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Local Bézout Theorem☆

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ABSTRACT

We give an elementary proof of what we call the Local Bézout Theorem. Given a system of n polynomials in n indeterminates with coefficients in a Henselian local domain, $(\mathbf{V}, \mathfrak{m}, \mathbf{k})$, which residually defines an isolated point in $\mathbf{k}^n$ of multiplicity r, we prove (under some additional hypothesis on $\mathbf{V}$) that there are finitely many zeroes of the system above the residual zero (i.e., with coordinates in $\mathfrak{m}$), and the sum of their multiplicities is r. Our proof is based on techniques of computational algebra.

1. Introduction

In this paper we use ideas from computer algebra to prove what we call *Local Bézout Theorem* (Theorem 11). It is a formal abstract algebraic version of a well known theorem in the complex case. This classical theorem says that, given an isolated point of multiplicity r as a zero of an algebraic complete intersection, after deforming the coefficients of these equations we find in a sufficiently small neighborhood of this point exactly r isolated zeroes counted with multiplicities. As far as we know there is a proof of this result by Arnold using powerfully the topology of $\mathbb{C}^n$, (Arnold et al., 1985), and another by Gunning and Rossi generalizing it to analytic functions using coverings of analytic spaces, (Gunning and Rossi, 1965). Here we state and prove an *algebraic version* of this theorem in the setting of Henselian rings and $\mathfrak{m}$-adic topology. Nevertheless, we do not discuss whether the classical result follows formally from our theorem.

Roughly speaking, given a basis of the local algebra of the isolated point, for instance by monomials (as a vector space over the ground field), as the point is a complete intersection, "by flatness",

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the same set of monomials are a basis of the algebra after deformation (Theorem 5). This algebra after deformation is named by Arnold the "multilocal ring". The consequence of this flatness in computational algebra is that, given the multiplication matrices in the local algebra, you can lift them to the multiplication matrices in the multilocal ring by the Implicit Function Theorem. In fact, what you obtain is a presentation of this multilocal ring by the so called "border basis".

This situation is discussed in the example of Section 5 where it is shown that the notion of border basis, Mourrain (1999), turns out to be the natural and computational efficient representation in the deformed algebra. In fact, it allows us to get exact results with few floating point computations, which is impossible using Groebner bases.

But our aim in this paper is to go the other way around; we profit these ideas to get a constructive and elementary proof of the Local Bézout Theorem in some abstract algebraic frame, avoiding the use of topological arguments. Namely, we work with Henselian rings and DVR (discrete valuation ring) so that we are dealing with the $\mathfrak{m}$-adic topology, *i.e.* the topology given by the valuation. In this way, we are able to describe the multilocal ring, and our Local Bézout Theorem becomes a purely algebraic theorem. Our results are summed up in Theorem 1, Corollary 8, and Theorem 11.

Despite our theoretical interest, our proofs are an invitation to study stability of symbolic algorithms and the possibility of combining numerical and symbolic techniques if some kind of algebraic stability, as flatness, is given, (Alonso et al., 2009).

We explain now the purely algebraic form of the Local Bézout Theorem we are interested in.

Let $\mathbf{A}_v$ be a valuation domain with maximal ideal $\mathfrak{m}_v$ and let

$$\mathbf{B} := (\mathbf{A}_v[X_1,\ldots,X_n]/(F_1,\ldots,F_n))_{(\mathfrak{m}_v,\underline{x})} = \mathbf{A}_v[x_1,\ldots,x_n]_{(\mathfrak{m}_v,\underline{x})},$$

where $(\mathfrak{m}_v,\underline{x})$ is a notation for $\mathfrak{m}_v + (x_1,\ldots,x_n)$, x_i is X_i mod $(F_1,\ldots,F_n)$, and $F_i(\underline{0}) \in \mathfrak{m}_v$, $i = 1,\ldots,n$.

Let $\mathbf{k}_v$ be the residue field of $\mathbf{A}_v$ and $\mathbf{K}_v$ the fraction field of $\mathbf{A}_v$.

We assume that $\mathbf{K}_v$ is algebraically closed.

We assume that

$$\overline{\mathbf{B}} := (\mathbf{k}_v[X_1,\ldots,X_n]/(f_1,\ldots,f_n))_{(\underline{x})} = \mathbf{k}_v[x_1,\ldots,x_n]_{(\underline{x})}$$

is zero-dimensional, where $f_i = \overline{F_i}$.

Since $\mathbf{K}_v$ is algebraically closed it is plausible to speak about the continuity of the roots.

The algebraic form of Local Bézout Theorem says that there are finitely many zeroes of $F_1,\ldots,F_n$ above the residual zero $(\underline{0},\ldots,0)$ (i.e., with coordinates in $\mathfrak{m}_v$), and the sum of their multiplicities equals the dimension of $\overline{\mathbf{B}}$ as $\mathbf{k}_v$-vector space, i.e., the multiplicity of the residual zero.

This theorem implies that, when one starts with a system having a strongly isolated zero $(\underline{\xi})$ at finite distance (i.e., there is no other zero in the infinitesimal neighborhood $(\xi_1 + \mathfrak{m}_v,\ldots,\xi_n + \mathfrak{m}_v)$ of $(\underline{\xi})$), after an infinitesimal perturbation, the system of equations remains "zero-dimensional in the infinitesimal neighborhood of $(\underline{\xi})$, with the same multilocal multiplicity"; in other words, the zeroes inside this neighborhood remain isolated zeroes and the sum of multiplicities does not change.

In this paper, we get an elementary proof in two important particular cases of this fact, namely in the case of a DVR, and for some Henselian rings (see Theorem 11).

2. Effective Mather's division theorem

Theorem 1. *Let* $\mathbf{V}$ *be a local Henselian ring with maximal ideal* $\mathfrak{m}$ *and residue field* $\mathbf{k}$, $F_1,\ldots,F_m \in \mathbf{V}[X_1,\ldots,X_n]$, $f_1,\ldots,f_m$ *their images in* $\mathbf{k}[X_1,\ldots,X_n]$. *We assume that the residual local algebra*

$$(\mathbf{k}[X_1,\ldots,X_n]/(f_1,\ldots,f_m))_{(\underline{x})} = \mathbf{k}[x_1,\ldots,x_n]_{(\underline{x})}$$

is zero-dimensional. Then

$$\mathbf{B} := (\mathbf{V}[X_1,\ldots,X_n]/(F_1,\ldots,F_m))_{(\mathfrak{m},\underline{x})}$$

is a finitely generated $\mathbf{V}$*-module.*

Remark. In case $\mathbf{V}$ is a ring of formal, algebraic, or analytic power series with coefficients in $\mathbf{k}$, Theorem 1 is a consequence of Mather's division theorem (cf. Ruiz, 1993), and also a particular case of the Weierstrass division with parameters, (cf. Hironaka, 1977).

In our abstract setting, without assuming any finiteness property on the coefficient ring, the fact that $\mathbf{B}/\mathfrak{m}\mathbf{B}$ is zero-dimensional, together with the Henselianity property allow us to make a constructive proof "ad hoc".

This section is devoted to the proof of this theorem. We need some preliminaries.

2.1. Standard border basis

In the sequel we shall identify the semi-group $\langle X_1, \ldots, X_n\rangle$ with $\mathbb{N}^n$. Let $<$ be a degree-compatible term ordering in the semi-group $\langle X_1, \ldots, X_n\rangle = \mathbb{N}^n$. For a nonzero element $f = \frac{g}{1+h} \in \mathbf{k}[\underline{X}]_{(\underline{X})}$ with $g, h \in \mathbf{k}[\underline{X}]$, $h(\underline{0}) = 0$, its *initial term or leading term* $\mathsf{T}(f)$, is the minimal term of g. Its *initial monomial or leading monomial* is the corresponding monomial, written as $\mathsf{M}(f) = \mathsf{lc}(f)\mathsf{T}(f)$.

Let us consider a finitely generated ideal $\mathcal{I} = (g_1, \ldots, g_m)$ of $\mathbf{k}[\underline{X}]_{(\underline{X})}$. The *monomial ideal* $\mathsf{in}_<(\mathcal{I}) \subseteq \mathbf{k}[\underline{X}]$ is defined as the ideal generated by the $\mathsf{T}(f)$'s for nonzero $f \in \mathcal{I}$. By Dickson's lemma, the corresponding "ideal" E of $\mathbb{N}^n$ is a finite union of orthants $u + \mathbb{N}^n$. This nonconstructive result became constructive by Mora's algorithm (Mora, 1982; Alonso et al., 1992), that computes the corresponding values of u's.

A *standard basis* of $\mathcal{I}$ is given by any finite subset of $\mathcal{I}$ whose leading terms generate $\mathsf{in}_<(\mathcal{I})$. It is known that a standard basis generate the ideal $\mathcal{I}$ in $\mathbf{k}[\underline{X}]_{(\underline{X})}$.

Let $\mathrm{F} = \mathbb{N}^n \setminus \mathrm{E}$, its elements, "the monomials under the staircase", are called *standard monomials*. The "border" of E is the set

$$\mathcal{B}(\mathrm{E}) = \left\{ \alpha \in \mathrm{E} \mid \exists i \in [\![1..n]\!] \ \frac{X^\alpha}{X_i} \in \mathrm{F} \right\}.$$

E.g., with $n = 2$, if E is generated by $(3, 0)$, $(1, 3)$, $(0, 5)$ we get

$$\mathcal{B}(\mathrm{E}) = \{(3, 0), (3, 1), (3, 2), (2, 3), (1, 3), (1, 4), (0, 5)\}.$$

In the following we restrict ourselves to zero-dimensional ideals. In this case F is a finite set providing a basis of the finite-dimensional $\mathbf{k}$-vector space $\mathbf{k}[\underline{X}]_{(\underline{X})}/\mathcal{I}$. If $f \in \mathbf{k}[\underline{X}]_{(\underline{X})}$, its expression modulo $\mathcal{I}$ as a $\mathbf{k}$-linear combination of standard monomials is called the canonical form of f and is denoted by $\mathrm{Can}_<(f, \mathcal{I})$.

The standard border basis of $\mathcal{I}$ (w.r.t. $<$) is the set

$$\{ X^\alpha - \mathrm{Can}_<(X^\alpha, \mathcal{I}) \mid \alpha \in \mathcal{B}(\mathrm{E}) \}.$$

The notion of border basis, without reference to a term ordering, has been introduced and studied by Mourrain (see Mourrain, 1999).

2.2. Finiteness generation

We go back to the hypotheses of Theorem 1, we call

$$\mathcal{J} = (F_1, \ldots, F_m) \subseteq \mathbf{C} = \mathbf{V}[\underline{X}]_{(\mathfrak{m},\underline{X})} \quad \text{and} \quad \mathcal{I} = \overline{\mathcal{J}} = (f_1, \ldots, f_m) \subseteq \mathbf{k}[\underline{X}]_{(\underline{X})}.$$

We consider the standard border basis of the ideal $\mathcal{I}$ w.r.t. a degree-compatible term ordering $<$

$$\Sigma = \{ g_\alpha = X^\alpha - \mathrm{Can}_<(X^\alpha, \mathcal{I}) \mid \alpha \in \mathcal{B}(\mathrm{E}) \}.$$

Claim 2. *We construct a lifting* $\{ H_\alpha \mid \alpha \in \mathcal{B}(\mathrm{E}) \}$ *of* Σ *in* $F_1\mathbf{V}[\underline{X}] + \cdots + F_m\mathbf{V}[\underline{X}]$*:*

$$\overline{H_\alpha} = g_\alpha, \quad H_\alpha \in \mathcal{J}, \ \forall \alpha \in \mathcal{B}(\mathrm{E}).$$

If moreover $m = n$ *then*

$$(H_\alpha; \alpha \in \mathcal{B}(\mathrm{E})) = \mathcal{J} \quad (\textit{ideals of } \mathbf{C}).$$

Proof. If $g_\alpha = \sum_{i=1}^m u_{\alpha,i} f_i$, with $u_{\alpha,i} \in \mathbf{k}[\underline{X}]_{(\underline{X})}$, we take $H_\alpha = \sum_{i=1}^m U_{\alpha,i} F_i$ where $U_{\alpha,i} \in \mathbf{C}$, and $\overline{U_{\alpha,i}} = u_{\alpha,i}$.
If moreover $m = n$ let us write

$$f_i = \sum_{\alpha\in\mathcal{B}(\mathrm{E})} v_{i,\alpha} g_\alpha = \sum_{\alpha\in\mathcal{B}(\mathrm{E}),j\in[\![1..n]\!]} v_{i,\alpha} u_{\alpha,j} f_j.$$

Since all syzygies for $(f_1, \ldots, f_n)$ are linear combinations of the trivial ones (Matsumura, 1986, Theorem 16.5)

$$s_{ij} = (0, \ldots, -f_j, \ldots, f_i, \ldots, 0), \quad 1 \le i < j \le n,$$

we get for each $k \in [\![1..n]\!]$

$$\Big(\sum_{\alpha\in\mathcal{B}(\mathrm{E})} v_{k,\alpha} u_{\alpha,1}, \ldots, -1 + \sum_{\alpha\in\mathcal{B}(\mathrm{E})} v_{k,\alpha} u_{\alpha,k}, \ldots, \sum_{\alpha\in\mathcal{B}(\mathrm{E})} v_{k,\alpha} u_{\alpha,n} \Big) = \sum_{i<j\in[\![1..n]\!]} c_{k,i,j} s_{ij}.$$

Let us prove that there exist $V_{i,\alpha}$'s in $\mathbf{C}$ such that $F_i = \sum_{\alpha\in\mathcal{B}(\mathrm{E})} V_{i,\alpha} H_\alpha$ for all $i \in [\![1..n]\!]$. This will be true if we find $C_{k,i,j}$'s such that for each $k \in [\![1..n]\!]$

$$\Big(\sum_{\alpha\in\mathcal{B}(\mathrm{E})} V_{k,\alpha} U_{\alpha,1}, \ldots, -1 + \sum_{\alpha\in\mathcal{B}(\mathrm{E})} V_{k,\alpha} U_{\alpha,k}, \ldots, \sum_{\alpha\in\mathcal{B}(\mathrm{E})} V_{k,\alpha} U_{\alpha,n} \Big) = \sum_{i<j\in[\![1..n]\!]} C_{k,i,j} S_{ij}$$

where

$$S_{ij} = (0, \ldots, -F_j, \ldots, F_i, \ldots, 0), \quad 1 \le i < j \le n.$$

Let us see this system of equations as a linear system over $\mathbf{C}$ with unknowns $V_{k,\alpha}$'s and $C_{k,i,j}$'s. This can be written $MA = \mathrm{I}_n$. E.g., with $\#\mathcal{B}(\mathrm{E}) = 5$ and $n = 3$ we get the matrices

$$M = \begin{bmatrix} U_{1,1} & U_{2,1} & U_{3,1} & U_{4,1} & U_{5,1} & F_2 & F_3 & 0 \\ U_{1,2} & U_{2,2} & U_{3,2} & U_{4,2} & U_{5,2} & -F_1 & 0 & F_3 \\ U_{1,3} & U_{2,3} & U_{3,3} & U_{4,3} & U_{5,3} & 0 & -F_1 & -F_2 \end{bmatrix}, \qquad A = \begin{bmatrix} V_{1,1} & V_{2,1} & V_{3,1} \\ V_{1,2} & V_{2,2} & V_{3,2} \\ V_{1,3} & V_{2,3} & V_{3,3} \\ V_{1,4} & V_{2,4} & V_{3,4} \\ V_{1,5} & V_{2,5} & V_{3,5} \\ C_{1,1,2} & C_{2,1,2} & C_{3,1,2} \\ C_{1,1,3} & C_{2,1,3} & C_{3,1,3} \\ C_{1,2,3} & C_{2,2,3} & C_{3,2,3} \end{bmatrix}.$$

Since the system $MA = \mathrm{I}_n$ admits a solution modulo $\mathfrak{m}$, the matrix $\overline{M}$ is residually surjective (i.e., the ideal of $n \times n$ minors equals (1) modulo $\mathfrak{m}$) and M is also surjective in $\mathbf{C}$. □

Claim 3. *We construct a lifting* $\{ G_\alpha \mid \alpha \in \mathcal{B}(\mathrm{E}) \}$ *of* Σ *in* $F_1\mathbf{V}[\underline{X}] + \cdots + F_m\mathbf{V}[\underline{X}]$ *such that*

$$\begin{array}{rcll} G_\alpha &=& X^\alpha + \sum_{\gamma\in\mathrm{F}} a_{\alpha,\gamma} X^\gamma, & a_{\alpha,\gamma} \in \mathbf{V},\ \overline{G_\alpha} = g_\alpha, \\ (G_\alpha; \alpha \in \mathcal{B}(\mathrm{E})) &=& (H_\alpha; \alpha \in \mathcal{B}(\mathrm{E})) & (\textit{ideals of } \mathbf{C}). \end{array}$$

Proof. Since the coefficient of X^α in H_α is a unity of $\mathbf{V}$, w.l.o.g., we may assume that it is 1. For $\gamma \in \mathrm{F}$ we call $h_{\alpha,\gamma}$ the coefficient of H_α for X^γ. Notice that our aim is to find $a_{\alpha,\gamma}$'s with $\overline{a_{\alpha,\gamma}} = \overline{h_{\alpha,\gamma}}$.

For this purpose, we consider new indeterminates $Z_{\alpha,\gamma}$ where $\alpha \in \mathcal{B}(\mathrm{E})$ and $\gamma \in \mathrm{F}$. We let

$$\widetilde{G_\alpha} = X^\alpha + \sum_{\gamma\in\mathrm{F}} (h_{\alpha,\gamma} - Z_{\alpha,\gamma}) X^\gamma.$$

So the formal $Z_{\alpha,\gamma}$'s have to be thought of as infinitesimals; since the values of the $Z_{\alpha,\gamma}$'s we are looking for must belong to $\mathfrak{m}$.
Let us call

$$\mathbf{V}' = \mathbf{V}[(Z_{\alpha,\gamma})_{\alpha\in\mathcal{B}(\mathrm{E}),\gamma\in\mathrm{F}}].$$

We are going to give a finite process that reduces an arbitrary monomial vX^{β}, where $v \in \mathbf{V}'$ and $\beta \in \mathrm{E}$, w.r.t. the $\widetilde{G_\alpha}$'s, producing a remainder with support in F.

The distance d of β to the border is defined as the minimal degree of a term τ such that there exists α in the border with $\tau \underline{X}^{\alpha} = X^{\beta}$. Our construction is by induction on this distance. If $\mathsf{d} = 0$ then the remainder is $vX^{\beta} - v\widetilde{G_\beta}$. If we know how to reduce monomials with distance d, let us consider the case where the distance of β to the border is $\mathsf{d}+1$. For some $i \in [\![1..n]\!]$ and some β' with distance d, we have $X^{\beta} = X_i X^{\beta'}$. If R is the remainder of $X^{\beta'}$, the monomials of $X_i R$ are either in F or in $\mathcal{B}(\mathrm{E})$.

We perform this reduction for all monomials of the H_α's which lie in E, beginning with the monomial X^{α} of H_α for each α. This first step of the process provides the coefficient 1 to each $Z_{\alpha,\gamma}$ in the remainder.

In the further steps other monomials of each H_α provide elements of $\mathfrak{m}[(Z_{\alpha,\gamma})]$ as coefficients of X^{γ} ($\gamma \in \mathrm{F}$) in the remainder. Finally we obtain the following form for each H_α ($\alpha \in \mathcal{B}(\mathrm{E})$):

$$H_\alpha = \sum_{\alpha' \in \mathcal{B}(\mathrm{E})} Q_{\alpha,\alpha'}(\underline{Z}, \underline{X})\, \widetilde{G_{\alpha'}} + \sum_{\gamma \in \mathrm{F}} R_{\alpha,\gamma}(\underline{Z})\, X^{\gamma} \quad \text{(coefficients in } \mathbf{V}).$$

Now we have to solve the system $R_{\alpha,\gamma}(\underline{Z}) = 0$ in $\mathbf{V}$. Clearly this system has a good shape which allows us to use the Multidimensional Hensel Lemma (see (Raynaud, 1970, Theorems 1 and 5, chap. 5), (Bochnak et al., 1998, Propositions 8.7.1 and 8.7.5)): the residual linear part is $\mathrm{Id}\, \underline{Z}^{\mathbf{t}} = 0$. The solution gives values $z_{\alpha,\gamma}$ in $\mathfrak{m}$ to all $Z_{\alpha,\gamma}$'s: we let $a_{\alpha,\gamma} = h_{\alpha,\gamma} - z_{\alpha,\gamma}$ and we get the claimed G_α's.

It remains to show that the G_α's we have constructed generate the same ideal as the H_α's. In fact one sees that $\det \big(Q_{\alpha,\alpha'}(\underline{z}, \underline{X})\big)_{\alpha,\alpha' \in \mathcal{B}(\mathrm{E})}$ is invertible in $\mathbf{C} = \mathbf{V}[\underline{X}]_{(\mathfrak{m},\underline{X})}$ since by construction $\overline{Q_{\alpha,\alpha'}(\underline{z}, \underline{X})} = \overline{Q_{\alpha,\alpha'}}(\underline{0}, \underline{0}) = \delta_{\alpha,\alpha'}$. □

Claim 4.

(1) *The reduction process we have described in the proof of Claim 3 works for any monomial X^{β}. This shows that any element of $\mathbf{V}[\underline{X}]$ is equivalent modulo $\mathcal{J}$ to a $\mathbf{V}$-linear combination of $\{ X^{\gamma} \mid \gamma \in \mathrm{F} \}$.*

(2) *Moreover any unit in the quotient ring $\mathbf{B} = (\mathbf{V}[\underline{X}]/(F_1, \ldots, F_m))_{(\mathfrak{m},\underline{X})}$ is equal to a $\mathbf{V}$-linear combination of $\{ x^{\gamma} \mid \gamma \in \mathrm{F} \}$.*

(3) *This shows that $\{ x^{\gamma} \mid \gamma \in \mathrm{F} \}$ generates the quotient ring $\mathbf{B}$ as a $\mathbf{V}$-module. So lifting a residual basis of the residual quotient ring $\mathbf{B}/\mathfrak{m}\mathbf{B} = \mathbf{k}[\underline{X}]_{(\underline{X})}/\mathcal{I} = \mathbf{k}[\underline{x}]_{(\underline{x})}$ gives a generating system of the $\mathbf{V}$-module $\mathbf{B}$.*

Proof. 1. Clear.

2. It is sufficient to show for $1 + Q(\underline{X}) \in \mathbf{V}[\underline{X}]$ with

$$Q(\underline{X}) = \sum_{\gamma \in \mathrm{F}, \gamma \neq 0} q_\gamma X^{\gamma},$$

that the element $1 + Q(\underline{x})$ has an inverse in $\mathbf{B}$ equal to some $\sum_{\gamma \in \mathrm{F}} y_\gamma x^{\gamma}$ ($y_\gamma \in \mathbf{V}$). We write the corresponding equation where the Y_γ's are unknowns in $\mathbf{V}$

$$\left(1 + \sum_{\gamma \in \mathrm{F}, \gamma \neq 0} q_\gamma x^{\gamma}\right) \left(\sum_{\gamma \in \mathrm{F}} Y_\gamma x^{\gamma}\right) = 1.$$

The reduction process of the left hand side provides a remainder. Writing this remainder as $1 + \sum_{\gamma \in \mathrm{F}, \gamma \neq 0} 0 \cdot x^{\gamma}$, we get a $\mathbf{V}$-linear system. If we can solve it we are done. We obtain a matrix equation (with a square matrix $M \in \mathbf{V}^{r \times r}$, $r = \#\mathrm{F}$)

$$M\, \underline{Y}^{\mathbf{t}} = [\, 1\ 0\ \cdots\ 0\,]^{\mathbf{t}}.$$

Since $\{ x^{\gamma} \mid \gamma \in \mathrm{F} \}$ is residually a basis of the $\mathbf{k}$-vector space $\mathbf{k}[\underline{x}]_{(\underline{x})}$, the residual linear system has a unique solution in $\mathbf{k}$. So $\overline{\det M}$ is nonzero in $\mathbf{k}$ and $\det M$ is invertible in $\mathbf{V}$.

3. Follows from 1 and 2. □

3. Zero-dimensional complete intersection

Theorem 5 (*Improvement of Theorem 1 in the Case of a DVR and a Complete Intersection*)**.** *Let $\mathbf{A}_v$ be a Henselian DVR with maximal ideal $\mathfrak{m}_v$ and residue field $\mathbf{k}_v$, $F_1, \ldots, F_n \in \mathbf{A}_v[X_1, \ldots, X_n]$, $f_1, \ldots, f_n$ their images in $\mathbf{k}_v[X_1, \ldots, X_n]$. We assume that*

$$(\mathbf{k}_v[X_1, \ldots, X_n]/(f_1, \ldots, f_n))_{(\underline{x})}$$

is zero-dimensional. Then

$$\mathbf{B} := (\mathbf{A}_v[X_1, \ldots, X_n]/(F_1, \ldots, F_n))_{(\mathfrak{m}_v, \underline{x})}$$

is a free $\mathbf{A}_v$-module and a basis is given by lifting any basis of the $\mathbf{k}_v$-vector space $\overline{\mathbf{B}} = \mathbf{k}_v[x_1, \ldots, x_n]_{(\underline{x})}$.

Proof. We continue using the notation of Claim 4. Let $(e_i)_{i \in [\![1..r]\!]}$ be an $\mathbf{A}_v$-linear generating system of $\mathbf{B}$ which is residually a basis. W.l.o.g. we assume that $e_i = e_i(\underline{x})$ comes from an element $e_i(\underline{X}) \in \mathbf{A}_v[\underline{X}]$. We have to show the $\mathbf{A}_v$-linear independence of this system. Let t be a generator of $\mathfrak{m}_v$. Let us consider an $\mathbf{A}_v$-linear dependence relation $\sum_{i=1}^{r} b_i e_i = 0$. Let k be the least t-order of the b_i's. Notice that $k \geq 1$, since residually the $(e_i)_{i \in [\![1..r]\!]}$'s are a basis. Hence it is sufficient to construct a new linear dependence relation with a smaller k. We have an equality

$$\sum_{i=1}^{r} b_i e_i(\underline{X}) = \sum_{j=1}^{n} F_j(\underline{X}) P_j(\underline{X}) \quad \text{with } P_j(\underline{X}) = \frac{Q_j(\underline{X})}{1 + R(\underline{X})} \text{ in } \mathbf{A}_v[\underline{X}]_{(\mathfrak{m}_v, \underline{X})}, \qquad \overline{R(\underline{0})} = 0. \tag{1}$$

Residually modulo $\mathfrak{m}_v$ we get in $\mathbf{k}[\underline{X}]_{(\underline{X})}$

$$\sum_{j=1}^{n} f_j(\underline{X}) p_j(\underline{X}) = 0.$$

Since $f_1(\underline{X}), \ldots, f_n(\underline{X})$ is a $\mathbf{k}[\underline{X}]_{(\underline{X})}$-regular sequence, the syzygies are generated by the trivial ones: $s_{ij} = (0, \ldots, -f_j, \ldots, f_i, \ldots, 0)$, $1 \leq i < j \leq n$ (Matsumura, 1986, Theorem 16.5). So

$$(p_1, \ldots, p_n) = \sum_{1 \leq i < j \leq n} v_{ij}(\underline{X}) s_{ij} \quad \text{in } \mathbf{k}[\underline{X}]_{(\underline{X})}.$$

Lifting this equality we get, with the trivial syzygies $S_{ij} = (0, \ldots, -F_j, \ldots, F_i, \ldots, 0)$

$$(P_1, \ldots, P_n) = \sum_{1 \leq i < j \leq n} V_{ij}(\underline{X}) S_{ij} \mod \mathfrak{m}_v \quad \text{in } \mathbf{A}_v[\underline{X}]_{(\underline{X})}.$$

We modify each P_i by subtracting the i-th coordinate in the right hand side, and we get

$$\widetilde{P_i} = P_i - \sum_{1 \leq \ell < i \leq n} V_{\ell i}(\underline{X}) F_\ell + \sum_{1 \leq i < k \leq n} V_{ik}(\underline{X}) F_k.$$

So $\widetilde{P_i} \in \mathfrak{m}_v \mathbf{A}_v[\underline{X}]_{(\underline{X})}$, Eq. (1) gives $\sum_{i=1}^{r} b_i e_i(\underline{X}) = \sum_{j=1}^{n} F_j(\underline{X}) \widetilde{P_j(\underline{X})}$ and both sides of the new equation can be divided by t. □

Claim 6 (*Multiplication Matrices*)**.** *With the same hypotheses as in Theorem 5, the reduction process described in the proof of Theorem 1 provides a monomial basis $\{ x^\gamma \mid \gamma \in \mathrm{F} \}$ for the $\mathbf{V}$-module $\mathbf{B}$ and the corresponding multiplication matrices w.r.t. this basis which have entries in $\mathbf{V}$. Residually this gives also the multiplication matrices w.r.t. the same basis viewed in the residual ring.*

Proof. Clear. □

We can generalize this theorem to some Henselian rings, as a consequence of the following proposition.

Proposition 7. *Let* $\mathbf{A}$ *be a local domain with maximal ideal* $\mathfrak{m}$, *containing a coefficient field* $\mathbf{k}$ *which is algebraically closed of characteristic zero. Let be* $F_1, \ldots, F_n \in \mathbf{A}[X_1, \ldots, X_n]$, *and* $f_1, \ldots, f_n$ *be their images in* $\mathbf{k}[X_1, \ldots, X_n]$, *and assume that*

$$(\mathbf{k}[X_1, \ldots, X_n]/(f_1, \ldots, f_n))_{(\underline{x})} = \mathbf{k}[x_1, \ldots, x_n]_{(\underline{x})}$$

is a finite-dimensional $\mathbf{k}$ *vector space with basis* $\overline{e_1}(\underline{x}), \ldots, \overline{e_r}(\underline{x})$. *Let* $e_1(\underline{X}), \ldots, e_r(\underline{X})$ *be any lifting of them to* $\mathbf{A}[X_1, \ldots, X_n]$. *Then they are* $\mathbf{A}$*-linear independent in* $\mathbf{B} := (\mathbf{A}[X_1, \ldots, X_n]/(F_1, \ldots, F_n))_{(\mathfrak{m},\underline{x})}$.

Proof. Notice that this statement has been already proved while proving Theorem 5 for DVR, without additional hypothesis over the residue field. We shall prove the proposition in some steps, from (i) to (iii).
(i) We assume that $\mathbf{A}$ is a Noetherian power series ring, that is, a quotient like $\mathbf{A} = \mathbf{k}[[T_1, \ldots, T_l]]/\mathfrak{J}$, where $\mathfrak{J}$ is a prime ideal. Following the proof of Theorem 5 we have to consider the equation like in (1), and we have to show that all the b_i's are zero in $\mathbf{A}$. Otherwise, we take some $b(\underline{T}) := b_j \in \mathbf{k}[[T_1, \ldots, T_l]]$ which is nonzero mod $\mathfrak{J}$. We apply the formal version of the Curve Selection Lemma (see Ruiz, 1993, Proposition IV.1.6), getting a local $\mathbf{k}$-homomorphism $\gamma : \mathbf{A} \to \mathbf{k}[[t]]$ such that $\gamma(b) \neq 0$. Here we have used that $\mathfrak{J}$ is a prime ideal and that the residue field $\mathbf{k}$ is algebraically closed of characteristic zero. We call $\gamma_i(t) := \gamma(T_i) : i = 1, \ldots, l)$, hence $\gamma(b) = b(\gamma_1(t), \ldots, \gamma_l(t))$. We set $A_0 := \mathbf{k}[[t]]$, and therefore we can apply the previous case of the DVR to the system obtained by specializing the coefficients of $F_i(\underline{X})$ in γ, that is $F_i^{\gamma}(\underline{X}) : i = 1, \ldots, n$. In fact, the same elements $\overline{e_i}(\underline{x})$ are a basis for the residual complete intersection: since putting $t = 0$ in $F_i^{\gamma}(\underline{X})$ gives the same as setting the T_i zero in $F_i(\underline{X})$. Hence $e_i^{\gamma}(\underline{X})$ are a lifting of this basis, and applying γ to the coefficients of Eq. (1), we get a contradiction.
(ii) Let $\mathbf{A}$ be verifying the hypothesis of the proposition and additionally we assume that $\mathbf{A}$ is Noetherian. Then $\mathbf{A}$ is contained in its completion $\mathbf{A}^{\nu}$, and $\mathbf{A}^{\nu}$ is a Noetherian complete ring with the same coefficient field $\mathbf{k}$. Hence, by the structure theorem of complete local rings (Nagata, 1962, Chapter V, page 106), $\mathbf{A}^{\nu}$ is a power series ring as in (i). Let $\mathfrak{m}_{\nu}$ be the maximal ideal of $\mathbf{A}^{\nu}$. Setting the $F_i(\underline{X})$ mod $\mathfrak{m}$ or mod $\mathfrak{m}_{\nu}$ gives the same ideal in $\mathbf{k}[\underline{X}]$. Since Eq. (1) implies that the b_i's are zero in $\mathbf{A}^{\nu}$, they are zero in $\mathbf{A}$.
(iii) Assume a general $\mathbf{A}$ under the hypothesis. Again we look at Eq. (1), and assume there is some $b_i \neq 0$ in $\mathbf{A}$. We consider the subring $\mathbf{A}_0$ of $\mathbf{A}$ defined as the smallest subring containing $\mathbf{k}$, the coefficients of the $e_i(\underline{X}) : i = 1, \ldots, r$, the b_i's and the coefficients of all polynomials involved in Eq. (1). We localize it at $\mathfrak{m} \cap \mathbf{A}_0$, getting a Noetherian local ring $\mathbf{A}_1$ with maximal ideal say $\mathfrak{m}_1$, and the same coefficient field $\mathbf{k}$ (as in (ii)). Notice that the equations $F_i(\underline{X})$ residually in $\mathbf{A}_1$ or in $\mathbf{A}$ give the same equations in $\mathbf{k}[X_1, \ldots, X_n]$. Since Eq. (1) with some $b_i \neq 0$ implies a contradiction in $\mathbf{A}_1$, the same will be true in $\mathbf{A}_0$. □

Corollary 8. *Let* $\mathbf{A}_{\upsilon}$ *be a Henselian ring, with maximal ideal* $\mathfrak{m}_{\upsilon}$ *and algebraically closed residue field* $\mathbf{k}_{\upsilon}$ *of characteristic zero,* $F_1, \ldots, F_n \in \mathbf{A}_{\upsilon}[X_1, \ldots, X_n]$, $f_1, \ldots, f_n$ *their images in* $\mathbf{k}_{\upsilon}[X_1, \ldots, X_n]$. *We assume that*

$$(\mathbf{k}_{\upsilon}[X_1, \ldots, X_n]/(f_1, \ldots, f_n))_{(\underline{x})}$$

is zero-dimensional. Then

$$\mathbf{B} := (\mathbf{A}_{\upsilon}[X_1, \ldots, X_n]/(F_1, \ldots, F_n))_{(\mathfrak{m}_{\upsilon},\underline{x})}$$

is a free $\mathbf{A}_{\upsilon}$*-module and a basis is given by lifting any basis of the* $\mathbf{k}_{\upsilon}$*-vector space* $\overline{\mathbf{B}} = \mathbf{k}_{\upsilon}[x_1, \ldots, x_n]_{(\underline{x})}$.

Proof. By Claim 4 any lifting of this basis is a system of generators of $\mathbf{B}$ as $\mathbf{A}_{\upsilon}$-module. Since the ring is Henselian and equicharacteristic it has $\mathbf{k}_{\upsilon}$ as a coefficient field, hence by Proposition 7 this lifting provides a set of elements of $\mathbf{B}$ that are $\mathbf{A}_{\upsilon}$-linearly independent. □

4. Multilocal ring and local Bézout Theorem

In this section $\mathbf{V}$ is a local domain with maximal ideal $\mathfrak{m}$ and quotient field $\mathbf{K}$. We denote by $\mathbf{V}^H$ its henselization (Nagata, 1962, Theorem 43.2, and 43.3). Assume that its henselization $\mathbf{V}^H$ is again a domain. Let $\mathbf{K}'$ be an algebraic closure of $\mathrm{Quot}(\mathbf{V}^H)$. Hence, it is also an algebraic closure of $\mathbf{K}$.

Claim 9. *Let $\mathbf{V}'$ be the integral closure of $\mathbf{V}^H$ in $\mathbf{K}'$. Then $\mathbf{V}'$ is a Henselian local ring with quotient field $\mathbf{K}'$. If $\mathfrak{m}'$ is its maximal ideal, we have $\mathbf{V} = \mathbf{K} \cap \mathbf{V}'$ and $\mathfrak{m} = \mathbf{V} \cap \mathfrak{m}'$. Inside $\mathbf{K}'$, $\mathbf{V}'$ is the unique Henselian local ring dominating $\mathbf{V}$ with quotient field $\mathbf{K}'$. We will say that $(\mathbf{K}', \mathbf{V}', \mathfrak{m}')$ is an algebraic closure of $(\mathbf{K}, \mathbf{V}, \mathfrak{m})$.*

Proof. See Nagata's book (Nagata, 1962, Theorem 43.12, Corollary 43.13, Theorem 43.5). More precisely by 43.12, $\mathbf{V}'$ is a local ring and dominates $\mathbf{V}$, by 43.13 it is Henselian. Finally 43.5 provides the uniqueness. □

Given polynomials $F_1, \ldots, F_n$ in $\mathbf{V}[X_1, \ldots, X_n]$ with $F_i(\underline{0}) \in \mathfrak{m}$, $i = 1, \ldots, n$, we define

$$\mathbf{B} := (\mathbf{V}[X_1, \ldots, X_n]/(F_1, \ldots, F_n))_{(\mathfrak{m}, \underline{x})} = \mathbf{V}[x_1, \ldots, x_n]_{(\mathfrak{m}, \underline{x})}.$$

Let $\mathcal{Z}_{\mathbf{K}'}(F_1, \ldots, F_n)$ the set of zeroes of $F_1, \ldots, F_n$ in $\mathbf{K}'$.

Claim 10. *With the above hypotheses, if the $\mathbf{V}$-module $\mathbf{B}$ is free of finite rank r, there are finitely many zeroes in $\mathcal{Z}_{\mathbf{K}'}(F_1, \ldots, F_n)$ with coordinates in $\mathfrak{m}'$ and the sum of their multiplicities is equal to r.*

Proof. Tensoring $\mathbf{B}$v by $\otimes_{\mathbf{V}} \mathbf{V}^H$ we get the same situation, replacing $\mathbf{V}$ by $\mathbf{V}^H$. This allows us to assume that $\mathbf{V} = \mathbf{V}^H$ and $\mathbf{V}'$ is integral over $\mathbf{V}$. Consider the quotient ring $\mathbf{V}[\underline{x}] = \mathbf{V}[\underline{X}]/(\underline{F})$. We have two multiplicative subsets $S = 1 + (\mathfrak{m}, \underline{x})$ and $U = \mathbf{V} \setminus \{0\}$. Localizing first at S we get $\mathbf{B}$. A new localization at U gives the same result as localizing at S the coordinate ring $\mathbf{K}[\underline{x}] = \mathbf{K}[\underline{X}]/(\underline{F})$. In the two cases, by the hypothesis, we get a finite $\mathbf{K}$-vector space of dimension r. We can now extend the scalars to $\mathbf{K}'$ in order to consider all zeroes of $\underline{F}$. Let $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in (\mathbf{K}')^n$ be such a zero. We have to see that this zero "remains alive" after localization in S, (i.e., the maximal ideal $(x_1 - \xi_1, \ldots, x_n - \xi_n)$ does not intersects S), if and only if all its coordinates belong to $\mathfrak{m}'$. Indeed, if $\xi_1 \in \mathbf{K}' \setminus \mathfrak{m}'$ then $\alpha = -1/\xi_1 \in \mathbf{V}'$, and the corresponding maximal ideal $(x_1 - \xi_1, \ldots, x_n - \xi_n)$ of $\mathbf{K}'[x]$ intersects S: $\alpha(x_1 - \xi_1) = 1 + \alpha x_1 \in 1 + (\mathfrak{m}', \underline{x})$. Let $\ell(T) \in \mathbf{V}[T]$ a monic polynomial of degree p such that $\ell(\alpha) = 0$. We get $T^p \ell(-1/T) = \ell_1(T) = 1 + \sum_{k=1}^{p} b_k T^k$ with $b_k \in \mathbf{V}$. Hence $S \ni \ell_1(x_1) \equiv \ell_1(\xi_1) = 0 \mod (x_1 - \xi_1, \ldots, x_n - \xi_n)$. The reciprocal is obvious. □

Claim 10 tells us that, under suitable hypotheses, the ring $\mathbf{K}' \otimes_{\mathbf{V}} \mathbf{B}$ (that is the localization $S^{-1}\mathbf{K}'[\underline{x}]$) represents a multilocal ring; namely the product of finitely many local rings corresponding to the zeroes "infinitely near to $(\underline{0})$".

We assume now that the residual algebra

$$\overline{\mathbf{B}} := (\mathbf{k}[X_1, \ldots, X_n]/(f_1, \ldots, f_n))_{(\underline{x})} = \mathbf{k}[x_1, \ldots, x_n]_{(\underline{x})}$$

is zero-dimensional, with $f_i = \overline{F_i}$.

Theorem 11 (*Bézout Local*)**.** *With the above hypotheses, and additionally $\mathbf{V}$ is either a DVR, or a Henselian domain with algebraically closed residue field of characteristic* 0, *there are finitely many zeroes of the system with coordinates in $\mathfrak{m}'$ and the sum of their multiplicities is equal to the dimension of the residual algebra as a $\mathbf{k}$-vector space.*

Proof. This follows from Theorem 5, Corollary 8, and Claim 10. □

5. Application to stability of the border basis in the local case

In this paragraph we discuss the problem of fast computing the zeroes of a polynomial system after a small perturbation.

We assume that the initial system is a system of n polynomials in n indeterminates which has an isolated zero. By translation we may assume this zero is $(\underline{0})$.

We are interested by the fast computation of the zeroes near $(\underline{0})$ after a small perturbation of the coefficients of the given polynomials. Namely, we are interested in computing the structure of the multilocal algebra of the zeroes close to $(\underline{0})$ after the deformation. Here "fast computation" means we use floating point arithmetic, with a fixed precision.

We show a simple example where Gröbner bases technique fails to find the zeroes, except when one uses a huge precision (many digits). On the contrary if we use the technique of proof of our Theorem 1, few digits are needed to obtain good accuracy.

Experimentation is made with Maple 11. We use the two following polynomials in x, y, with a perturbation parameter t.

$$F_1 := y^2 + tyx^2 + t$$
$$F_2 := x^3 + yx^2 - t^2.$$

When $t = 0$, setting $f_i = F_i(x, y, 0)$, the system $f_1 = 0, f_2 = 0$ has an isolated zero $(0, 0)$ of multiplicity 6. After a small perturbation ($t = \epsilon \in \mathbb{R}$), an exact computation of Gröbner basis of the system ($F_1 = 0, F_2 = 0$) shows that the system has 7 roots. We know that 6 ones must be very near $(0, 0)$: a cluster of six roots.

To simplify computations, instead of using border basis, we use a subset of it which in the notation of Alonso et al. (2006) is called a *Janet system of generators*. In our case a Janet system of generators for the system $f_1 = 0, f_2 = 0$ is given by f_1, f_2 and $f_3 := yf_2$. We slightly change the reduction process. Namely, for the reduction of a term in $\langle x, y\rangle$, only subtractions of f_1 times a power products of x, y's, and subtractions of f_2 and f_3 by power products of x's are allowed. Notice that in this way the reduction process is unique. Following the proof of Claim 3, we introduce the formal polynomials $\{\widetilde{F_i} : i = 1, 2, 3\}$. In this way, the number of polynomials to be considered is less. Therefore the number of coefficients of the $\widetilde{F_i}$'s is less than the one we would need following the same argument given in the proof using the whole border basis. The reduction process we perform is similar to the one in Claim 3. Reductions with F_2, and F_3 are made by subtracting a multiple of these polynomials by monomials in $\langle x\rangle$, while reductions with F_1 can be done multiplying it by any monomial in $\langle x, y\rangle$.

We introduce the $\widetilde{F_i}$'s:

$$\begin{aligned}
\widetilde{F_1} &:= y^2 + a_0 + a_1\, x + a_2\, x^2 + a_3\, y + a_4\, yx + a_5\, yx^2\\
\widetilde{F_2} &:= x^3 + b_0 + b_1\, x + b_2\, x^2 + b_3\, y + b_4\, yx + b_5\, yx^2\\
\widetilde{F_3} &:= x^3y + c_0 + c_1\, x + c_2\, x^2 + c_3\, y + c_4\, xy + c_5\, x^2y.
\end{aligned}$$

Next we reduce this Janet set in order to get polynomials with only monomials under the staircase, and we equal to zero the coefficients of these monomials, to obtain the Hensel equations. We get

$$a_1 = a_2 = a_3 = a_4 = 0, \qquad b_1 = b_2 = b_3 = b_4 = 0,$$
$$a_0 = t, \qquad b_0 = -t^2, \qquad a_5 = t, \qquad b_5 = 1,$$
$$c_2 = \frac{t(1 + tc_5)^2}{-1 - 3\,tc_5 - 3\,t^2{c_5}^2 - t^3{c_5}^3 + t^5}, \qquad c_0 = \frac{t^3c_2}{1 + tc_5},$$
$$c_1 = \frac{t^4c_2}{(1 + tc_5)^2}, \qquad c_3 = \frac{-t^2}{1 + tc_5}, \qquad c_4 = \frac{-t^3}{(1 + tc_5)^2}$$
$$- t^6\, c_5^7 - 6t^5\, c_5^6 - 15t^4\, c_5^5 + (-20t^3 + t^6 + t^8)\, c_5^4 + (-15t^2 + 4t^5 + 2t^7)\, c_5^3$$
$$+ (-6t + 6t^4)\, c_5^2 + (-1 + 4t^3 - 2t^5)\, c_5 + (t^2 - t^4 + t^9) = 0.$$

The zero near 0 of the last equation gives with Newton's method, the following successive approximations:

$$t^2 - t^4 + O\left(t^5\right)$$
$$t^2 - t^4 - 2\,t^5 + 6\,t^7 + 7\,t^8 - 3\,t^9 + O\left(t^{10}\right)$$
$$t^2 - t^4 - 2\,t^5 + 6\,t^7 + 7\,t^8 - 3\,t^9 - 35\,t^{10} - 30\,t^{11} + 45\,t^{12} + \cdots + O\left(t^{20}\right).$$

Using $c_5 = t^2 - t^4 - 2\,t^5$, or $c_5 = t^2$, leads to the two following (approximating) matrices for the multiplication by y in the basis of monomials under the staircase:

$$aprMy := \begin{bmatrix} 0 & 0 & -t & 0 & -t^3 - t^5 & -t^4 + 2\,t^7 \\ 0 & 0 & 0 & 0 & -t - t^6 & -t^5 \\ 1 & 0 & 0 & 0 & -t^3 + t^6 & t^5 - t^7 \\ 0 & 0 & 0 & 0 & -t^2 + t^5 & -t + t^4 - 2\,t^6 - 3\,t^7 \\ 0 & 1 & 0 & 0 & -t^4 + 2\,t^7 & -t^3 + 2\,t^6 \\ 0 & 0 & -t & 1 & t^3 - t^5 - 2\,t^6 & t^2 - t^4 - 2\,t^5 + 4\,t^7 \end{bmatrix},$$

$$AprMy := \begin{bmatrix} 0 & 0 & -t & 0 & -t^3 & -t^4 \\ 0 & 0 & 0 & 0 & -t & -t^5 \\ 1 & 0 & 0 & 0 & -t^3 & t^5 \\ 0 & 0 & 0 & 0 & -t^2 & -t \\ 0 & 1 & 0 & 0 & -t^4 & -t^3 \\ 0 & 0 & -t & 1 & t^3 & t^2 \end{bmatrix}.$$

We may use the characteristic polynomial of one of these matrices. For small values of t, say $|t| \leq 10^{-2}$, the second one is sufficient:

$$\begin{aligned} Gy &= y^6 + \left(-t^2 + t^4\right) y^5 + \left(t^6 + 3\,t\right) y^4 + \left(t^7 + t^8 + t^{10} - 2\,t^3\right) y^3 \\ &\quad + \left(t^7 + 3\,t^2\right) y^2 + \left(-t^6 + 2\,t^9 - t^4\right) y + (t^3 - t^8). \end{aligned}$$

Computing with 12 digits we get correct answers up to many digits for the cluster of six roots.

On the other hand, a floating point Gröbner basis computation fails to give a polynomial in y of degree 7 unless we use a huge precision. Here are some results: we indicate on the same row first the value of t, second the multiset-distance between the "exact" zero cluster and the one obtained by our fast computation, third the precision needed to get the correct degree for the y-polynomial when using a floating point Gröbner basis computation: for small precision this degree is 5, increasing the precision it becomes 0, finally with high precision it gets the correct value 7. Let us note also that the size of the cluster is $\sim\sqrt{|t|}$.

t	$*$	$**$	
-10^{-10}	5.10^{-29}	50	
-10^{-16}	2.10^{-45}	80	
-10^{-22}	5.10^{-62}	111	
-10^{-28}	2.10^{-78}	140	
-10^{-34}	5.10^{-95}	170	NB : a bug with 238 digits
-10^{-40}	2.10^{-111}	200	
-10^{-46}	5.10^{-128}	230	
-10^{-52}	2.10^{-144}	260	
-10^{-58}	5.10^{-161}	290	
-10^{-76}	2.10^{-210}	380	

* : precision of the result obtained through our fast floating point computation with 12 digits.
** : needed digits to get a correct answer with the technique of floating point GB computations.

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