ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 312 (2007) 733–754

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Non-abelian Hopf cohomology

Philippe Nuss *, Marc Wambst

Institut de Recherche Mathématique Avancée, Université Louis-Pasteur et CNRS, 7, rue René-Descartes, 67084 Strasbourg Cedex, France

Received 8 March 2006

Available online 15 November 2006

Communicated by Michel Van den Bergh

Abstract

We introduce non-abelian cohomology sets of Hopf algebras with coefficients in Hopf modules. We prove that these sets generalize Serre's non-abelian group cohomology theory. Using descent techniques, we establish that our construction enables to classify as well twisted forms for modules over Hopf-Galois extensions as torsors over Hopf-modules.

Keywords: Non-abelian cohomology; Non-commutative descent theory; Hopf-Galois extension; Hopf-module; Twisted form; Torsor; Hilbert's Theorem 90

Introduction

The aim of this article is to extend to Hopf algebras the concept of non-abelian cohomology of groups. Introduced in 1958 by Lang and Tate [8] for Galois groups with coefficients in an algebraic group, the non-abelian cohomology theory in degree 0 and 1 was formalized by Serre [12,13]. For an arbitrary group G acting on a group A which is not necessarily abelian, Serre constructs a 0-cohomology group $\mathrm{H}^0(G, A)$ and a 1-cohomology pointed set $\mathrm{H}^1(G, A)$. These objects generalize the two first groups of the classical Eilenberg–MacLane cohomology sequence $\mathrm{H}^*(G, A) = \mathrm{Ext}^*_{\mathbf{Z}[G]}(\mathbf{Z}, A)$, defined only when A is abelian. It is well known that the non-abelian cohomology set $\mathrm{H}^1(G, A)$ classifies the torsors on A (see [13]).

* Corresponding author.
E-mail addresses: nuss@math.u-strasbg.fr (P. Nuss), wambst@math.u-strasbg.fr (M. Wambst).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.10.005

The non-abelian cohomology theory of groups comes naturally into play in the particular case where S/R is a G-Galois extension of rings in the sense of [9]. The situation is the following: a finite group G acts on a ring extension S/R and, in a compatible way, on an S-module M. The coefficient group is then the group of S-automorphisms $A = \mathrm{Aut}_S(M)$ of M. In [10], one of the authors showed that the set $\mathrm{H}^1(G, \mathrm{Aut}_S(M))$ classifies as well descent cocycles on M as twisted forms of M.

Galois extensions of rings may be viewed as particular cases of Hopf-Galois extensions defined by Kreimer–Takeuchi [7], where a Hopf algebra H (non-necessarily commutative nor cocommutative) plays the rôle of the Galois group. Indeed, given a group G, a G-Galois extension of rings is nothing but a $\mathbf{Z}^G$-Hopf-Galois extension of rings, where $\mathbf{Z}^G$ stands for the dual Hopf algebra of the group ring $\mathbf{Z}[G]$.

Suppose now fixed a ground ring k, a Hopf algebra H over k, and an H-comodule algebra S (for instance, any H-Hopf-Galois extension S/R is based on such a datum). For any (H, S)-Hopf module M, that is an abelian group M endowed with an S-action and a compatible H-coaction, we define in the cosimplicial spirit a 0-cohomology group $\mathrm{H}^0(H, M)$ and a 1-cohomology pointed set $\mathrm{H}^1(H, M)$.

The philosophy behind the construction is the following (precise definitions will be given in the core of the paper). Start with a G-Galois extension S/R, where G is a finite group, and with M a (G, S)-Galois module, i.e. an abelian group M endowed with two compatible S- and G-actions. The group $\mathrm{Aut}_S(M)$ inherits a G-action by conjugation. Let k^G be the dual Hopf algebra of the group ring $k[G]$. A 1-cocycle in the sense of Serre is represented by a certain map $\alpha : G \to \mathrm{Aut}_S(M)$. By duality, α formally defines an element in $M \otimes_k M^* \otimes_k k^G$, which can also be seen as a map $\Phi_\alpha : M \to M \otimes_k k^G$ satisfying some conditions. Assume now given, instead of G, a Hopf-algebra H coacting on a ring S. Let M be an (H, S)-Hopf module, that is a module on which both H and S act in a compatible way. We replace the former map $\Phi_\alpha : M \to M \otimes_k k^G$ by a map $\Phi : M \to M \otimes_k H$ and state general requirements—the cocycle conditions—which reflect the group-cocycle condition on α. This construction gives rise to a 1-cohomology pointed set $\mathrm{H}^1(H, M)$.

We establish two mains results. The first theorem shows that the 1-cohomology set $\mathrm{H}^1(H, M)$ generalizes the non-abelian group 1-cohomology set of Serre. The second one relates $\mathrm{H}^1(H, M)$ to $\mathrm{Twist}(S/R, N_0)$, the isomorphy class of the twisted forms of an extended module $M = N_0 \otimes_R S$. More precisely, we prove the two following statements:

Theorem A. *For a group G and a (k^G, S)-Hopf module M, there is an isomorphism of pointed sets*

$$\mathrm{H}^1\big(k^G, M\big) \cong \mathrm{H}^1\big(G, \mathrm{Aut}_S(M)\big).$$

Theorem B. *For a Hopf-algebra H and an (H, S)-Hopf module M of the form $M = N_0 \otimes_R S$, there is an isomorphism of pointed sets*

$$\mathrm{H}^1(H, M) \cong \mathrm{Twist}(S/R, N_0).$$

The precise wording of Theorem A will be found in Corollary 3.2, and that of Theorem B in Theorem 1.2. As a consequence of Theorem B, we deduce (Corollary 1.3) a Hopf version of the celebrated Theorem 90 stated in 1897 by Hilbert in his *Zahlbericht*.

In order to prove these two results, we bring in an auxiliary cohomology theory $\mathrm{D}^i(H, M)$ $(i = 0, 1)$ related to descent theory. The pointed set $\mathrm{D}^1(H, M)$ classifies the (H, S)-Hopf module structures on M and, in the case of a Hopf-Galois extension, the descent data on M. Moreover, it may be viewed as torsors on M (Proposition 2.8).

We mention here that A. Blanco Ferro [1], generalizing a construction due to M. Sweedler [14], defined a 1-cohomology set $\mathrm{H}^1(H, A)$, where H is a Hopf-algebra and A is an H-module algebra. He applied his theory, which is in some sense dual to ours, to a commutative particular case: not only does H have to be a commutative finitely generated k-projective Hopf algebra, but S/k is a commutative Hopf-Galois extension. For any k-module N, setting $A = \mathrm{End}_S(N \otimes_k S)$, Blanco Ferro showed in this particular case that his set $\mathrm{H}^1(H^*, A)$ classifies the twisted forms of $N \otimes_k S$ where H^* stands for the dual Hopf algebra of H.

0. Conventions

Let k be a fixed commutative and unital ring. The unadorned symbol $\otimes$ between a right k-module and a left k-module stands for $\otimes_k$. By *algebra* we mean a unital associative k-algebra. A *division algebra* is either a commutative field or a skew-field. By *module* over a ring R, we always understand a right R-module unless otherwise stated. Denote by $\mathfrak{Mod}_R$ the category of R-modules and by $\mathfrak{Set}$ the category of sets.

Let H be a Hopf-algebra over k with multiplication μ_H, unity map η_H, comultiplication Δ_H, counity map ε_H, and antipode σ_H. Let S be an algebra, μ_S its multiplication, η_S its unity map. We assume that S is a right H-comodule algebra, in other words that S is equipped with an H-coaction map $\Delta_S : S \to S \otimes H$ which is a morphism of algebras. Let M be both an S-module and an H-comodule with the H-coaction map $\Delta_M : M \to M \otimes H$. If Δ_M verifies the equality

$$\Delta_M(ms) = \Delta_M(m)\Delta_S(s), \tag{1}$$

for any $m \in M$ and $s \in S$, we say that M is an (H, S)-*Hopf module* (also called a *relative Hopf module* in the literature) and that $\Delta_M : M \to M \otimes H$ is (H, S)-*linear*. A morphism $f : M \to M'$ of (H, S)-Hopf modules is an S-linear map f such that $(f \otimes \mathrm{id}_M) \circ \Delta_M = \Delta_{M'} \circ f$. To denote the coactions on elements, we use the Sweedler–Heyneman convention, that is, for $m \in M$, we write $\Delta_M(m) = m_0 \otimes m_1$, with summation implicitly understood. More generally, when we write down a tensor we usually omit the summation sign $\sum$.

Denote by R the algebra of H-coinvariants of S, that is $R = \{s \in S \mid \Delta_S(s) = s \otimes 1\}$. An S-module M is said to be *extended* if there exists an R-module N_0 such that M is equal to $N_0 \otimes_R S$. The inclusion map $\psi : R \hookrightarrow S$ is a (*right*) *H-Hopf-Galois extension* if ψ is faithfully flat and the map $\Gamma_\psi : S \otimes_R S \to S \otimes H$, called *Galois map*, given on an indecomposable tensor $s \otimes t \in S \otimes_R S$ by

$$\Gamma_\psi(s \otimes t) = s\Delta_S(t),$$

is a k-linear isomorphism. By Hopf-Galois descent theory [5,11], every (H, S)-Hopf module is isomorphic to an extended S-module. Conversely, an extended S-module $M = N_0 \otimes_R S$ owns an (H, S)-Hopf module structure with the canonical coaction $\Delta_M = \mathrm{id}_{N_0} \otimes \Delta_S : N_0 \otimes_R S \to N_0 \otimes_R S \otimes H$.

Let G be a finite group. Denote by k^G the k-free Hopf algebra over the k-basis $\{\delta_g\}_{g\in G}$, with the following structure maps: the multiplication is given by $\delta_g \cdot \delta_{g'} = \partial_{g,g'}\delta_g$, where $\partial_{g,g'}$ stands for the Kronecker symbol of g and g'; the comultiplication Δ_{k^G} is defined by $\Delta_{k^G}(\delta_g) = \sum_{ab=g} \delta_a \otimes \delta_b$; the unit in k^G is the element $1 = \sum_{g\in G} \delta_g$; the counit ε_{k^G} is defined by $\varepsilon_{k^G}(\delta_g) = \partial_{g,e}1$; the antipode σ_{k^G} sends δ_g on $\delta_{g^{-1}}$. When k is a field, then k^G is the dual of the usual group Hopf-algebra $k[G]$. It is easy to see that a k^G-Hopf-Galois extension is the same as a G-Galois extension of k-algebras in the sense of [9]. To give an action of G on S is equivalent to give a coaction map of k^G on S, the two structures being related by the equality

$$\Delta_S(s) = \sum_{g\in G} g(s) \otimes \delta_g.$$

An S-module M will be called a (G, S)*-Galois module* if it is endowed with a (G, S)-action, that is a G-action $\gamma : G \to \mathrm{Aut}_k(M)$ such that following twisted S-linearity condition:

$$g(ms) = g(m)g(s) \tag{2}$$

holds for any $g \in G$, $m \in M$, and $s \in S$ (when no confusion about γ is possible, we denote for simplicity $g(m)$ instead of $\gamma(g)(m)$). When γ verifies (2), we say that the morphism γ is (G, S)*-linear*. Denote by $\mathrm{Aut}^{\gamma}_S(M)$ the subgroup of $\mathrm{Aut}_k(M)$ which is the image of γ.

To give a (G, S)-Galois module structure on M is equivalent to give a (k^G, S)-Hopf module structure on S. By Galois descent theory, a (G, S)-Galois module is isomorphic to an extended module $N \otimes_R S$.

1. Non-abelian Hopf cohomology theory

In this section we define a non-abelian Hopf cohomology theory, and state our main result, Theorem 1.2, which compares in the Hopf-Galois context the 1-Hopf cohomology set with twisted forms. We deduce a Hopf-Galois version of Hilbert's Theorem 90.

1.1. Definition of the non-abelian Hopf cohomology sets

Let H be a Hopf-algebra and S be an H-comodule algebra. For any S-module M, we endow $M \otimes H^{\otimes n}$ with an S-module structure given by

$$(m \otimes \underline{h})s = ms \otimes \underline{h},$$

for $m \in M$, $\underline{h} \in H^{\otimes n}$, and $s \in S$.

Set $\mathrm{W}^n_k(M) = \mathrm{Hom}_k(M, M \otimes H^{\otimes n})$ and $\mathrm{W}^n_S(M) = \mathrm{Hom}_S(M, M \otimes H^{\otimes n})$. We equip the k-module $\mathrm{W}^n_k(M)$ with a composition-type product $\odot : \mathrm{W}^n_k(M) \otimes \mathrm{W}^n_k(M) \to \mathrm{W}^n_k(M)$, defined by

$$\begin{cases} \varphi \odot \varphi' = \varphi \circ \varphi' & \text{if } n = 0, \\ \varphi \odot \varphi' = (\mathrm{id}_M \otimes \mu_H^{\otimes n}) \circ (\mathrm{id}_M \otimes \chi_n) \circ (\varphi \otimes \mathrm{id}_H^{\otimes n}) \circ \varphi' & \text{if } n > 0, \end{cases}$$

for $\varphi, \varphi' \in \mathrm{W}^n_k(M)$; here $\chi_n : H^{\otimes n} \otimes H^{\otimes n} \to (H \otimes H)^{\otimes n}$ denotes the intertwining operator given by

$$\chi_n\big((a_1 \otimes \cdots \otimes a_n) \otimes (b_1 \otimes \cdots \otimes b_n)\big) = (a_1 \otimes b_1) \otimes \cdots \otimes (a_n \otimes b_n).$$

It restricts to a product still denoted $\odot$ on $\mathrm{W}_S^n(M)$. Thanks to the product $\odot$, the modules $\mathrm{W}_k^n(M)$ and $\mathrm{W}_S^n(M)$ become a monoid: the associativity of $\odot$ is a direct consequence of the coassociativity of Δ_H and the neutral element is $\upsilon_n = \mathrm{id}_M \otimes \eta_H^{\otimes n}$. Further we shall use that the group of invertible elements of the monoid $\mathrm{W}_S^0(M)$ is $\mathrm{Aut}_S(M)$.

Suppose that M is an H-comodule. Denote by T the *flip* of $H \otimes H$, the automorphism of $H \otimes H$ which sends an indecomposable tensor $h \otimes h'$ to $h' \otimes h$. We define two maps $d^i : \mathrm{W}_k^0(M) \to \mathrm{W}_k^1(M)$ $(i = 0, 1)$ and three maps $d^i : \mathrm{W}_k^1(M) \to \mathrm{W}_k^2(M)$ $(i = 0, 1, 2)$ by the formulae

$$\begin{aligned}
d^0\varphi &= (\mathrm{id}_M \otimes \mu_H) \circ (\Delta_M \otimes \mathrm{id}_H) \circ (\varphi \otimes \sigma_H) \circ \Delta_M, \\
d^1\varphi &= (\mathrm{id}_M \otimes \eta_H) \circ \varphi, \\
d^0\Phi &= (\mathrm{id}_M \otimes \mu_H \otimes \mathrm{id}_H) \circ (\Delta_M \otimes T) \circ (\Phi \otimes \sigma_H) \circ \Delta_M, \\
d^1\Phi &= (\mathrm{id}_M \otimes \Delta_H) \circ \Phi, \\
d^2\Phi &= (\mathrm{id}_M \otimes \mathrm{id}_H \otimes \eta_H) \circ \Phi = \Phi \otimes \eta_H,
\end{aligned}$$

where $\varphi : M \to M$ and $\Phi : M \to M \otimes H$ are k-linear morphisms.

Lemma 1.1. *Let M be an (H, S)-Hopf-module. The restriction of the above defined maps to the corresponding monoids $\mathrm{W}_S^0(M)$ and $\mathrm{W}_S^1(M)$ are morphims of monoids which may be organized in the following cosimplicial diagram*:

$$\mathrm{W}_S^0(M) \underset{d^1}{\overset{d^0}{\rightrightarrows}} \mathrm{W}_S^1(M) \overset{d^0,\, d^1,\, d^2}{\rightrightarrows\!\!\to} \mathrm{W}_S^2(M). \tag{3}$$

Proof. We adopt the Sweedler–Heyneman convention and use the Hopf yoga, for instance, the fact that for any $x, y \in H$, one has $x_0 \otimes \sigma_H(x_1)x_2y = x_0 \otimes \varepsilon_H(x_1)y = x \otimes y$. First one has to show that $d^i\varphi$ and $d^i\Phi$ are S-linear. This assertion is obvious for $d^1\varphi$. Let us prove it for $d^0\varphi$. We get, for any $m \in M$ and $s \in S$, the equalities

$$\begin{aligned}
d^0\varphi(ms) &= \big[(\mathrm{id}_M \otimes \mu_H) \circ (\Delta_M \otimes \mathrm{id}_H) \circ (\varphi \otimes \sigma_H) \circ \Delta_M\big](ms) \\
&= \big[(\mathrm{id}_M \otimes \mu_H) \circ (\Delta_M \otimes \mathrm{id}_H)\big]\big(\varphi(m_0)s_0 \otimes \sigma_H(m_1 s_1)\big) \\
&= (\mathrm{id}_M \otimes \mu_H)\big[\varphi(m_0)_0 s_0 \otimes \varphi(m_0)_1 s_1 \otimes \sigma_H(s_2)\sigma_H(m_1)\big] \\
&= \varphi(m_0)_0 s_0 \otimes \varphi(m_0)_1\big(s_1\sigma_H(s_2)\big)\sigma_H(m_1) \\
&= \varphi(m_0)_0 s \otimes \varphi(m_0)_1 \sigma_H(m_1) \\
&= d^0\varphi(m)s.
\end{aligned}$$

The S-linearity of $d^1\Phi$ and $d^2\Phi$ is obvious. We prove it for $d^0\Phi$. For any $m \in M$ and $s \in S$, set $\Phi(m) = m' \otimes m''$. We have $d^0\Phi(m) = ((m_0)')_0 \otimes ((m_0)')_1\sigma_H(m_1) \otimes (m_0)''$, hence

$$\begin{aligned}
d^0\Phi(ms) &= \big[(\mathrm{id}_M \otimes \mu_H \otimes \mathrm{id}_H) \circ (\varDelta_M \otimes T) \circ (\Phi \otimes \sigma_H) \circ \varDelta_M\big](ms) \\
&= \big[(\mathrm{id}_M \otimes \mu_H \otimes \mathrm{id}_H) \circ (\varDelta_M \otimes T)\big]\big((m_0)' s_0 \otimes (m_0)'' \otimes \sigma_H(m_1 s_1)\big) \\
&= (\mathrm{id}_M \otimes \mu_H \otimes \mathrm{id}_H)\big[\big((m_0)'\big)_0 s_0 \otimes \big((m_0)'\big)_1 s_1 \otimes \sigma_H(s_2)\sigma_H(m_1) \otimes (m_0)''\big] \\
&= \big((m_0)'\big)_0 s \otimes \big((m_0)'\big)_1 \sigma_H(m_1) \otimes (m_0)'' \\
&= d^0\Phi(m)s.
\end{aligned}$$

We prove now that d^i respects the monoid structures on $\mathrm{W}_S^k(M)$, that is

$$d^i\varphi \odot d^i\varphi' = d^i(\varphi \odot \varphi'), \qquad d^i\Phi \odot d^i\Phi' = d^i(\Phi \odot \Phi'), \quad \text{and} \quad d^i(\upsilon_k) = \upsilon_{k+1}$$

for any $\varphi, \varphi' \in \mathrm{W}_S^0(M)$, any $\Phi, \Phi' \in \mathrm{W}_S^1(M)$, $k \in \{0, 1\}$, and any appropriate index i. Let us prove this on the 0-level for φ and φ' in $\mathrm{W}^0(M)$. For any $m \in M$, we have:

$$\begin{aligned}
\big(d^0\varphi' \odot d^0\varphi\big)(m) &= (\mathrm{id}_M \otimes \mu_H)\big(d^0\varphi' \otimes \mathrm{id}_H\big)\big(d^0\varphi(m)\big) \\
&= (\mathrm{id}_M \otimes \mu_H)\big(d^0\varphi' \otimes \mathrm{id}_H\big)\big(\varphi(m_0)_0 \otimes \varphi(m_0)_1\sigma_H(m_1)\big) \\
&= \varphi'\big(\varphi(m_0)_0\big)_0 \otimes \varphi'\big(\varphi(m_0)_0\big)_1 \sigma_H\big(\varphi(m_0)_1\big)\varphi(m_0)_2\sigma_H(m_1) \\
&= \varphi'\big(\varphi(m_0)_0\big)_0 \otimes \varphi'\big(\varphi(m_0)_0\big)_1 \varepsilon_H\big(\varphi(m_0)_1\big)\sigma_H(m_1) \\
&= (\mathrm{id}_M \otimes \mu_H)\big((\varDelta_M \circ \varphi') \otimes \mathrm{id}_H\big)\big[\varphi(m_0)_0 \otimes \varepsilon_H\big(\varphi(m_0)_1\big)\sigma_H(m_1)\big] \\
&= (\mathrm{id}_M \otimes \mu_H)\big((\varDelta_M \circ \varphi') \otimes \mathrm{id}_H\big)\big[\varphi(m_0) \otimes \sigma_H(m_1)\big] \\
&= (\mathrm{id}_M \otimes \mu_H)\big((\varDelta_M \circ \varphi' \circ \varphi) \otimes \sigma_H\big)\varDelta_M(m) \\
&= d^0(\varphi' \odot \varphi)(m)
\end{aligned}$$

and

$$\begin{aligned}
d^1\varphi \odot d^1\varphi'(m) &= (\mathrm{id}_M \otimes \mu_H)\big(d^1\varphi' \otimes \mathrm{id}_H\big)\big(d^1\varphi(m)\big) \\
&= (\mathrm{id}_M \otimes \mu_H)\big(d^1\varphi' \otimes \mathrm{id}_H\big)\big(\varphi(m) \otimes 1\big) \\
&= (\mathrm{id}_M \otimes \mu_H)\big(\varphi'\big(\varphi(m)\big) \otimes 1 \otimes 1\big) \\
&= \varphi'\big(\varphi(m)\big) \otimes 1 \\
&= d^1(\varphi' \odot \varphi)(m).
\end{aligned}$$

We do not write down the computations on the 1-level, which are very similar to the previous ones. We leave to the reader the straightforward proof of $d^i(\upsilon_k) = \upsilon_{k+1}$ and also the easy checking of the following three formulae

$$d^2d^0 = d^0d^1, \qquad d^1d^0 = d^0d^0, \qquad d^2d^1 = d^1d^1,$$

which mean that the diagram (3) is precosimplicial. □

We define the 0-cohomology group $\mathrm{H}^0(H,M)$ and the 1-cohomology set $\mathrm{H}^1(H,M)$ in the following way. Let

$$\mathrm{H}^0(H,M)=\{\varphi\in \mathrm{Aut}_S(M)\;|\; d^1\varphi=d^0\varphi\}$$

be the equalizer of the pair (d^0,d^1). It is obviously a group since d^i is a morphism of monoids.

The set $\mathrm{Z}^1(H,M)$ of 1-*Hopf cocycles of H with coefficients in M* is the subset of $\mathrm{W}^1_S(M)$ defined by

$$\mathrm{Z}^1(H,M)=\left\{\Phi\in \mathrm{W}^1_k(M)\;\middle|\;\begin{array}{l}(\mathrm{ZC}_1)\;\; \Phi(ms)=\Phi(m)s,\ \text{ for all } m\in M \text{ and } s\in S\\ (\mathrm{ZC}_2)\;\; (\mathrm{id}_M\otimes\varepsilon_H)\circ\Phi=\mathrm{id}_M\\ (\mathrm{ZC}_3)\;\; d^2\Phi\odot d^0\Phi=d^1\Phi\end{array}\right\}.$$

The group $\mathrm{Aut}_S(M)$ acts on the right on $\mathrm{Z}^1(H,M)$ by

$$(\Phi\leftharpoondown f)=d^1f^{-1}\odot\Phi\odot d^0 f,$$

where $\Phi\in \mathrm{Z}^1(H,M)$ and $f\in\mathrm{Aut}_S(M)$. Two 1-Hopf cocycles Φ and Φ' are said to be *cohomologous* if they belong to the same orbit under the action of $\mathrm{Aut}_S(M)$ on $\mathrm{Z}^1(H,M)$. We denote by $\mathrm{H}^1(H,M)$ the quotient set $\mathrm{Aut}_S(M)\backslash \mathrm{Z}^1(H,M)$; it is pointed with distinguished point the class of the map $\upsilon_1=\mathrm{id}_M\otimes\eta_H$.

For $i=0,1$, we call $\mathrm{H}^i(H,M)$ the *ith-Hopf cohomology set of H with coefficients in the (H,S)-Hopf module M.*

1.2. The main theorem: Comparison of the 1-Hopf cohomology set with twisted forms in the Hopf-Galois context

Let H be a Hopf-algebra, $\psi: R\to S$ be an H-Hopf-Galois extension, and $M=N_0\otimes_R S$ be the extended S-module of an R-module N_0. We endow M with the canonical (H,S)-Hopf module structure given by the coaction $\Delta_M=\mathrm{id}_{N_0}\otimes\Delta_S$. The central result of this paper asserts that the Hopf 1-cohomology set $\mathrm{H}^1(H,M)$ is isomorphic to the pointed set $\mathrm{Twist}(S/R,N_0)$ of twisted forms of N_0 up to isomorphisms.

Let $\psi: R\to S$ be any extension of rings and N_0 be an R-module. Recall that a *twisted form of N_0 (over S/R)* is a pair (N,φ), where N is an R-module and $\varphi: N\otimes_R S\to N_0\otimes_R S$ is an S-linear isomorphism. Let $\mathrm{twist}(S/R,N_0)$ be the set of twisted forms of N_0. Two twisted forms (N,φ) and (N',φ') of N_0 are *isomorphic* if N and N' are isomorphic as R-modules. Following [6], we denote by $\mathrm{Twist}(S/R,N_0)$ the pointed set of isomorphism classes of twisted forms of N_0, the distinguished point being the class of $(N_0,\mathrm{id}_{N_0}\otimes\mathrm{id}_S)$. We mention here that all the results of [10] involving equivalence classes of twisted forms are actually proven for this definition of $\mathrm{Twist}(S/R,N_0)$ and not for the one given in [10, §6.3], where the equivalence relation is too restrictive.

Theorem 1.2. *Let H be a Hopf-algebra, $\psi: R\to S$ be an H-Hopf-Galois extension, and $M=N_0\otimes_R S$ be the extended S-module of an R-module N_0. There is an isomorphism of pointed sets*

$$\mathrm{H}^1(H,M)\cong \mathrm{Twist}(S/R,N_0).$$

Theorem 1.2 allows us to state the following non-commutative generalization of Noether's cohomological form of Hilbert's Theorem 90.

Corollary 1.3. *Let H be a Hopf-algebra and $\psi : K \to L$ be an H-Hopf-Galois extension of division algebras. Then, for any positive integer n, we have*

$$\mathrm{H}^1\big(H, L^n\big) = \{1\}.$$

Here we denote by 1 the distinguished point of $\mathrm{H}^1(H, L^n)$.

Proof of Corollary 1.3. Observe that L^n is isomorphic to the extended L-module $K^n \otimes_K L$. By Theorem 1.2, the pointed set $\mathrm{H}^1(H, L^n)$ is isomorphic to $\mathrm{Twist}(L/K, K^n)$, which is known to be trivial [10, Corollary 6.21]. □

The rest of the paper is mainly devoted to the proof of Theorem 1.2. This is done in two steps. At first we introduce a non-abelian cohomology theory $\mathrm{D}^i(H, M)$, for $i = 0, 1$, which is related to non-commutative descent theory. In Theorem 2.6, we prove the isomorphism $\mathrm{D}^1(H, M) \cong \mathrm{Twist}(S/R, N_0)$. Subsequently we show that the Hopf cohomology sets $\mathrm{H}^i(H, M)$ are isomorphic to the descent cohomology sets $\mathrm{D}^i(H, M)$.

2. Descent cohomology sets

In this section we introduce two descent cohomology sets. We compute them in the Galois case and relate them to the usual non-abelian group cohomology theory. In addition, in the Hopf-Galois context, we prove that the 1-descent cohomology set classifies twisted forms and interpret it in terms of torsors on the module of coefficients.

2.1. *Definition of descent cohomology sets*

Let H be a Hopf-algebra, S be an H-comodule algebra, and M be an (H, S)-Hopf module with coaction $\Delta_M : M \to M \otimes H$. We define the 0-cohomology group $\mathrm{D}^0(H, M)$ by

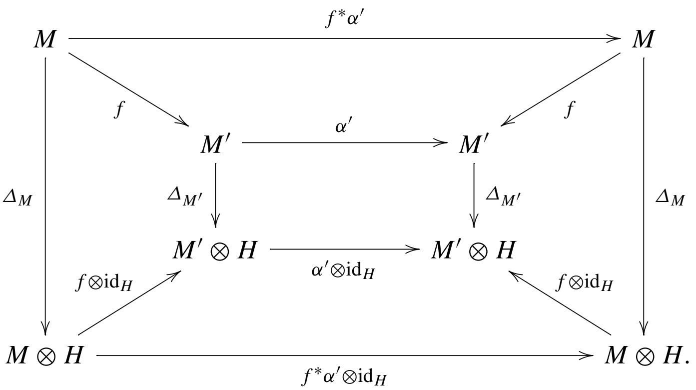
$$\mathrm{D}^0(H, M) = \big\{\alpha \in \mathrm{Aut}_S(M) \;\big|\; (\alpha \otimes \mathrm{id}_H) \circ \Delta_M = \Delta_M \circ \alpha\big\}.$$

It is the set of the S-linear automorphisms of M which are maps of H-comodules. This set obviously carries a group structure given by the composition of automorphisms.

Lemma 2.1. *Let H be a Hopf-algebra and S be an H-comodule algebra. Any isomorphism $f : M \to M'$ of (H, S)-Hopf modules induces an isomorphism of groups $f^* : \mathrm{D}^0(H, M') \to \mathrm{D}^0(H, M)$ given on $\alpha' \in \mathrm{D}^0(H, M')$ by*:

$$f^*\alpha' = f^{-1} \circ \alpha' \circ f.$$

Proof. The S-linearity of $f^*\alpha'$ immediately follows from the S-linearity of f and that of α'. In order to prove that $f^*\alpha'$ belongs to $\mathrm{D}^0(H, M)$, it is sufficient to observe that the following diagram is commutative:



□

We introduce now a 1-cohomology set $\mathrm{D}^1(H, M)$ in the following way. The set $\mathrm{C}^1(H, M)$ of 1-*descent cocycles of H with coefficients in M* is defined to be the set of all k-linear H-coactions $F : M \to M \otimes H$ on M making M an (H, S)-Hopf module. In other words, one has:

$$\mathrm{C}^1(H,M) = \left\{ F : M \to M \otimes H \;\middle|\; \begin{array}{l} (\mathrm{CC}_1)\ \ F(ms) = F(m)\Delta_S(s), \ \text{for all } m \in M \text{ and } s \in S \\ (\mathrm{CC}_2)\ \ (\mathrm{id}_M \otimes \varepsilon_H) \circ F = \mathrm{id}_M \\ (\mathrm{CC}_3)\ \ (F \otimes \mathrm{id}_H) \circ F = (\mathrm{id}_M \otimes \Delta_H) \circ F \end{array} \right\}.$$

Notice that $\mathrm{C}^1(H, M)$ is pointed (hence not empty) with the coaction map Δ_M as distinguished point.

Lemma 2.2. *Let H be a Hopf-algebra and S be an H-comodule algebra. Any isomorphism $f : M \to M'$ of S-modules induces a bijection $f^* : \mathrm{C}^1(H, M') \to \mathrm{C}^1(H, M)$ given on $F' \in \mathrm{C}^1(H, M')$ by*

$$f^*F' = \big(f^{-1} \otimes \mathrm{id}_H\big) \circ F \circ f.$$

For any S-module M, one has

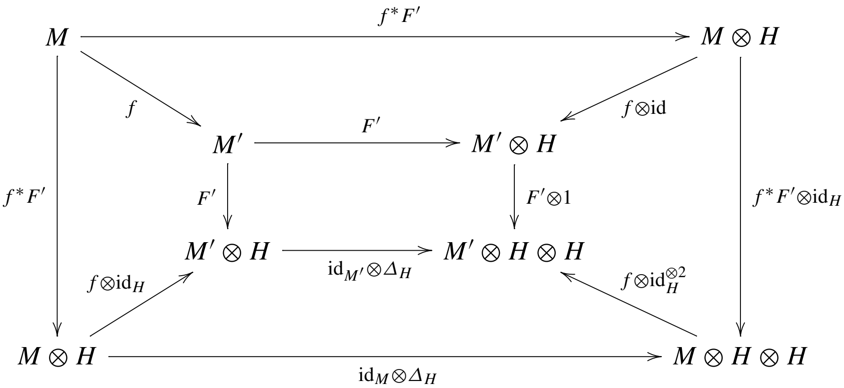
$$(\mathrm{id}_M)^* = \mathrm{id}_{\mathrm{C}^1(H,M)}.$$

For any composable isomorphisms of S-modules $f : M \to M'$ and $f' : M' \to M''$, the following equality holds

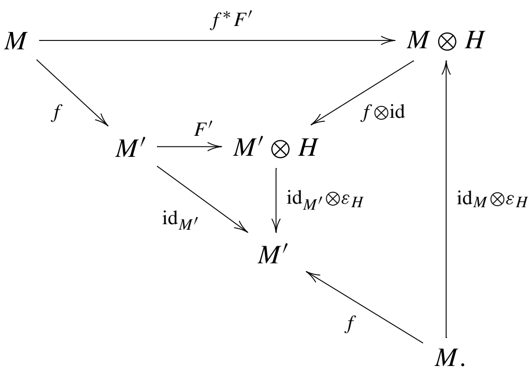
$$(f' \circ f)^* = f^* \circ f'^*.$$

If moreover $f : M \to M'$ is an isomorphism of (H, S)-Hopf modules, then f^* realizes an isomorphism of pointed sets between $\mathrm{C}^1(H, M')$ and $\mathrm{C}^1(H, M)$.

Proof. Let $f : M \to M'$ be an isomorphism of S-modules. The (H, S)-linearity of f^*F' immediately follows from the S-linearity of f and from the (H, S)-linearity of F'. The coassociativity of f^*F' comes from the commutativity of the diagram



whereas the compatibility of f^*F' with the counity of H is expressed by the commutativity of the diagram



Hence we have shown that f^*F' belongs to $\mathrm{C}^1(H, M)$. By the very definition, f^*F' is bijective and $(\mathrm{id}_M)^* = \mathrm{id}_{\mathrm{C}^1(H,M)}$.

Let $f : M \to M'$ and $f' : M' \to M''$ be two isomorphisms of S-modules. One has, for any $F' \in \mathrm{C}^1(H, M')$, the following equalities

$$\begin{aligned}(f' \circ f)^*(F') &= \big((f' \circ f)^{-1} \otimes \mathrm{id}_H\big) \circ F' \circ (f' \circ f) \\ &= \big(\big(f^{-1} \circ f'^{-1}\big) \otimes \mathrm{id}_H\big) \circ F' \circ (f' \circ f) = f^*\big(f'^*F'\big).\end{aligned}$$

Moreover, if f is an isomorphism of (H, S)-Hopf modules, the map f^* preserves the distinguished points: indeed, the equality $f^*\Delta_{M'} = \Delta_M$ is equivalent to the fact that f is a morphism of (H, S)-Hopf modules. $\square$

From Lemma 2.2, one readily obtains the following result.

Corollary 2.3. *Let H be a Hopf-algebra, S be an H-comodule algebra, and M be an (H,S)-Hopf module. The group $\mathrm{Aut}_S(M)$ acts on the right on $\mathrm{C}^1(H,M)$ by*

$$(F \leftharpoonup f) = f^*F = \left(f^{-1} \otimes \mathrm{id}_H\right) \circ F \circ f,$$

where $F \in \mathrm{C}^1(H,M)$ and $f \in \mathrm{Aut}_S(M)$.

Two 1-descent cocycles F and F' are said to be *cohomologous* if they belong to the same orbit under the action of $\mathrm{Aut}_S(M)$ on $\mathrm{C}^1(H,M)$. We denote by $\mathrm{D}^1(H,M)$ the quotient set $\mathrm{Aut}_S(M)\backslash \mathrm{C}^1(H,M)$; it is pointed with distinguished point the class of the coaction Δ_M.

For $i = 0, 1$, we call $\mathrm{D}^i(H,M)$ the *ith-descent cohomology set of H with coefficients in M*. The choice of this name finds its motivation in the following observation. Suppose that $\psi : R \to S$ is an H-Hopf-Galois extension. As shown in [11], an (H,S)-Hopf module may always be descended to an R-module N_0, that is M is isomorphic to an extended S-module $N_0 \otimes_R S$. The set $\mathrm{C}^1(H,M)$ is exactly those of all descent data on M described in [10].

Corollary 2.4. *Let H be a Hopf-algebra and S be an H-comodule algebra.*

- *Any isomorphism $f : M \to M'$ of S-modules induces a bijection $f^* : \mathrm{D}^1(H,M') \to \mathrm{D}^1(H,M)$.*
- *Any isomorphism $f : M \to M'$ of (H,S)-Hopf modules induces an isomorphism of pointed sets $f^* : \mathrm{D}^1(H,M') \to \mathrm{D}^1(H,M)$.*

Proof. Suppose that F_1 and F_2 are two cohomologous 1-cocycles of $\mathrm{C}^1(H,M')$, with $g \in \mathrm{Aut}_S(M')$ such that $F_1 = g^*F_2$. Then $f^*F_2 = f^*g^*F_1 = f^*g^*(f^{-1})^*f^*F_1 = (f^{-1}gf)^*(f^*F_1)$, so f^*F_1 and f^*F_2 are cohomologous in $\mathrm{C}^1(H,M)$. □

2.2. Application to the Galois case

We work now with the Hopf algebra k^G dual to the group algebra $k[G]$ for G a finite group. Let $\psi : R \to S$ be a k^G-Galois extension and M a (G,S)-Galois module. We may assume that M is already extended, so that M is equal to $N_0 \otimes_R S$ for an R-module N_0. Endow M with the canonical (H,S)-Hopf module structure given by the coaction $\Delta_M = \mathrm{id}_{N_0} \otimes \Delta_S$. In this paragraph, we compute the descent cohomology set of k^G with coefficients in $M = N_0 \otimes_R S$ in terms of the Galois 1-cohomology set of G with coefficients in $\mathrm{Aut}_S(M)$.

Recall that for any group G and any (left) G-group A, one classically defines two non-abelian cohomology sets of G with coefficients in A (see [12,13]). This is done in the following way. The 0-cohomology group $\mathrm{H}^0(G,A)$ is the group A^G of invariant elements of A under the action of G. The set $\mathrm{Z}^1(G,A)$ of 1-cocycles is given by

$$\mathrm{Z}^1(G,A) = \left\{\alpha \in \mathfrak{Set}(G,A) \,\middle|\, \alpha(gg') = \alpha(g)^g\left(\alpha(g')\right),\ \forall g, g' \in G\right\}.$$

It is pointed with distinguished point the constant map $1 : G \to A$. The group A acts on the right on $\mathrm{Z}^1(G,A)$ by

$$(\alpha \leftharpoonup a)(g) = a^{-1}\alpha(g)^g a,$$

where $a \in A$, $\alpha \in \mathrm{Z}^1(G, A)$, and $g \in G$. Two 1-cocycles α and α' are *cohomologous* if they belong to the same orbit under this action. The non-abelian 1-cohomology set $\mathrm{H}^1(G, A)$ is the left quotient $A\backslash \mathrm{Z}^1(G, A)$. Then $\mathrm{H}^1(G, A)$ is pointed with distinguished point the class of the constant map $1 : G \to A$.

Let G be a finite group, $\psi : R \to S$ be a G-Galois extension, and $M = N_0 \otimes_R S$ be the extended S-module of an R-module N_0. The S-module M is a (G, S)-Galois module by the canonical action given on an indecomposable tensor $n \otimes s \in N_0 \otimes_R S$ by

$$g(n \otimes s) = n \otimes g(s),$$

where $g \in G$, $n \in N_0$, and $s \in S$. The group G acts by automorphisms on $\mathrm{Aut}_S(M)$ by

$${}^g f = (\mathrm{id}_{N_0} \otimes g) \circ f \circ \big(\mathrm{id}_{N_0} \otimes g^{-1}\big),$$

where $g \in G$ and $f \in \mathrm{Aut}_S(M)$. Hence $\mathrm{Aut}_S(M)$ becomes a G-group and we get at our disposal the two non-abelian cohomology sets $\mathrm{H}^0(G, \mathrm{Aut}_S(M))$ and $\mathrm{H}^1(G, \mathrm{Aut}_S(M))$.

Proposition 2.5. *Let G be a finite group, $\psi : R \to S$ be a G-Galois extension, and $M = N_0 \otimes_R S$ be the extended S-module of an R-module N_0. There is the equality of groups*

$$\mathrm{D}^0\big(k^G, M\big) = \mathrm{H}^0\big(G, \mathrm{Aut}_S(M)\big)$$

and an isomorphism of pointed sets

$$\mathrm{D}^1\big(k^G, M\big) \cong \mathrm{H}^1\big(G, \mathrm{Aut}_S(M)\big).$$

Proof. Let us prove the equality between the groups. It is sufficient to show that for any $f \in \mathrm{Aut}_S(M)$, the condition $(f \otimes \mathrm{id}_{kG}) \circ \Delta_M = \Delta_M \circ f$ is equivalent to the fact that f is G-invariant. Indeed, the first condition reflects that f belongs to $\mathrm{D}^0(k^G, M)$, whereas $\mathrm{H}^0(G, \mathrm{Aut}_S(M))$ is precisely the group $\mathrm{Aut}_S(M)^G$ of G-invariant automorphisms in $\mathrm{Aut}_S(M)$. Pick $f \in \mathrm{Aut}_S(M)$, $n \in N_0$, and $s \in S$. One has

$$\begin{aligned}\big((f \otimes \mathrm{id}_{kG}) \circ \Delta_M\big)(n \otimes s) &= \sum_{g \in G} (f \otimes \mathrm{id}_{kG})\big(n \otimes g(s) \otimes \delta_g\big) \\ &= \sum_{g \in G} \big(f \circ (\mathrm{id}_{N_0} \otimes g)\big)(n \otimes s) \otimes \delta_g.\end{aligned}$$

On the other hand, setting $f(n \otimes s) = n' \otimes s'$, one gets

$$(\Delta_M \circ f)(n \otimes s) = \Delta_M(n' \otimes s') = \sum_{g \in G} \big(n' \otimes g(s')\big) \otimes \delta_g = \sum_{g \in G} \big((\mathrm{id}_{N_0} \otimes g) \circ f\big)(n \otimes s) \otimes \delta_g.$$

Since $\{\delta_g\}_{g \in G}$ is a basis of k^G, the relation $(f \otimes \mathrm{id}_{kG}) \circ \Delta_M = \Delta_M \circ f$ is equivalent to the set of equalities $f \circ (\mathrm{id}_{N_0} \otimes g) = (\mathrm{id}_{N_0} \otimes g) \circ f$, with g running through G. This exactly means that f is G-invariant in $\mathrm{Aut}_S(M)$.

We prove now the isomorphism on the 1-cohomology level. Let us show that any $F \in \mathrm{C}^1(k^G, M)$ induces a (G, S)-Galois module action $\gamma : G \to \mathrm{Aut}_S^\gamma(M)$ defined by

$$F(m) = \sum_{g \in G} \big(\gamma(g)\big)(m) \otimes \delta_g.$$

For simplicity denote $\gamma(g)(m)$ by $g(m)$. The k-linearity of F tells us that $g(m + m') = g(m) + g(m')$, for any $g \in G$ and $m, m' \in M$; the equality $(\mathrm{id}_M \otimes \varepsilon_{k^G}) \circ F = \mathrm{id}_M$ implies that $1(m) = m$; the coassociativity condition of F says that $(gg')(m) = g(g'(m))$, for any $g, g' \in G$ and $m \in M$; finally the (k^G, S)-linearity of F is equivalent to the (G, S)-linearity of γ. As shown in [10], the action map γ gives rise to the 1-Galois cocycle $\alpha : G \to \mathrm{Aut}_S^\gamma(M)$ defined by

$$\alpha(g) = \gamma(g) \circ \big(\mathrm{id}_{N_0} \otimes g^{-1}\big).$$

It is easy to check that the correspondence between F and α is bijective. Thus already at the 1-cocycle level there exists a bijection between $\mathrm{Z}^1(G, \mathrm{Aut}_S(M))$ and $\mathrm{C}^1(k^G, M)$.

Take two cocycles F and F' in $\mathrm{C}^1(k^G, M)$. Denote by γ (respectively γ') the corresponding Galois actions and by α (respectively α') the Galois cocycles associated with γ (respectively γ'). Suppose that the cocycles F and F' are cohomologous, with $f \in \mathrm{Aut}_S(M)$ such that $(f \otimes \mathrm{id}_{k^G}) \circ F = F' \circ f$. Then $f \circ \gamma(g) = \gamma'(g) \circ f$, for all $g \in G$, or equivalently $\gamma(g) = f^{-1} \circ \gamma'(g) \circ f$. Therefore

$$\begin{aligned}
\alpha(g) &= f^{-1} \circ \gamma'(g) \circ f \circ \big(\mathrm{id}_{N_0} \otimes g^{-1}\big) \\
&= f^{-1} \circ \gamma'(g) \circ \big(\mathrm{id}_{N_0} \otimes g^{-1}\big) \circ (\mathrm{id}_{N_0} \otimes g) \circ f \circ \big(\mathrm{id}_{N_0} \otimes g^{-1}\big) \\
&= f^{-1} \circ \alpha'(g) \circ {}^g f,
\end{aligned}$$

which means that α and α' are Galois-cohomologous. Conversely, the previous equalities show that two cohomologous Galois cocycles α and α' give rise to two cohomologous cocycles F and F' in $\mathrm{C}^1(k^G, M)$. □

2.3. Comparison between the 1-descent cohomology set and the set of twisted forms in the Hopf-Galois context

Let H be a Hopf-algebra, $\psi : R \to S$ be an H-Hopf-Galois extension, and $M = N_0 \otimes_R S$ be the extended S-module of an R-module N_0. We endow M with the canonical (H, S)-Hopf module structure given by the coaction $\Delta_M = \mathrm{id}_{N_0} \otimes \Delta_S$. The main result of this paragraph asserts that the descent 1-cohomology set $\mathrm{D}^1(H, M)$ is isomorphic to the pointed set $\mathrm{Twist}(S/R, N_0)$ of twisted forms of N_0 up to isomorphisms.

Theorem 2.6. *Let H be a Hopf-algebra, $\psi : R \to S$ be an H-Hopf-Galois extension, and $M = N_0 \otimes_R S$ be the extended S-module of an R-module N_0. Then there is an isomorphism of pointed sets*

$$\mathrm{D}^1(H, M) \cong \mathrm{Twist}(S/R, N_0).$$

In order to prove Theorem 2.6, we need an intermediate result. For any $F \in \mathrm{C}^1(H, M)$ denote by N_F the R-module of F-coinvariants, that is $N_F = \{m \in M \mid F(m) = m \otimes 1\}$. We state the following lemma.

Lemma 2.7. *Under the same hypotheses as in Theorem* 2.6*, for any $F \in \mathrm{C}^1(H, M)$, there exists an isomorphism*

$$\varphi_F : N_F \otimes_R S \xrightarrow{\sim} M$$

given by $\varphi_F(m \otimes s) = ms$, for any $m \in N_F$ and $s \in S$.

Proof. The existence of the isomorphism φ_F results from Hopf-Galois descent theory [11, Theorem 3.7] (see also [5]). Indeed, consider the functor "*restriction of scalars*" $\psi^* : \mathfrak{Mod}_S \to \mathfrak{Mod}_R$ and its left adjoint functor $\psi_! : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$, the functor "*extension of scalars*." Then φ_F is nothing but a counit for the comonad on $\mathfrak{Mod}_S$ induced by the adjunction $\psi_! \dashv \psi^*$ (see, e.g., [4]).

We explicit now the expression of φ_F. By arguments stemming from descent theory [3,10], the S-module M is isomorphic to $N_d \otimes_R S$, where N_d is the R-module deduced from the Cipolla descent data d on M associated to the (H, S)-Hopf module structure of M. By [10, Proposition 4.10], d is the map given by the composition

$$M \xrightarrow{F} M \otimes H \xrightarrow{\beta} M \otimes_S (S \otimes H) \xrightarrow{\mathrm{id}_M \otimes \Gamma_\psi^{-1}} M \otimes_S (S \otimes_R S) \xrightarrow{\beta'} M \otimes_R S,$$

where β (respectively β') is the obvious k-linear (respectively S-linear) isomorphism and Γ_ψ is the Galois isomorphism mentioned in the Conventions.

Let us now compute d. For $m \in M$, set $F(m) = \sum_i m_i \otimes h_i \in M \otimes H$. For any fixed index i, set $\Gamma_\psi^{-1}(1 \otimes h_i) = \sum_j s_{ij} \otimes t_{ij}$, or equivalently $\sum_j s_{ij} \Delta_S(t_{ij}) = 1 \otimes h_i$. So

$$d(m) = \sum_i \sum_j m_i s_{ij} \otimes t_{ij}.$$

According to [10, Corollary 4.11], we have $N_d = \{m \in M \mid \sum_i \sum_j m_i s_{ij} \Delta_S(t_{ij}) = m \otimes 1\}$, therefore

$$N_d = \left\{m \in M \;\middle|\; \sum_i m_i \otimes h_i = m \otimes 1\right\} = \left\{m \in M \;\middle|\; F(m) = m \otimes 1\right\} = N_F.$$

It is proven in [3] that the descent isomorphism from $N_d \otimes_R S$ to M is given by the correspondence $m \otimes s \mapsto ms$ for $m \in N_d$ and $s \in S$. □

Proof of Theorem 2.6. Let F be an element of $\mathrm{C}^1(H, M)$ and φ_F be the isomorphism from $N_F \otimes_R S$ to $M = N_0 \otimes_R S$ given by the previous lemma. The datum (N_F, φ_F) is a twisted form of N_0. Denote by $\tilde{\mathcal{T}}$ the map from $\mathrm{C}^1(H, M)$ to the set $\mathrm{twist}(S/R, N_0)$ defined by

$$\tilde{\mathcal{T}}(F) = (N_F, \varphi_F).$$

The map $\tilde{\mathcal{T}}$ obviously sends the distinguished point Δ_M of $\mathrm{C}^1(H, M)$ to the distinguished point $(N_0, \mathrm{id}_{N_0 \otimes_R S})$ of twist$(S/R, N_0)$.

Suppose that F and F' are cohomologous in $\mathrm{C}^1(H, M)$. We claim that the corresponding descended modules N_F and $N_{F'}$ are isomorphic in $\mathfrak{Mod}_R$. Indeed, let $f \in \mathrm{Aut}_S(M)$ such that $(f \otimes \mathrm{id}_H) \circ F = F' \circ f$. For any $n \in N_F$, the image $f(n)$ belongs to $N_{F'}$, since

$$F'\big(f(n)\big) = (f \otimes \mathrm{id}_H)\big(F(n)\big) = (f \otimes \mathrm{id}_H)(n \otimes 1) = f(n) \otimes 1.$$

So the automorphism f induces an isomorphism from N_F to $N_{F'}$. From this fact we deduce a quotient map

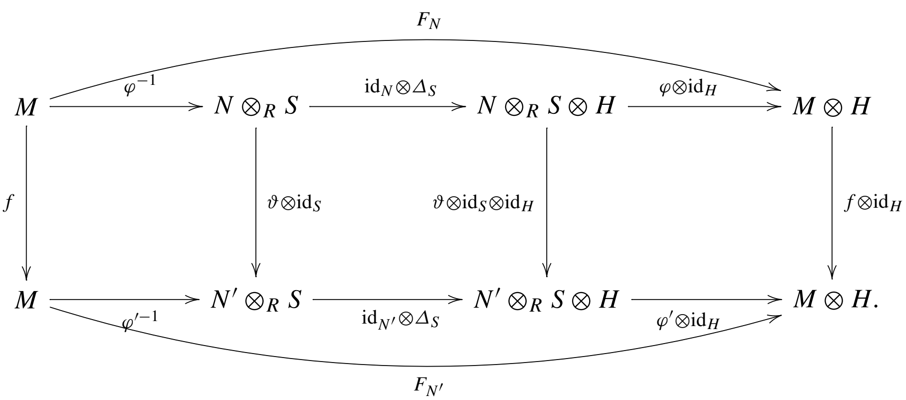
$$\mathcal{T} : \mathrm{D}^1(H, M) \to \mathrm{Twist}(S/R, N_0).$$

We now prove that $\mathcal{T}$ is an isomorphism of pointed sets. In order to do this, we introduce the map $\tilde{\mathcal{D}} : \mathrm{twist}(S/R, N_0) \to \mathrm{C}^1(H, M)$ which associates to any twisted form (N, φ) of M the map $F_N : M \to M \otimes H$ defined by

$$F_N = \big(\varphi^{-1}\big)^*(\mathrm{id}_N \otimes \Delta_S) = (\varphi \otimes \mathrm{id}_H) \circ (\mathrm{id}_N \otimes \Delta_S) \circ \varphi^{-1}.$$

Since $(\mathrm{id}_N \otimes \Delta_S)$ is the canonical (H, S)-Hopf module structure on $N \otimes_R S$, by Lemma 2.2, the map F_N belongs to $\mathrm{C}^1(H, M)$.

Suppose that (N, φ) and (N', φ') are two equivalent twisted forms of M via $\vartheta \in \mathrm{Aut}_S(M)$. Set $f = \varphi' \circ (\vartheta \otimes \mathrm{id}_S) \circ \varphi^{-1}$. Observe that the following diagram commutes:



So $F_{N'}$ equals $f^* F_N$ and therefore $\tilde{\mathcal{D}}$ induces a quotient map

$$\mathcal{D} : \mathrm{Twist}(S/R, N_0) \to \mathrm{D}^1(H, M).$$

It remains to prove that $\mathcal{T} \circ \mathcal{D}$ and $\mathcal{D} \circ \mathcal{T}$ are the identity maps.

The composition $\mathcal{T} \circ \mathcal{D}$ is the identity. Let (N, φ) be a twisted form of N_0. Since $N_{F_N} \otimes_R S$ is isomorphic to $N \otimes_R S$ (Lemma 2.7), we deduce from Hopf-Galois descent theory [11, Theorem 3.7] the existence of an isomorphism $\vartheta : N \to N_{F_N}$. So the twisted form $\tilde{\mathcal{T}}(\tilde{\mathcal{D}}(N, \varphi))$ is equivalent to (N, φ). In concrete terms, ϑ fits into the following commutative diagram of R-modules with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & N \otimes_R S & \overset{\mathrm{id}_N \otimes \Delta_S}{\underset{(\mathrm{id}_{N \otimes_R S}) \otimes \eta_H}{\rightrightarrows}} & N \otimes_R S \otimes H \\
 & & \wr \downarrow \vartheta & & \wr \downarrow \varphi & & \wr \downarrow \varphi \otimes \mathrm{id}_H \\
0 & \longrightarrow & N_{F_N} & \hookrightarrow & M & \overset{F_N}{\underset{\mathrm{id}_M \otimes \eta_H}{\rightrightarrows}} & M \otimes H.
\end{array}$$

Hence one gets $\mathcal{T} \circ \mathcal{D} = \mathrm{id}$.

The composition $\mathcal{D} \circ \mathcal{T}$ is the identity. Let F be an element of $\mathrm{C}^1(M, H)$. Consider the following diagram:

$$\begin{array}{ccccc}
M & & \xrightarrow{\quad F_{N_F} \quad} & & M \otimes H \\
\| & \searrow^{\varphi_F^{-1}} & & \swarrow^{\varphi_F^{-1} \otimes \mathrm{id}_H} & \| \\
 & N_F \otimes_R S & \xrightarrow{\mathrm{id}_{N_F} \otimes \Delta_S} & N_F \otimes_R S \otimes H & \\
\| & \swarrow_{\varphi_F} & & \searrow_{\varphi_F \otimes \mathrm{id}_H} & \| \\
M & & \xrightarrow[\quad F \quad]{} & & M \otimes H.
\end{array}$$

The left and right triangles are trivially commutative. The upper trapezium commutes by the definition of F_{N_F}. Let us show the commutativity of the lower trapezium. Pick an indecomposable tensor $m \otimes s$ in $N_F \otimes_R S$. Setting $\Delta_S(s) = s_0 \otimes s_1$, we have

$$(\varphi_F \otimes \mathrm{id}_H) \circ (\mathrm{id}_{N_F} \otimes \Delta_S)(m \otimes s) = \varphi_F(m \otimes s_0) \otimes s_1 = ms_0 \otimes s_1.$$

The latter equality comes from Lemma 2.7. On the other hand, using the (H, S)-linearity of F, one has

$$(F \circ \varphi_F)(m \otimes s) = F(ms) = F(m)\Delta_S(s) = ms_0 \otimes s_1.$$

So the whole diagram is commutative. Hence we obtain $F = F_{N_F}$, which means $\tilde{\mathcal{D}} \circ \tilde{\mathcal{T}} = \mathrm{id}$. Therefore we conclude $\mathcal{D} \circ \mathcal{T} = \mathrm{id}$. □

2.4. The 1-descent cohomology set and torsors

Let G be a finite group and A be a G-group. Recall that an *A-torsor* (or *A-principal homogeneous space*) is a non-empty G-set P on which A acts on the right in a compatible way with the G-action and such that P is an affine space over A (see [13]). Pursuing our analogy between non-abelian group- and Hopf-cohomology theories, we are led to state the following definition.

Let H be a Hopf algebra and M be an (H,S)-Hopf module. An *M-torsor* is a triple (X,Δ_X,β), where $\Delta_X : X \to X \otimes H$ is a map conferring X a structure of (H,S)-Hopf module and $\beta : M \to X$ is an S-linear isomorphism.

Denote by $\mathrm{tors}(M)$ the set of M-torsors. It is pointed with distinguished point $(M,\Delta_M,\mathrm{id}_M)$. We say that two M-torsors (X,Δ_X,β) and $(X',\Delta_{X'},\beta')$ are *equivalent* if there exists $f \in \mathrm{Aut}_S(M)$ such that the composition $\beta \circ f \circ \beta'^{-1} : X' \to X$ is a morphism of (H,S)-Hopf modules. Denote by $\mathrm{Tors}(M)$ the set of equivalence classes of M-torsors; it is pointed with distinguished point the class of $(M,\Delta_M,\mathrm{id}_M)$. We have the following result.

Proposition 2.8. *Let H be a Hopf algebra and M be an (H,S)-Hopf module. There is an isomorphism of pointed sets*

$$\mathrm{D}^1(H,M) \cong \mathrm{Tors}(M).$$

Proof. Define $\tilde{\mathcal{U}} : \mathrm{C}^1(H,M) \to \mathrm{tors}(M)$ and $\tilde{\mathcal{V}} : \mathrm{tors}(M) \to \mathrm{C}^1(H,M)$ by

$$\tilde{\mathcal{U}} : F \longmapsto (M,F,\mathrm{id}_M) \quad \text{and} \quad \tilde{\mathcal{V}} : (X,\Delta_X,\beta) \longmapsto \beta^*\Delta_X.$$

We set here $\beta^*\Delta_X = (\beta^{-1} \otimes \mathrm{id}_H) \circ \Delta_X \circ \beta$, which, following Lemma 2.2, is an element of $\mathrm{C}^1(H,M)$ since Δ_X belongs to $\mathrm{C}^1(H,X)$. Using again Lemma 2.2, it is easy to check that $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ define maps $\mathcal{U} : \mathrm{D}^1(H,M) \to \mathrm{Tors}(M)$ and $\mathcal{V} : \mathrm{Tors}(M) \to \mathrm{D}^1(H,M)$ on the quotients.

It is straightforward to prove $\tilde{\mathcal{V}} \circ \tilde{\mathcal{U}} = \mathrm{id}_{\mathrm{C}i^1(H,M)}$. Moreover, the torsor $(\tilde{\mathcal{U}} \circ \tilde{\mathcal{V}})(X,\Delta_X,\beta)$ equals $(M,\beta^*\Delta_X,\mathrm{id}_M)$, which, via $f = \mathrm{id}_M$, is equivalent to (X,Δ_X,β) in $\mathrm{tors}(M)$. □

3. The isomorphism between Hopf cohomology sets and descent cohomology sets

In this section, we interpret the non-commutative Hopf cohomology sets in terms of the descent cohomology sets.

Let H be a Hopf-algebra and $(M,\Delta_M : M \to M \otimes H)$ be an H-comodule. We define a map $\tilde{\kappa}$ from $\mathrm{W}^1_k(M)$ to itself by the formula

$$\tilde{\kappa}(\Phi) = \Phi \odot \Delta_M,$$

for any $\Phi \in \mathrm{W}^1_k(M)$. The map $\tilde{\kappa}$ is a bijection. Indeed, denote by Δ'_M the map $(\mathrm{id}_M \otimes \sigma_H) \circ \Delta_M$, which is easily seen to be the $\odot$-inverse of Δ_M in $\mathrm{W}^1_k(M)$. The inverse map of $\tilde{\kappa}$ is therefore given by

$$\tilde{\kappa}^{-1}(\Phi) = \Phi \odot \Delta'_M.$$

Theorem 3.1. *Let H be a Hopf-algebra, S be an H-comodule algebra, and M be an (H,S)-Hopf module with coaction $\Delta_M : M \to M \otimes H$. The identity map $\mathrm{id}_{\mathrm{Aut}_S(M)}$ realizes the equality of groups*

$$\mathrm{H}^0(H,M) = \mathrm{D}^0(H,M).$$

The translation map $\tilde{\kappa}$ induces an isomorphism of pointed sets

$$\kappa : \mathrm{H}^1(H,M) \to \mathrm{D}^1(H,M).$$

As a consequence of this result and of Proposition 2.5, one immediately gets the following corollary which relates non-abelian Hopf-cohomology objects to non-abelian group-cohomology objects.

Corollary 3.2. *Let G be a finite group, $\psi : R \to S$ be a G-Galois extension, and $M = N_0 \otimes_R S$ be the extended S-module of an R-module N_0. There is the equality of groups*

$$\mathrm{H}^0\big(k^G, M\big) = \mathrm{H}^0\big(G, \mathrm{Aut}_S(M)\big)$$

and an isomorphism of pointed sets

$$\mathrm{H}^1\big(k^G, M\big) \cong \mathrm{H}^1\big(G, \mathrm{Aut}_S(M)\big).$$

Proof of Theorem 3.1.

0-*level.* Let φ be an element of $\mathrm{H}^0(H, M)$. Then, by definition we have $d^0\varphi = d^1\varphi$. This equality implies

$$(\mathrm{id}_M \otimes \mu_H)\big(d^0\varphi \otimes \mathrm{id}_H\big)\Delta_M = (\mathrm{id}_M \otimes \mu_H)\big(d^1\varphi \otimes \mathrm{id}_H\big)\Delta_M.$$

Let us compute the left-hand side on an element $m \in M$. We get

$$\begin{aligned}(\mathrm{id}_M \otimes \mu_H)\big(d^0\varphi \otimes \mathrm{id}_H\big)\Delta_M(m) &= \varphi(m_0)_0 \otimes \varphi(m_0)_1\sigma_H(m_1)m_2 = \varphi(m_0)_0 \otimes \varphi(m_0)_1\varepsilon_H(m_1)\\ &= (\Delta_M \circ \varphi)\big(m_0\varepsilon_H(m_1)\big) = (\Delta_M \circ \varphi)(m).\end{aligned}$$

The right-hand side applied to $m \in M$ is equal to

$$(\mathrm{id}_M \otimes \mu_H)\big(d^1\varphi \otimes \mathrm{id}_H\big)\Delta_M(m) = \varphi(m_0) \otimes 1_H m_1 = \varphi(m_0) \otimes m_1 = (\varphi \otimes \mathrm{id}_H)\Delta_M(m).$$

Thus, one has $\Delta_M \circ \varphi = (\varphi \otimes \mathrm{id}_H) \circ \Delta_M$, and therefore f belongs to $\mathrm{D}^0(H, M)$.

Conversely, let f be an element of $\mathrm{D}^0(H, M)$. It satisfies the relation $(f \otimes \mathrm{id}_H) \circ \Delta_M = \Delta_M \circ f$. Compose each term of this equality on the left with $(\mathrm{id}_M \otimes \mu_H) \circ (\Delta_M \otimes \sigma_H)$. The left-hand side becomes then exactly $d^0 f$. Apply the right-hand side on $m \in M$. Setting $m' = f(m)$, we get

$$m'_0 \otimes m'_1\sigma_H\big(m'_2\big) = m'_0 \otimes \varepsilon_H\big(m'_1\big)1_H = m' \otimes 1_H = f(m) \otimes 1_H = d^1 f(m).$$

Therefore $d^0 f$ equals $d^1 f$, hence f belongs to $\mathrm{H}^0(H, M)$.

1-*level.* We begin to prove that $\tilde\kappa$ restricts to a bijection, still denoted by $\tilde\kappa$, from $\mathrm{Z}^1(H, M)$ to $\mathrm{C}^1(H, M)$. With the aim to do that, we shall show that via $\tilde\kappa$, for any $i = 1, 2, 3$, Condition ZC_i of Section 1.1 is equivalent to Condition CC_i of Section 2.1. We then prove that the bijection $\tilde\kappa$ induces a quotient map $\kappa : \mathrm{H}^1(H, M) \to \mathrm{D}^1(H, M)$ which is an isomorphism. Adopt the following notations. For $\Phi \in \mathrm{Z}^1(H, M)$ and $m \in M$, we denote the tensor $\Phi(m) \in M \otimes H$ by $m_{[0]} \otimes m_{[1]}$. Similarly, for $F \in \mathrm{C}^1(H, M)$ and $m \in M$, we set $F(m) = m_{(0)} \otimes m_{(1)}$.

– *Equivalence of Conditions* ZC_1 *and* CC_1. Fix $\Phi \in \mathrm{Z}^1(H, M)$ and set $F = \tilde{\kappa}(\Phi) = \Phi \odot \Delta_M$. So, for any $m \in M$, we have $F(m) = (m_0)_{[0]} \otimes (m_0)_{[1]} m_1$. Pick now $s \in S$. Condition ZC_1 on Φ means $(ms)_{[0]} \otimes (ms)_{[1]} = m_{[0]} s \otimes m_{[1]}$. Let us compute $F(ms)$:

$$\begin{aligned} F(ms) &= \big((ms)_0\big)_{[0]} \otimes \big((ms)_0\big)_{[1]} (ms)_1 \\ &= (m_0 s_0)_{[0]} \otimes (m_0 s_0)_{[1]} m_1 s_1 \\ &= (m_0)_{[0]} s_0 \otimes (m_0)_{[1]} m_1 s_1 \\ &= F(m) \Delta_S(s). \end{aligned}$$

We use here the fact that Δ_M is twisted S-linear (second equality). Hence F verifies Condition CC_1.

Conversely, fix $F \in \mathrm{C}^1(H, M)$. Condition CC_1 on F means $(ms)_{(0)} \otimes (ms)_{(1)} = m_{(0)} s_0 \otimes m_{(1)} s_1$, for any s in S. Set $\Phi = \tilde{\kappa}^{-1}(F) = F \odot \Delta'_M$, so $\Phi(m) = (m_0)_{(0)} \otimes (m_0)_{(1)} \sigma_H(m_1)$. Compute $\Phi(ms)$:

$$\begin{aligned} \Phi(ms) &= \big((ms)_0\big)_{(0)} \otimes \big((ms)_0\big)_{(1)} \sigma_H\big((ms)_1\big) \\ &= (m_0 s_0)_{(0)} \otimes (m_0 s_0)_{(1)} \sigma_H(s_1) \sigma_H(m_1) \\ &= (m_0)_{(0)} s_0 \otimes (m_0)_{(1)} s_1 \sigma_H(s_2) \sigma_H(m_1) \\ &= (m_0)_{(0)} s_0 \otimes (m_0)_{(1)} \varepsilon_H(s_1) \sigma_H(m_1) \\ &= (m_0)_{(0)} s \otimes (m_0)_{(1)} \sigma_H(m_1) \\ &= \Phi(m)(s \otimes 1). \end{aligned}$$

Thus Φ verifies Condition ZC_1.

– *Equivalence of Conditions* ZC_2 *and* CC_2. We still take $\Phi \in \mathrm{Z}^1(H, M)$ and set $F = \tilde{\kappa}(\Phi) = \Phi \odot \Delta_M$, so $F(m) = (m_0)_{[0]} \otimes (m_0)_{[1]} m_1$, for any $m \in M$. Pick $s \in S$. Condition ZC_2 on Φ is given by the relation $m_{[0]} \varepsilon_H(m_{[1]}) = m$. Let us verify Condition CC_2 for F:

$$\begin{aligned} (\mathrm{id}_M \otimes \varepsilon_H) F(m) &= (m_0)_{[0]} \varepsilon_H\big((m_0)_{[1]}\big) \varepsilon_H(m_1) \\ &= (m_0) \varepsilon_H(m_1) \\ &= (\mathrm{id}_M \otimes \varepsilon_H) \Delta_M(m) \\ &= m. \end{aligned}$$

Conversely, if F verifies Condition CC_2, an easy computation shows that $\Phi = F \odot \Delta'_M$ fulfils Condition ZC_2.

– *Equivalence of Conditions* ZC_3 *and* CC_3. We introduce the deformed differential map $\delta : \mathrm{W}^1_S(M) \to \mathrm{W}^2_S(M)$ defined on $\Phi \in \mathrm{W}^1_S(M)$ by the formula

$$\delta \Phi = (\mathrm{id}_M \otimes T) \circ d^2 \Phi$$

(recall that T is the flip of $H \otimes H$, see Section 1.1). We prove now that Condition CC_3 on $F \in \mathrm{W}^1_S(M)$ may be translated into the equality

$$d^2 F \odot \delta F = d^1 F. \tag{4}$$

Indeed, as a consequence of the definitions of $\odot$ and of d^2, one gets

$$\begin{aligned} d^2 F \odot \delta F &= \big(\mathrm{id}_M \otimes \mu_H^{\otimes 2}\big)(\mathrm{id}_M \otimes \chi_2)\big(d^2 F \otimes \mathrm{id}_H^{\otimes 2}\big)\delta F \\ &= \big(\mathrm{id}_M \otimes \mu_H^{\otimes 2}\big)(\mathrm{id}_M \otimes \chi_2)\big(F \otimes \eta_H \otimes \mathrm{id}_H^{\otimes 2}\big)\delta F. \end{aligned}$$

Take $m \in M$ and observe that we have $\delta F(m) = m_{(0)} \otimes 1 \otimes m_{(1)}$. Let us compute $(d^2 F \odot \delta F)(m)$:

$$\begin{aligned} \big(d^2 F \odot \delta F\big)(m) &= \big(\mathrm{id}_M \otimes \mu_H^{\otimes 2}\big)(\mathrm{id}_M \otimes \chi_2)\big(F \otimes \eta_H \otimes \mathrm{id}_H^{\otimes 2}\big)\delta F(m) \\ &= \big(\mathrm{id}_M \otimes \mu_H^{\otimes 2}\big)(\mathrm{id}_M \otimes \chi_2)\big(F \otimes \eta_H \otimes \mathrm{id}_H^{\otimes 2}\big)(m_{(0)} \otimes 1 \otimes m_{(1)}) \\ &= \big(\mathrm{id}_M \otimes \mu_H^{\otimes 2}\big)(\mathrm{id}_M \otimes \chi_2)\big(F(m_{(0)}) \otimes 1 \otimes 1 \otimes m_{(1)}\big) \\ &= F(m_{(0)}) \otimes m_{(1)} \\ &= \big((F \otimes \mathrm{id}_H) \circ F\big)(m). \end{aligned}$$

Since $d^1 F = (\mathrm{id}_M \otimes \Delta_H) \circ F$, Condition CC_3 is equivalent to equality (4).

Let Φ be an element of $\mathrm{W}^1_S(M)$. Set $F = \tilde{\kappa}(\Phi) = \Phi \odot \Delta_M$. We write down a sequence of equivalent assertions which begins with Condition ZC_3 on Φ and ends with an avatar of (4).

$$\begin{aligned} d^2\Phi \odot d^0\Phi = d^1\Phi \quad &\Longleftrightarrow \quad d^2\big(F \odot \Delta'_M\big) \odot d^0\big(F \odot \Delta'_M\big) = d^1\big(F \odot \Delta'_M\big) \\ &\Longleftrightarrow \quad d^2 F \odot d^2\Delta'_M \odot d^0 F \odot d^0\Delta'_M = d^1 F \odot d^1\Delta'_M \\ &\Longleftrightarrow \quad d^2 F \odot \big(d^2\Delta'_M \odot d^0 F \odot d^0\Delta'_M \odot d^1\Delta_M\big) = d^1 F. \end{aligned}$$

It suffices now to prove $d^2\Delta'_M \odot d^0 F \odot d^0\Delta'_M \odot d^1\Delta_M = \delta F$. For any $m \in M$, one has the two equalities $d^0\Delta'_M(m) = m_0 \otimes m_1\sigma_H(m_3) \otimes \sigma_H(m_2)$ and $d^1\Delta_M(m) = m_0 \otimes m_1 \otimes m_2$. Thus one gets

$$\begin{aligned} \big(d^0\Delta'_M \odot d^1\Delta_M\big)(m) &= m_0 \otimes m_1\sigma_H(m_3)m_4 \otimes \sigma_H(m_2)m_5 \\ &= m_0 \otimes m_1 \otimes \sigma_H(m_2)m_3 = m_0 \otimes m_1 \otimes 1. \end{aligned} \tag{5}$$

It remains to compute $(d^2\Delta'_M \odot d^0 F)(m)$. Denote the tensor $d^0 F(m_0) \in M \otimes H$ by $x \otimes y$, the summation being implicitly understood. Then $d^0 F(m)$ is given by $x_0 \otimes x_1\sigma_H(m_1) \otimes y$. We also have $d^2\Delta'_M(m) = m_0 \otimes \sigma_H(m_1) \otimes 1$. Therefore we get

$$\big(d^2\Delta'_M \odot d^0 F\big)(m) = x_0 \otimes \sigma_H(x_1)x_2\sigma_H(m_1) \otimes 1y = x \otimes \sigma_H(m_1) \otimes y. \tag{6}$$

Combining (5) and (6), one obtains

$$\begin{aligned}\big(\big(d^2\Delta'_M \odot d^0 F\big) \odot \big(d^0\Delta'_M \odot d^1\Delta_M\big)\big)(m) &= x \otimes \sigma_H(m_1)m_2 \otimes y1\\ &= x \otimes \varepsilon_H(m_1)1 \otimes y\\ &= (\mathrm{id}_M \otimes T)\big(x \otimes y \otimes \varepsilon_H(m_1)1\big)\\ &= (\mathrm{id}_M \otimes T)(F \otimes \mathrm{id}_H)\big(m_0 \otimes \varepsilon_H(m_1)1\big)\\ &= (\mathrm{id}_M \otimes T)\big(F(m) \otimes 1\big)\\ &= (\mathrm{id}_M \otimes T)\big(d^2 F\big)(m)\\ &= (\delta F)(m).\end{aligned}$$

– *Factorization of* $\tilde{\kappa}$. We claim that the bijection $\tilde{\kappa}$ factorizes through an isomorphism from $\mathrm{H}^1(H, M)$ to $\mathrm{D}^1(H, M)$. Indeed, take Φ and Φ' two cohomologous 1-Hopf cocycles and $f \in \mathrm{Aut}_S(M)$ satisfying the equality $d^1 f^{-1} \odot \Phi \odot d^0 f = \Phi'$. Set $F = \tilde{\kappa}(\Phi)$ and $F' = \tilde{\kappa}(\Phi')$. One has then the equivalences

$$\begin{aligned} d^1 f^{-1} \odot \Phi \odot d^0 f = \Phi' \quad &\Longleftrightarrow \quad d^1 f^{-1} \odot \big(F \odot \Delta'_M\big) \odot d^0 f = F' \odot \Delta'_M\\ &\Longleftrightarrow \quad F \odot \Delta'_M \odot d^0 f \odot \Delta_M = d^1 f \odot F'\\ &\Longleftrightarrow \quad F \odot d^1 f = d^1 f \odot F'\\ &\Longleftrightarrow \quad F \circ f = (f \otimes \mathrm{id}_H) \circ F'.\end{aligned}$$

The last equality means that F and F' are descent-cohomologous. Observe that the third equivalence is a consequence of the equality $d^0 f = \Delta_M \odot d^1 f \odot \Delta'_M$, which may be easily checked by the reader. □

Post-scriptum. The present work in its first preprint version led T. Brzeziński to generalize the descent cohomology to the coring framework [2]. For any coring C and any C-comodule M, this author defines two descent cohomology sets $\mathcal{D}^0(C, M)$ and $\mathcal{D}^1(C, M)$, which coincide respectively with $\mathcal{D}^0(H, M)$ and $\mathcal{D}^1(H, M)$ (notations of Section 2) when C is the coring $S \otimes H$.

References

[1] A. Blanco Ferro, Hopf algebras and Galois descent, Publ. Sec. Mat. Universitat Autònoma Barcelona 30 (1) (1986) 65–80.

[2] T. Brzeziński, Descent cohomology and corings, preprint, arXiv: math.RA/0601491, 2006.

[3] M. Cipolla, Discesa fedelmente piatta dei moduli, Rend. Circ. Mat. Palermo (2) 25 (1976).

[4] P. Deligne, Catégories tannakiennes, in: The Grothendieck Festschrift, vol. II, in: Progr. Math., vol. 87, Birkhäuser, Boston, MA, 1990, pp. 111–195.

[5] Y. Doi, M. Takeuchi, Hopf-Galois extensions of algebras, the Miyashita–Ulbrich action, and Azumaya algebras, J. Algebra 121 (2) (1989) 488–516.

[6] M.A. Knus, Quadratic and Hermitians Forms over Rings, Grundlehren Math. Wiss., vol. 294, Springer, Berlin, 1991.

[7] H.F. Kreimer, M. Takeuchi, Hopf algebras and Galois extensions of an algebra, Indiana Univ. Math. J. 30 (5) (1981) 675–692.

[8] S. Lang, J. Tate, Principal homogeneous spaces over abelian varieties, Amer. J. Math. 80 (1958) 659–684.

[9] L. le Bruyn, M. Van den Bergh, F. van Oystaeyen, Graded Orders, Birkhäuser, Boston, MA, 1988.

[10] P. Nuss, Noncommutative descent and non-abelian cohomology, K-Theory 12 (1) (1997) 23–74.

[11] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Israel J. Math. 72 (1–2) (1990) 167–195.
[12] J.-P. Serre, Corps locaux, troisième édition corrigée, Hermann, Paris, 1968.
[13] J.-P. Serre, Galois Cohomology, Springer, Berlin, 1997;
Translated from Cohomologie galoisienne, Lecture Notes in Math., vol. 5, Springer, Berlin, 1973.
[14] M.E. Sweedler, Cohomology of algebras over Hopf algebras, Trans. Amer. Math. Soc. 133 (1968) 205–239.