k})
\]

\[
+ \frac{1}{4} \prod_{i=1}^{\rho} (1 - x_i)((1 - x_i)^k(1 - \overline{x})^{s - \rho - k} + (1 - y_i)^k(1 - \overline{y})^{s - \rho - k})
\]

\[
+ \frac{1}{2} \prod_{i=1}^{\rho} (1 - x_i)(1 - y_i)^k(1 - \overline{x})^{s - \rho - k}
\]

\[
= g(x, \overline{x}) + g(y, \overline{y}) + \frac{1}{2} \prod_{i=1}^{\rho} (1 - x_i)(1 - y_i)^k(1 - \overline{x})^{s - \rho - k},
\]

where

\[
g(a, b) := -\frac{1}{2^s} \prod_{i=1}^{\rho} (1 - x_i^2)(1 - a^2)^k(1 - b^2)^{s - \rho - k} + \frac{1}{4} \prod_{i=1}^{\rho} (1 - x_i)(1 - a)^k(1 - b)^{s - \rho - k}.
\]

Considering the partial derivatives of the function \( g \) one can show that \( g(a, b) \) can only be minimal for \( a = b \).

Therefore, the minimum of \( f(x, y) \) can only be attained for the points \( x = (x_1, \ldots, x_{\rho}, x, \ldots, x) \) and \( y = (x_1, \ldots, x_{\rho}, y, \ldots, y) \). W.l.o.g. assume \( x < y \).

Hence, it follows that the minimum of \( f(x, y) \) can only be attained for the points \( x = (x_1, \ldots, x_s) \) and \( y = (y_1, \ldots, y_s) \) with \( x_i \leq y_i \) for all \( i \in \{1, \ldots, s\} \). In this case our function \( f(x, y) \) becomes

\[
f(x, y) = -\frac{1}{2^s} \left( \prod_{i=1}^{s} (1 - x_i^2) + \prod_{i=1}^{s} (1 - y_i^2) \right) + \frac{1}{4} \left( \prod_{i=1}^{s} (1 - x_i) + 3 \prod_{i=1}^{s} (1 - y_i) \right).
\]
We can now proceed as before. Setting the partial derivatives of \( f \) zero we find that we must have \( x_1 = \cdots = x_s =: x \) and \( y_1 = \cdots = y_s =: y \) where
\[
x = \frac{1}{8} \frac{2^s}{(1 + x)^{s-1}} \quad \text{and} \quad y = \frac{3}{8} \frac{2^s}{(1 + y)^{s-1}}.
\]
In this case we have
\[
L^2(\mathcal{P}) - \frac{1}{3^s} = \frac{1}{4} (1 - x)^s - \frac{1}{2^s} (1 - x^2)^s + \frac{3}{4} (1 - y)^s - \frac{1}{2^s} (1 - y^2)^s
\]
\[
= (1 - x)^s \left( \frac{1}{4} - \frac{1}{2^s} (1 + x)^s \right) + (1 - y)^s \left( \frac{3}{4} - \frac{1}{2^s} (1 + y)^s \right)
\]
\[
= (1 - x)^s \left( \frac{1}{4} - \frac{1}{2^s} \frac{2^s (1 + x)}{8x} \right) + (1 - y)^s \left( \frac{3}{4} - \frac{1}{2^s} 3 \cdot 2^s (1 + y) \frac{8y}{8y} \right)
\]
\[
= - \left( \frac{(1 - x)^{s+1}}{8x} + 3 \frac{(1 - y)^{s+1}}{8y} \right).
\]
Finally, we show, that the squared \( L_2 \)-discrepancy of a 2-element point set in \([0, 1]^s\) with at least one of the points on the boundary of the unit cube is not smaller than this value.

Assume that both points have a component which is equal to 1. Then we have \( L^2(\mathcal{P}) = 1/3^s \) and thus this cannot give a minimum.

Assume that only one point, say \( y \), has a component which is equal to 1 and that no component of \( x \) is zero. Then we find that
\[
L^2(\mathcal{P}) = \frac{1}{3^s} - \frac{1}{2^s} \prod_{i=1}^s (1 - x_i^2) + \frac{1}{4} \prod_{i=1}^s (1 - x_i).
\]
In the same way as above this becomes minimal for \( x = (x, x, \ldots, x) \) with \( x = (2^s/8) (1/(1 + x)^{s-1}) \). Inserting this in the formula for the \( L_2 \)-discrepancy we obtain
\[
L^2(\mathcal{P}) - \frac{1}{3^s} = - \frac{(1 - x)^{s+1}}{8x}
\]
and thus this cannot give a minimum.

If one component of \( x \) is zero, say \( x_j \) and the other components are not 1, then \( \partial_x f(x, y)/\partial x_j < 0 \) and thus this cannot give a minimum. If \( x \) has zero and one components, then
\[
L^2(\mathcal{P}) = \frac{1}{3^s} - \frac{1}{2^s} \prod_{i=1}^s (1 - y_i^2) + \frac{1}{4} \prod_{i=1}^s (1 - y_i).
\]
Again here we cannot attain a minimum.

Similar for \( y \). \( \square \)

In the sequel we use the following notation: for sequences \((a_s)_{s \geq 1}\) and \((b_s)_{s \geq 1}\) we write \( a_s \sim b_s \) if \( a_s/b_s \to 1 \) as \( s \to \infty \).

**Corollary 2.** We have \( L_2^{(s)} \sim 3^{-s/2} \).

**Proof.** This follows from Theorem 1 and some simple calculations. \( \square \)

We turn to the star discrepancy.
Theorem 3. We have
\[ D_2^{\ast}(s) = \delta_s, \]
where \( \delta_s \) is the unique positive real solution of \( x^s + x - \frac{1}{2} = 0 \).

Proof. Let \( \mathcal{P} = \{x, y\} \) with \( x = (x_1, \ldots, x_s) \) and \( y = (y_1, \ldots, y_s) \). First we show that we can assume \( x_i \leq y_i \) for all \( i = 1, \ldots, s \).

If there exist indices \( 1 \leq i < j \leq s \) (w.l.o.g. \( i = 1 \) and \( j = 2 \)) such that \( x_1 \leq y_1 \) and \( x_2 \geq y_2 \), then consider two boxes
\[ B_1 := [0, y_1) \times [0, x_2) \times [0, 1)^{s-2} \quad \text{and} \quad B_2 := [0, y_1] \times [0, x_2] \times [0, 1)^{s-2}. \]

For the discrepancy function \( R(B) := \frac{1}{2} A_2(B) - \lambda(B) \) of these sets we have \( R(B_1) = -y_1 x_2 \) and \( R(B_2) = 1 - y_1 x_2 \), hence \( D^\ast(\mathcal{P}) \geq \max(y_1 x_2, 1 - y_1 x_2) \geq \frac{1}{2} > \delta_s \).

So let in the following \( x_i \leq y_i \) for all \( i = 1, \ldots, s \).

Further we may assume \( x_i < \frac{1}{2} \) for all \( 1 \leq i \leq s \), since otherwise \( D^\ast(\mathcal{P}) \geq \frac{1}{2} > \delta_s \) (consider the box \( B = [0, 1)^{j-1} \times [0, \frac{1}{2}) \times [0, 1)^{s-j} \)) and \( y_i > \frac{1}{2} \) for all \( 1 \leq i \leq s \), since otherwise \( D^\ast(\mathcal{P}) \geq \frac{1}{2} > \delta_s \) (consider the box \( B = [0, y_1] \times \cdots \times [0, y_s] \) with \( R(B) = 1 - \prod_{j=1}^{s} y_j \geq \frac{1}{2} \)).

To determine \( D^\ast(\mathcal{P}) \) we have to consider the following boxes:

(a) as large as possible, containing neither \( x \) nor \( y \), i.e., the boxes
\[ B_i = [0, 1)^{j-1} \times [0, x_i) \times [0, 1)^{s-i} \]

with \( R(B_i) = -x_i \).

(b) as small as possible, containing only \( x \), i.e., the box
\[ B = \prod_{i=1}^{s} [0, x_i] \]

with \( R(B) = \frac{1}{2} - \prod_{i=1}^{s} x_i \).

(c) as large as possible, containing only \( x \), i.e., the boxes
\[ B_i = [0, 1)^{j-1} \times [0, y_i) \times [0, 1)^{s-i} \]

with \( |R(B_i)| = y_i - \frac{1}{2} \).

(d) as small as possible containing both \( x \) and \( y \), i.e., the box
\[ B = \prod_{i=1}^{s} [0, y_i] \]

with \( R(B) = 1 - \prod_{i=1}^{s} y_i \).

Hence,
\[ D^\ast(\mathcal{P}) \geq \max \left\{ \max_{1 \leq i \leq s} \max \left\{ x_i, \frac{1}{2} - \prod_{j=1}^{s} x_j \right\}, \max_{1 \leq i \leq s} \max \left\{ y_i, \frac{1}{2} - \prod_{j=1}^{s} y_j \right\} \right\} \]
\[ = \max \left\{ \delta_s, \delta'_s - \frac{1}{2} \right\}, \]

(Note that \( \max\{x_i, \frac{1}{2} - \prod_{j=1}^{s} x_j\} \) is symmetric in \( x_1, \ldots, x_s \) and the maximum is attained for \( x_1 = \cdots = x_s = x \) and \( x = \frac{1}{2} - x^s \). Similar arguments hold for the “\( y \)-part”) where \( \delta'_s \) is the only positive real solution of \( x^s + x - \frac{3}{2} = 0 \). It is easily shown that \( \delta_s > \delta'_s - \frac{1}{2} \), hence \( D_2^{\ast}(s) \geq \delta_s \).
Of course (consider again the boxes from items (a)–(d)) for the set
\[ \mathcal{P} = \{ (\delta_s, \ldots, \delta_s), (\delta'_s, \ldots, \delta'_s) \} \]
we have \( D^*(\mathcal{P}) = \delta_s \) and the result follows. \( \square \)

**Remark 1.** Note that \( \lim_{s \to \infty} D^*_2(s) = \frac{1}{2} \).

For the extreme discrepancy we have the following result.

**Theorem 4.** We have
\[ D^*_2 = \tilde{\delta}_s, \]
where \( \tilde{\delta}_s \) is the unique positive real solution of \( x^s + x - 1 = 0 \).

**Proof.** To show that \( D^*_2 \geq \tilde{\delta}_s \) it suffices to consider \( \mathcal{P} \subseteq (0, 1)^s \) and intervals contained in \((0, 1)^s\).

Let
\[ \vec{x}^{(i)} := (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_s) \]
and
\[ \vec{y}^{(i)} := (y_1, \ldots, y_{i-1}, 1 - y_i, y_{i+1}, \ldots, y_s) \]
and for \( B = \prod_{j=1}^s [a_j, b_j] \) let
\[ \vec{B}^{(i)} := \prod_{j=1}^{i-1} [a_j, b_j] \times [1 - a_i, 1 - b_i] \times \prod_{j=i+1}^s [a_i, b_i]. \]

We use an analogous notation for other, i.e., open, closed, \ldots intervals as well.

Then for the set \( \{ x, y \} \) the expression \( \frac{1}{2} A_2(B) - \lambda(B) \) has the same value as the expression \( \frac{1}{2} A_2(\vec{B}^{(i)}) - \lambda(\vec{B}^{(i)}) \) for the set \( \{ \vec{x}^{(i)}, \vec{y}^{(i)} \} \). Hence, we may restrict to consider point sets \( \mathcal{P} = \{ x, y \} \) with \( x_i \leq y_i \) for all \( i = 1, \ldots, s \).

Consider now the intervals
\[ B_0 := \prod_{j=1}^s [x_j, y_j] \quad \text{and} \quad B_i := (0, 1)^{i-1} \times (x_i, y_i) \times (0, 1)^{s-i} \]
for \( i = 1, \ldots, s \). We have
\[ R(B_0) = 1 - \prod_{j=1}^s (y_j - x_j) \]
and
\[ R(B_i) = y_i - x_i = d_i \quad \forall i = 1, \ldots, s. \]

Hence,
\[ D(P) \geq \max \left\{ d_1, \ldots, d_s, 1 - \prod_{j=1}^s d_j \right\} \geq \tilde{\delta}_s. \]

(Note that again because of symmetry reasons the maximum is attained for \( d_1 = \cdots = d_s = d \) and \( d = 1 - d^s \).) We now consider the point set \( \mathcal{P} = \{ x, y \} \) with \( x = (0, \ldots, 0) \) and \( y = (\tilde{\delta}s, \ldots, \tilde{\delta}s) \). It is easy to see, that for this point set we have \( D(\mathcal{P}) = \tilde{\delta}_s \) and the result follows. \( \square \)
Corollary 5. We have $D_2^{(s)} = 1 - \varepsilon(s)$ with $\varepsilon(s) \sim \log s/s$.

Proof. This follows from Theorem 4 and simple analysis of the equation $x^4 + x - 1 = 0$. □

For the analysis of the isotropic discrepancy we will make use of the following fact.

Lemma 6. Let $\mathcal{P}$ be a set of $s + 1$ points in $[0, 1]^s$. Let $V(\mathcal{P})$ denote the $s$-dimensional volume of the simplex $S(\mathcal{P})$ generated by $\mathcal{P}$ and let $W(\mathcal{P})$ denote the $s$-dimensional volume of the largest convex set $C(\mathcal{P})$ contained in $[0, 1]^s$ which contains no point of $\mathcal{P}$. Then we have

$$J(\mathcal{P}) = \max \left\{ \frac{s}{s+1}, 1 - V(\mathcal{P}), W(\mathcal{P}) \right\} \text{ for } s = 2, 3$$

and

$$J(\mathcal{P}) = \max \{1 - V(\mathcal{P}), W(\mathcal{P})\} \text{ for } s \geq 4.$$

Proof. Of course $J(\mathcal{P}) \geq \max \{1 - V(\mathcal{P}), W(\mathcal{P})\}$. From [2, Theorem 2.8] we obtain that in arbitrary dimension we always have

$$V(\mathcal{P}) \leq \frac{(s + 1)(s + 1)^{s+1/2}}{2^s \cdot s!}$$

and this right-hand side is at most $1/(s + 1)$ for $s \geq 4$.

Let $C \subseteq [0, 1]^s$ be any convex subset of $[0, 1]^s$ containing exactly $i$ points from $\mathcal{P}$ with $1 \leq i \leq s$, then

$$|R(C)| = \left| \frac{i}{s+1} - \lambda(C) \right| \leq \max \left\{ \frac{i}{s+1}, 1 - \frac{i}{s+1} \right\} \leq \frac{s}{s+1}.$$

If $H$ is a hyperplane containing (at least) $s$ points of $\mathcal{P}$, then we have $R(H) = s/(s+1)$ (or even $R(H) = 1$, when $H = S(\mathcal{P})$). Hence $|R|$ attains its maximum for $H$, $S(\mathcal{P})$ or $C(\mathcal{P})$. For $s \geq 4$ (as mentioned above) we have $V(\mathcal{P}) \leq 1/(s + 1)$, hence $H = S(\mathcal{P})$ or $1 - V(\mathcal{P}) \geq R(H)$ and so the maximum if $|R|$ is attained by $S(\mathcal{P})$ or $C(\mathcal{P})$. The result follows. □

Theorem 7. We have

$$J_3^{(2)} = \frac{2}{3}.$$

Proof. It follows from Lemma 6, that $J_3^{(2)} \geq \frac{2}{3}$. Consider the point set

$$\mathcal{P} = \{(0, 0), (1, 0), (\frac{1}{2}, \frac{2}{3})\}.$$

Here $V(\mathcal{P}) = \frac{1}{3}$ and $W(\mathcal{P}) = \frac{2}{3}$ which is attained for the grey trapezoid in Fig. 1. Using Lemma 6 again we find that $J(\mathcal{P}) = \frac{2}{3}$ and the result follows. □

Fig. 1. Three points with isotropic discrepancy $\frac{2}{3}$. 

Theorem 8. For every $s \geq 2$ we have
\[
1 - \frac{(s + 1)^{s+1}/2}{2^s \cdot s!} \leq J_s(s) \leq 1 - \varepsilon(s)(\max\{0, 1 - 2\varepsilon(s)\})^s,
\]
where
\[
\varepsilon(s) := \left(\frac{(s + 1)^{(s+1)/2}(1-(\log(4/3)/\log(s+1)))}{2^s \cdot s!}\right)^{1/s}.
\]

Proof. By [2, Theorem 2.8] the largest simplex generated by $s + 1$ points in $[0, 1]^s$ has volume at most
\[
\frac{(s + 1)^{(s+1)/2}}{2^s \cdot s!},
\]
hence the lower bound holds by Lemma 6.

If $\varepsilon(s) \geq \frac{1}{2}$, then the upper bound is trivial. Let $s$ such that $\varepsilon(s) < \frac{1}{2}$ (this is satisfied for $s \geq 5$). Consider the interval
\[
I_s(s) := [\varepsilon(s), 1 - \varepsilon(s)]^s.
\]

Let $\tilde{\mathcal{P}}$ be any set of $s + 1$ points in $[0, 1]^s$ such that the volume of the simplex $S(\tilde{\mathcal{P}})$ spanned by $\tilde{\mathcal{P}}$ is maximal. Then again by [2, Theorem 2.8] we have that the volume of $S(\tilde{\mathcal{P}})$ is at least $\varepsilon(s)$. Hence, if $\mathcal{P}$ is any set of $s + 1$ points in $I_s(s)$ such that the volume of the simplex $S(\mathcal{P})$ spanned by $\mathcal{P}$ is maximal we have that the volume of $S(\mathcal{P})$ is at least $\varepsilon(s)(1 - 2\varepsilon(s))^s$.

Let $C(\mathcal{P})$ be the largest convex set in $[0, 1]^s$ containing none of the points of $\mathcal{P}$. Consider all the cubes $T_1 \times \cdots \times T_s$ with $T_i \in [[0, \varepsilon(s)), [1 - \varepsilon(s), 1]]$ for all $1 \leq i \leq s$. There must be at least one of these cubes such that $(T_1 \times \cdots \times T_s) \cap C(\mathcal{P})$ is empty, otherwise $C(\mathcal{P})$ would contain $I_s(s)$ and hence $\mathcal{P}$, a contradiction.

Therefore the volume of $C(\mathcal{P})$ is at most $1 - \varepsilon(s)$. From Lemma 6 we obtain
\[
J(\mathcal{P}) \leq 1 - \varepsilon(s)(1 - 2\varepsilon(s))^s.
\]
The result follows. □

Corollary 9. For every $\kappa > 0$ and every $s$ large enough we have
\[
1 - \frac{(s + 1)^{(s+1)/2}}{2^s \cdot s!} \leq J_{s+1}(s) \leq 1 - \frac{(s + 1)^{(s+1)/2}(1-\kappa)}{2^s \cdot s!}.
\]

Proof. This follows from Theorem 8 and simple analysis. □

References