ELSEVIER
Theoretical Computer Science 219 (1999) 169–184
Theoretical Computer Science
www.elsevier.com/locate/tcs

Computable Banach spaces via domain theory[1]

Abbas Edalat, Philipp Sünderhauf *

Department of Computing, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK

Abstract

This paper extends the order-theoretic approach to computable analysis via continuous domains to complete metric spaces and Banach spaces. We employ the domain of formal balls to define a computability theory for complete metric spaces. For Banach spaces, the domain specialises to the domain of closed balls, ordered by reversed inclusion. We characterise computable linear operators as those which map computable sequences to computable sequences and are effectively bounded. We show that the domain-theoretic computability theory is equivalent to the well-established approach by Pour-El and Richards. © 1999 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

This paper is part of a programme to introduce the theory of continuous domains as a new approach to computable analysis. Initiated by the various applications of continuous domain theory to modelling classical mathematical spaces and performing computations as outlined in the recent survey paper by Edalat [6], the authors started this work with [9] which was concerned with computability on the real line. Here we will deal with computability on metric and Banach spaces.

Continuous domains were introduced independently by Dana Scott [20] and Yuri Ershov [10]. Their traditional use is in programming semantics, but recently they have been successfully employed as computational models for spaces in classical mathematics.

A continuous domain is a partially ordered set equipped with the notions of *completeness* and *approximation*. The completeness axiom requires existence of least upper bounds for all directed subsets, and the approximation axiom implies that all elements arise as directed suprema of their essential parts or approximants. These axioms pro-

* Corresponding author. Tel.: +44 171 594 8253; fax: +44 171 581 8024; e-mail: ps15@doc.ic.ac.uk.

[1] Work supported by the EPSRC project *Foundational Structures for Computer Science* at Imperial College.

PII: S0304-3975(98)00288-6

vide the link to the machine-based level of recursion theory or Turing machines: We enumerate a convenient set of approximants and let the machine operate on this set.

In our earlier paper [9], we started a systematic exploration of the use of continuous domains for computable analysis by considering computability on the real line, the complex plane, and $\mathbb{R}^n$. It was shown that the resulting notion of computability is equivalent to the work of Pour-El and Richards [18]. Stoltenberg-Hansen and Tucker [22] had already shown this equivalence when one uses a so-called algebraic domain to model the real numbers.

The present paper extends the continuous domain-theoretic framework to deal with metric spaces and Banach spaces. The domain employed for this purpose is the *domain of formal balls*, defined in [7], which, for Banach spaces, coincides with the collection of all closed balls ordered by reversed set inclusion. Hence every point is represented as intersection of a shrinking sequence of balls, a straightforward generalisation of the well-established representation of real numbers by shrinking sequences of intervals.

Our main result is that the continuous domain-theoretic notion of computability structure for a Banach space coincides with the notion of computability structure in terms of computable sequences suggested by Pour-El and Richards [18]. The domain-theoretic setting gives rise in a natural way to a notion of *computable operator*, which coincides with the corresponding notion implicitly given in the work of Pour-El and Richards.

While this paper was being prepared, Stoltenberg-Hansen and Tucker independently gave in [23] an approach to computability for Banach spaces using algebraic domains and proved that, under an additional technical condition which is satisfied in all known cases, it is equivalent with the notion of Pour-El and Richards. In *loc.cit.* they also show the effective equivalence of algebraic and continuous domain representability.

Just as one finds different approaches to *discrete* computability theory, e.g. recursive functions, the λ-calculus, Turing machines, Post systems, the RAM machine, there are also various different approaches to *continuous* computability theory or computable analysis, each focusing on different aspects. To name but a few, there are the so-called Russian [5] and Polish [11, 14] recursion theoretic approaches, Weihrauch's Turing machine approach also called Type 2 Theory of Effectivity (TTE) [13, 24, 26], the work of Stoltenberg-Hansen and Tucker which employ algebraic domains [21–23, 3, 4], and, finally, the axiomatic approach by Hauck [12] and Pour-El and Richards [18].

With this work, we provide the working mathematician and computer scientist with continuous domains as another choice to approach continuous computability theory. In this approach the computational structure, i.e. the continuous domain, serves a multiple purpose. First, it provides the means to supply the classical space with an effective structure. Second, it may be used to perform actual computations. And third, it is a rich mathematical object and can possibly be used to re-develop the classical mathematical theory with the domain replacing the classical space, thus incorporating effectivity for free.

In the case of the interval domain modelling the real line, the first of these points is addressed in our previous paper [9]. An example for the second point is the theoretical framework and implementation of a package providing exact real arithmetic by Edalat and Potts [17, 8]. The last point can be seen as partly covered by the well-established theory of interval analysis [15, 2]. For Banach spaces as dealt with in the present paper, the last two points have yet to be tackled in future work.

Note that we deliberately do not give a formal definition when a domain is a *computational model* for a space. We rather propose to use this term informally: A computational model for a space is a domain which is constructed in a simple way to represent the space and is useful in performing various computations on the space.

We will use the terminology and results of [9]. Our main reference for recursion theory is Rogers [19] and for domain theory is Abramsky and Jung [1].

2. Computability on metric spaces

In this section, we briefly show how to extend the results of [9] to cover computability on metric spaces, based on the *domain of formal balls* which was introduced by Edalat and Heckmann in [7].

2.1. The domain of formal balls

Suppose (X,d) is a complete metric space. Let $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$ denote the set of nonnegative real numbers. The *domain of formal balls* is the set

$$B(X) = \{\bot\} \cup (X \times \mathbb{R}_0^+)$$

endowed with the order defined by $\bot \sqsubseteq A$ for all $A \in B(X)$ and

$$(x,r) \sqsubseteq (y,s) \;\Leftrightarrow\; d(x,y) + s \leqslant r.$$

The poset $(B(X), \sqsubseteq)$ is a continuous domain. The way-below relation turns out to be $\bot \ll A$ for all $A \in B(X)$ and

$$(x,r) \ll (y,s) \;\Leftrightarrow\; d(x,y) + s < r.$$

An element is maximal iff it is of the form $(x,0)$ for some $x \in X$. The space X in the metric topology is via $x \mapsto (x,0)$ homeomorphic to the set of maximal elements with the relative Scott-topology. If $X_0 \subseteq X$ is a dense subset of X then the set $\{\bot\} \cup \{(x,r) \mid x \in X_0,\ r \in \mathbb{Q},\ r \geqslant 0\}$ is a basis for $(B(X), \sqsubseteq)$.

For every Lipschitz-continuous function $f : X \to Y$ between the metric spaces (X,d) and (Y,d) with a Lipschitz-constant M, satisfying $d(f(x), f(x')) \leqslant M d(x,x')$ for all $x, x' \in X$, there is a Scott-continuous extension $g : B(X) \to B(Y)$ to the domains of formal balls. It is defined by $g(\bot) = \bot$ and $g((x,r)) = (f(x), Mr)$ [7, Theorem 3.1].

2.2. Effective metric spaces

If (X,d) is a separable metric space, then the domain of formal balls is ω-continuous and we are in the position to endow it with an effective structure.

Definition 1. An effectively *given metric space* is a complete metric space with a countable dense subset $\{x_0, x_1, x_2, \ldots\}$ such that the domain of formal balls is effectively given with respect to the basis $\{B_0, B_1, \ldots\}$ with $B_0 = \bot$ and

$$B_{\langle n,m\rangle+1} = (x_n, |q_m|),$$

where $\langle \cdot, \cdot \rangle : \mathbb{N} \to \mathbb{N}$ is a standard pairing function.

Theorem 2. *A dense sequence $x_0, x_1, x_2, \ldots$ in a complete metric space gives rise to an effective representation if and only if the relation*

$$d(x_n, x_m) < q_k$$

is r.e. in n, m, k.

Proof. The domain of formal balls is effectively given with respect to the basis $\{B_0, B_1, \ldots\}$ as defined above iff the order of approximation is r.e. This is the case iff the relation $d(x_n, x_{n'}) + |q_{m'}| < |q_m|$ is r.e. As addition of rational numbers and the absolute value function are recursive in the indices, this proves the theorem. □

In [25] Weihrauch defines a computable metric space to be a metric space with a dense sequence $(a_n)_{n\in\mathbb{N}}$ such that the relation $d(a_n, a_m) < q_k$ is r.e. in n, m, k. So Theorem 2 means, in particular, that a complete metric space is effectively given in our sense if and only if it is a computable metric space in the sense of *loc.cit.* The stronger assumption of both $q_k < d(a_n, a_m)$ and $d(a_n, a_m) < q_k$ being r.e. is part of Moschovakis's definition of a recursive metric space [16], the other part being countability of X. This stronger condition is implied by the requirement of $\ll$ being recursive rather than merely r.e. Blanck investigates in [3, 4] metric spaces with a countable dense subspace on which the distance function takes values in a computable ordered field and can be tracked by a recursive function. He constructs effective domain representations of these spaces in the spirit of Stoltenberg-Hansen and Tucker by using the ideal completion of the set of finite consistent sets of formal balls.

Note that our model for metric spaces is quite similar to our model for the reals [9]. The dense sequence $(x_n)_{n\in\mathbb{N}}$ takes over the role of the rationals $(q_n)_{n\in\mathbb{N}}$. Writing the intervals in the form $[x-r, x+r]$ makes this similarity even more apparent. In fact, the interval domain is isomorphic to the domain of formal balls of the real line. So it is not surprising that many of the definitions and results from [9] easily transfer to the more general setting of complete metric spaces. If (X, d) is an effective metric space we denote the resulting enumeration of computable elements of $B(X)$ by $\xi_X : \mathbb{N} \to B(X)$. We say that an element $x \in X$ is *computable* if $\xi_X(n) = (x, 0)$ for some $n \in \mathbb{N}$. A sequence $(y_n)_{n\in\mathbb{N}}$ in X is *computable* if there is a total recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $(y_n, 0) = \xi_X(f(n))$.

Theorem 3. *An element $x \in X$ is computable iff it is the effective limit of a sequence $(x_{n_i})_{i\in\mathbb{N}}$, i.e. iff there is $f : \mathbb{N} \to \mathbb{N}$ recursive such that $d(x, x_{f(n)}) \leqslant 1/2^n$ for all $n \in \mathbb{N}$. This equivalence is effective.*

Proof. The proof of the (only if) part is identical to that of Theorem 15 of [9]. For the (if) part, we need to change the proof of Theorem 18 of *loc.cit.* as follows.

Assume $f:\mathbb{N}\to\mathbb{N}$ is such that $d(x_{f(n)},x)\leqslant 2^{-n}$ for all $n\in\mathbb{N}$. Define $h:\mathbb{N}\to\mathbb{N}$ total recursive such that

$$B_{h(n)}=\left(x_{f(n+2)},\frac{1}{2^n}\right).$$

Then $B_{h(n)}\sqsubseteq(x,0)$ and the sequence $(B_{h(n)})_{n\in\mathbb{N}}$ is increasing as $d(x_{f(n+2)},x_{f(n+3)})+1/(2^{n+1})\leqslant d(x_{f(n+2)},x)+d(x,x_{f(n+3)})+1/(2^{n+1})\leqslant 1/(2^{n+2})+1/(2^{n+3})+1/(2^{n+1})\leqslant 1/2^n$. Since $\lim_{n\to\infty}1/2^n=0$, the lub of this chain is maximal, hence this lub must be $(x,0)$. Therefore, an index for the function h is an index for $(x,0)$. □

As in the case of the real line, effectivity allows us to characterise the computable sequences [9, Theorem 20].

Corollary 4. *A sequence $(y_n)_{n\in\mathbb{N}}$ is computable if and only if there is a total recursive function $r:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that $d(x_{r(n,k)},y_n)\leqslant 2^{-k}$ for all $n,k\in\mathbb{N}$. This equivalence is effective.*

2.3. Computable functions

Definition 5. A function $f:X\to Y$ between effectively given metric spaces is *computable* iff there is an extension $g:B(X)\to B(Y)$ (i.e. $g((x,0))=(f(x),0)$ for all $x\in X$) which is computable in the sense of effective domain theory.

Theorem 6. *A function with a given Lipschitz constant between effectively given metric spaces is computable if and only if it maps computable sequences of points to computable sequences.*

Proof. Necessity of this condition is immediate. For sufficiency, assume that $f:X\to Y$ is Lipschitz-continuous and maps computable sequences to computable sequences. Denote the dense sequences specifying the effective structure of X and Y by $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively. As f maps computable sequences to computable sequences, there is $h:\mathbb{N}\to\mathbb{N}$ recursive such that $(f(x_n),0)=\xi_Y(h(n))$ for all $n\in\mathbb{N}$. Assume $M\in\mathbb{R}$ is a computable Lipschitz constant for f. Then the extension $g:B(X)\to B(Y)$ of f is defined by $g(\perp)=\perp$ and $g((x,r))=(f(x),Mr)$. It remains to show computability of this function. Observe that

$$\begin{aligned}(y_i,|q_j|)\ll g((x_n,|q_m|)) &\Leftrightarrow d(y_i,f(x_n))+M|q_m|<|q_j|\\ &\Leftrightarrow d(y_i,f(x_n))<\max(0,|q_j|-M|q_m|)\\ &\Leftrightarrow (y_i,\max(0,|q_j|-M|q_m|))\ll\xi_Y(h(n)).\end{aligned}$$

As h is recursive and as the arithmetic operations on the rationals are computable, this relation is r.e. in the indices n,m,i,j. □

Remark. Note that for a Lipschitz function to be computable it is essential to explicitly have a Lipschitz constant. If we only know that some Lipschitz constant exists, but do not have one explicitly, we cannot construct the extension to the domain of balls.

3. Computability on Banach spaces

A *Banach space* is a complete normed vector space. For simplicity, we only consider Banach spaces over the reals. The origin in a Banach space is denoted by $\mathbf{0}$.

3.1. Balls in Banach spaces

With the definition $d(x,y)=\|x-y\|$, every Banach space is a complete metric space. Hence, a natural candidate for endowing Banach spaces with a computability structure is the domain of formal balls which we met in Section 2. For normed vector spaces, in particular for Banach spaces, the theory simplifies significantly: For these spaces the poset of formal balls coincides with the set of closed balls, ordered by reversed set inclusion [7, Theorem 2.14]. We use the notation

$$C(x,r)=\{y\in X \mid \|x-y\|\leqslant r\}$$

for the closed ball of radius r around the point $x\in X$. So the *domain of balls* is the set

$$B(X)=\{\bot\}\cup\{C(x,r)\mid x\in X,\ r\geqslant 0\}$$

endowed with the order $\bot\sqsubseteq A$ for all $A\in B(X)$ and

$$\begin{aligned} C(x,r)\sqsubseteq C(y,s) &\Leftrightarrow C(x,r)\supseteq C(y,s)\\ &\Leftrightarrow \|x-y\|+s\leqslant r. \end{aligned}$$

Recall that the order of approximation $\ll$ is given by $\bot\ll A$ for all $A\in B(X)$ and

$$C(x,r)\ll C(y,s) \Leftrightarrow \|x-y\|+s<r.$$

We are now going to extend the vector space operations and the norm to the balls. Addition $+: B(X)\times B(X)\to B(X)$ is defined by

$$C(x,r)+C(y,s):=C(x+y,r+s)$$

and $\bot+B=B+\bot:=\bot$ for all $B\in B(X)$.

Scalar multiplication in this setting is of type $\mathscr{I}\times B(X)\to B(X)$ and is defined by

$$[\alpha-\varepsilon,\alpha+\varepsilon]\cdot C(x,r):=C(\alpha x,\varepsilon\|x\|+|\alpha|r+\varepsilon r)$$

and $\bot\cdot B=I\cdot\bot:=\bot$ for all $B\in B(X)$ and $I\in\mathscr{I}$. This definition might seem rather involved, but we will see in Proposition 8 below that this choice of extension of scalar multiplication is canonical. Observe that the special cases $\varepsilon=0$ and $r=0$ give results for the scalar product $[\alpha-\varepsilon,\alpha+\varepsilon]\cdot C(x,r)$ which are intuitive, namely $\{\alpha\}\cdot C(x,r)=C(\alpha x,|\alpha|r)$ and $[\alpha-\varepsilon,\alpha+\varepsilon]\cdot\{x\}=C(\alpha x,\varepsilon\|x\|)$.

Finally, for the norm, we set

$$\|C(x,r)\| := [\|x\|-r,\|x\|+r]$$

and $\|\bot\| := \bot$ to get $\|\cdot\| : B(X)\to\mathscr{I}$.

Proposition 7. *Addition, scalar multiplication, and the norm as defined above are Scott-continuous. Restricted to maximal elements, they yield the usual operations.*

Proof. Monotonicity of the operations is easily verified from the definitions. For scalar multiplication, we have to use the fact that $C(x,r)\sqsubseteq C(y,s)$ implies $\|x\|+r\geqslant\|y\|+s$. In order to verify that $[\alpha-\varepsilon,\alpha+\varepsilon]C(x,r)\sqsubseteq[\alpha-\varepsilon,\alpha+\varepsilon]C(y,s)$ we calculate

$$\begin{aligned}\|\alpha x-\alpha y\|+\varepsilon\|y\|+|\alpha|s+\varepsilon s &= |\alpha|(\|x-y\|+s)+\varepsilon(s+\|y\|)\\ &\leqslant |\alpha|r+\varepsilon(r+\|x\|)\end{aligned}$$

by the above observation. Monotonicity in the first argument is shown in a similar fashion. Preservation of directed suprema for all three operations is immediate by the characterisation in terms of limits: We have $C(x,r)=\bigvee^{\uparrow}_{n\in\mathbb{N}}C(x_n,r_n)$ iff $x=\lim_{n\to\infty}x_n$ and $r=\lim_{n\to\infty}r_n$. Thus $x+y=\lim_{n\to\infty}(x_n+y)$ and $r+s=\lim_{n\to\infty}(r_n+s)$, so $C(x,r)+C(y,s)=\bigvee^{\uparrow}_{n\in\mathbb{N}}(C(x_n,r_n)+C(y,s))$. Hence addition is Scott-continuous. (Recall that separate continuity is equivalent to joint continuity for functions $f:D\times D'\to E$ between domains.) It is obvious that we get the usual operations in the case that $r=s=\varepsilon=0$. $\square$

The extension of the norm can also be seen as a special case of the extension of Lipschitz-continuous functions to the domain of balls for metric spaces (cf. the end of Section 2.1, and, for computability, Theorem 17). For every Banach space X, the norm $\|\cdot\|:X\to\mathbb{R}$ is Lipschitz-continuous with constant $M=1$. The extension to the balls is exactly the norm for $B(X)$ as defined above.

We defined the operations on balls by referring to centres and radii. As we are dealing with subsets, another possibility would be to consider the pointwise operations. For addition, the two approaches coincide as the next proposition shows. In the case of scalar multiplication, however, the pointwise operation does not necessarily yield a ball as result. But we have made the best possible choice:

Proposition 8. *Suppose* $(X,\|\cdot\|)$ *is a Banach space. Then*
(1) $C(x,r)+C(y,s)=\{x'+y'\mid x'\in C(x,r)\ \&\ y'\in C(y,s)\}$.
(2) $[\alpha-\varepsilon,\alpha+\varepsilon]C(x,r)\supseteq\{\beta y\mid\beta\in[\alpha-\varepsilon,\alpha+\varepsilon]\ \&\ y\in C(x,r)\}$.
(3) *There is no smaller ball than* $[\alpha-\varepsilon,\alpha+\varepsilon]C(x,r)$ *which is centred at* αx *and contains the pointwise scalar product as in* (2).
(4) $\|C(x,r)\|\supseteq\{\|y\|\mid y\in C(x,r)\}$.

(5) *There is no smaller interval than* $\|C(x,r)\|$ *which is centred at* $\|x\|$ *and contains the pointwise norm as in* (4).

Proof. Note that $y \in C(x,r)$ iff there is x' with $\|x'\| \leqslant r$ and $y = x + x'$. Using this, one readily sees that the right-to-left inclusions hold. As an example, we show (2), the most involved case. Suppose $y \in C(x,r)$ and $\beta \in [\alpha - \varepsilon, \alpha + \varepsilon]$. Write $y = x + x'$ with $\|x'\| \leqslant r$. Then $\beta y = \beta x + \beta x' = \alpha x + (\beta - \alpha)x + \beta x'$ and $\|(\beta - \alpha)x + \beta x'\| \leqslant |\beta - \alpha|\|x\| + |\beta| r \leqslant \varepsilon\|x\| + (|\alpha| + \varepsilon)r$ since $|\alpha - \beta| \leqslant \varepsilon$. For $\subseteq$ in (1) assume that $z \in C(x,r) + C(y,s)$, i.e. that $z = x + y + w$ with $\|w\| \leqslant r + s$. We split w as $w = [r/(r+s)]w + [s/(r+s)]w$ and observe that $\|[r/(r+s)]w\| = [r/(r+s)]\|w\| \leqslant r$ which implies $z \in \{x' + y' \mid x' \in C(x,r) \ \&\ y' \in C(y,s)\}$. To see (3) for the case of $\alpha \geqslant 0$ note that $x + (r/\|x\|)x \in C(x,r)$ and $(\alpha + \varepsilon)(x + (r/\|x\|)x) = \alpha x + (\varepsilon\|x\| + \varepsilon r + \alpha r)x/\|x\|$ so the radius could not be smaller. Similarly for $\alpha \leqslant 0$, with $-\varepsilon$ and $-(r/\|x\|)x$ replacing ε and $(r/\|x\|)x$. Finally (5). It suffices to note that $x + (r/\|x\|)x$ is an element of $C(x,r)$ and has norm $\|x\| + r$. $\square$

For the case of singleton intervals, the scalar multiplication is indeed the pointwise defined scalar product as the latter is a ball. We will write $\alpha C(x,r)$ to abbreviate $\{\alpha\}C(x,r) = C(\alpha x, |\alpha| r)$. Also, as usual, $A - B$ stands for $A + (-B)$, i.e. $A + \{-1\}B$.

3.2. Effectively given Banach spaces

Definition 9. An *effectively given Banach space* is a Banach space X with an effective structure for the continuous domain $B(X)$ of balls such that addition, scalar multiplication and the norm are computable functions. We denote the effective basis by $\{A_0, A_1, A_2, \ldots\}$ and require further that the set $\{n \in \mathbb{N} \mid A_n = \bot\}$ be recursive.

For simplicity, we denote the resulting enumeration of computable elements of $B(X)$ with ξ_X rather than $\xi_{B(X)}$. Note that the term "effectively given Banach space" always refers to a given enumeration of a basis for the domain of balls. Hence a Banach space may be effectively given in more than one way.

Recall that for an interval $[a,b] \in \mathscr{I}$, we denote its length $b - a$ by $|[a,b]|$. Hence for a ball $A \in B(X)$, the number $|\|A\||$ denotes its diameter. In fact, we calculate $|\|C(x,r)\|| = |[\|x\| - r, \|x\| + r]| = 2r$. Note that by [9, Lemma 14] and computability of the norm, the relations $|\|A_n\|| < q_m$ and $|\|A_n\|| > q_m$ are r.e. in n, m.

Proposition 10. *Suppose that* $(X, \|\cdot\|)$ *is an effectively given Banach space. Then* $\{\mathbf{0}\}$ *and the unit ball* $C(\mathbf{0},1)$ *are computable elements. Moreover, indices for these elements can be obtained from the effective structure.*

Proof. To get an index for the unit ball, we make use of the fact that $A_n \ll C(\mathbf{0},1)$ iff there is $r > 1$ with $A_n \ll C(\mathbf{0},r)$ iff there is $m \in \mathbb{N}$ with $1 < |\|A_m\||$ such that $A_n \ll (A_m - A_m)$. (Note that $C(x,r) - C(x,r) = C(\mathbf{0},2r)$.) These relations are r.e. in n, m by the above remark and the fact that the operations are assumed computable.

With the unit ball at hand, we readily get an index for the origin as $\{\mathbf{0}\} = \bigvee^{\uparrow}_{n\in\mathbb{N}} C(\mathbf{0}, 1/n) = \bigvee^{\uparrow}_{n\in\mathbb{N}} (1/n)C(\mathbf{0}, 1)$. □

Let us now turn our interest to the Banach space X, i.e. the set of maximal elements of $B(X)$. By Proposition 10 the origin $\mathbf{0}$ is a computable element of X. Are there any other computable elements in the Banach space? Can we lay our hands on them? The following proposition is the answer to these questions. We are able to find sufficiently many computable elements in X.

Proposition 11. *There is a total recursive function $f:\mathbb{N}\to\mathbb{N}$ such that $\xi_X(f(n))$ is a maximal element above A_n for every $n\in\mathbb{N}$. The resulting sequence $(x_n)_{n\in\mathbb{N}}$ of vectors, where $C(x_n, 0) = \xi_X(f(n))$, is dense in X.*

Proof. Fix $n\in\mathbb{N}$. We construct a recursive function $g:\mathbb{N}\to\mathbb{N}$ which gives an increasing sequence $A_{g(0)} \sqsubseteq A_{g(1)} \sqsubseteq \cdots$ with lub in the maximal elements above A_n. Then $f(n)$ is defined to be the derived index of the function g and the first part of the proposition is proved. For the construction of g, set $g(0) = n$. To define $g(m+1)$ for $m\in\mathbb{N}$, consider the set $M_m = \{g(m)\} \cup \{i\in\mathbb{N} \mid A_{g(m)} \ll A_i\}$. This certainly is r.e. and hence so is $M'_m = \{i\in M_m \mid |||A_i||| < 1/2^m\}$. Note that the set M'_m is not empty: If $g(m)\notin M'_m$ then $A_{g(m)}$ is not maximal and so there is a maximal element of $B(X)$ way-above it. This certainly implies the existence of a basis element A_i with $|||A_i||| < 1/2^m$ way-above $A_{g(m)}$, then $i\in M'_m$. So, using the Selection Theorem, we may define $g(m+1)$ to be some element of M'_m. It is clear that the total recursive function g defined by this procedure has the desired property.

To see that the sequence $(x_n)_{n\in\mathbb{N}}$ is dense in X, suppose $x\in X$ and $\varepsilon>0$. Now $C(x,\varepsilon) \ll \{x\}$, hence there is $n\in\mathbb{N}$ with $C(x,\varepsilon) \ll A_n \ll \{x\}$. This implies $A_n \subseteq C(x,\varepsilon)$ and as $x_n\in A_n$ we certainly have $\|x - x_n\| \leqslant \varepsilon$. □

Observe that this construction is not possible if the basis elements A_n are replaced by the computable elements $\xi_X(n)$ as the relation $\xi_X(n) \ll A_m$ is *not* r.e. In fact, there is no recursive enumeration of all computable singletons.

The above constructed sequence of computable maximal elements allows us to pass to a more convenient basis for $B(X)$. Let $B_0 = \bot$ and

$$B_{\langle n,m\rangle+1} = C(x_n, |q_m|)$$

with the sequence $(x_n)_{n\in\mathbb{N}}$ from Proposition 11.

Theorem 12. *Every effectively given Banach space is an effectively given metric space. In particular, the above constructed set $\{B_0, B_1, B_2, \ldots\}$ is an effective basis for $B(X)$ which makes the operations computable.*

Proof. The sequence $(x_n)_{n\in\mathbb{N}}$ is dense in X. Furthermore, the relation

$$\|x_n - x_m\| < q_k$$

is r.e. in n, m, k as the norm is computable. Thus we are dealing with an effectively given metric space by Theorem 2. It remains to show computability of the operations. This is an easy consequence of computability with respect to the given basis $\{A_0, A_1, \dots\}$, e.g., for addition we have

$$\begin{aligned} & C(x_i, |q_j|) \ll C(x_n, |q_m|) + C(x_{n'}, |q_{m'}|) \\ & \quad \Leftrightarrow \|x_i - (x_n + x_{n'})\| + |q_m| + |q_{m'}| < |q_j| \end{aligned}$$

which is clearly r.e. in i, j, n, m, n', m'. □

So the set comprising all the B_n does, in fact, constitute a computability structure for X. How does it relate to the original structure? Let us denote the two resulting effective domains by $(B(X), A)$ and $(B(X), B)$, respectively.

Proposition 13. *The identity* $\mathsf{id} : B(X) \to B(X)$ *is B-A computable.*

Proof. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is the function defined in Proposition 11 and that $n_0 \in \mathbb{N}$ is the code for $C(\mathbf{0}, 1)$ from Proposition 10. Then

$$\begin{aligned} B_{\langle n,m \rangle + 1} &= C(x_n, |q_m|) \\ &= C(x_n, 0) + C(\mathbf{0}, |q_m|) \\ &= \xi_X(f(n)) + |q_m| \cdot \xi_X(n_0). \end{aligned}$$

As the operations are computable and f is recursive, this shows that the relation $A_k \ll B_\ell$ is r.e. by [9, Lemma 5]. □

This need not be true for the other direction, in particular, there might be basis elements A_n which are not computable w.r.t. to the B-structure. But we are not really interested in the balls, we are interested in the Banach space. The following proposition shows that the identity $\mathsf{id} : X \to X$ has an extension $B(X) \to B(X)$ which is A-B computable. Hence, the two computability theories for $(X, \|\cdot\|)$ coincide and we may pass to the more convenient basis $\{B_0, B_1, \dots\}$.

Proposition 14. *There is a total recursive function* $f : \mathbb{N} \to \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *the function* $\phi_{f(n)}$ *gives an increasing chain* $B_{\phi_{f(n)}(0)} \sqsubseteq B_{\phi_{f(n)}(1)} \sqsubseteq \cdots$ *with least upper bound* $C_n := \bigvee^{\uparrow}_{i \in \mathbb{N}} B_{\phi_{f(n)}(i)}$ *below* $\xi_X(n)$. *If* $\xi_X(n)$ *is maximal then* $C_n = \xi_X(n)$.

Proof. The idea how to define f is essentially the same as in the proof of of [9, Theorem 15]. Employing Proposition 11, we can avoid partiality of f. In detail we proceed as follows:

(1) Start with a natural number n.

(2) Recall from [9, Section 3.1] that the enumeration ξ_X of the set of all computable elements of $B(X)$ is defined by constructing a total recursive function $\eta_X : \mathbb{N} \to \mathbb{N}$

with the property that for each natural number n the function $\phi_{\eta_X(n)}$ is total and such that we get a chain $A_{\phi_{\eta_X(n)}(0)} \sqsubseteq A_{\phi_{\eta_X(n)}(1)} \sqsubseteq \cdots$ with least upper bound $\xi_X(n)$.

(3) Apply Proposition 11 to get a sequence $(x_{h(i)})_{i\in\mathbb{N}}$ with $x_{h(i)} \in A_{\phi_{\eta_X(n)}(i)}$ for all $i \in \mathbb{N}$.
(4) Define $g : \mathbb{N} \to \mathbb{N}$ recursive such that $|||A_{\phi_{\eta_X(n)}(i)}||| < q_{g(i)} < |||A_{\phi_{\eta_X(n)}(i)}||| + 1/2^i$.
(5) Then $B_{\langle h(i),g(i)\rangle+1} = C(x_{h(i)}, q_{g(i)}) \supseteq A_{\phi_{\eta_X(n)}(i)}$.
(6) Now we have to make the sequence $B_0, B_1, \ldots$ increasing. We use the same method as in the algorithm for obtaining the enumeration of computable elements in an effectively given domain described in [9] just after Proposition 3:
   (a) Define $j(0) = \langle h(0), g(0)\rangle + 1$.
   (b) Start with $i = 0$ and $k = 1$.
   (c) The set $P = \{\ell \geqslant k \mid B_{j(i)} \ll B_{\langle h(\ell),g(\ell)\rangle+1}\} \subseteq \mathbb{N}$ is r.e.
   (d) Write $P = \bigcup_{m\in\mathbb{N}} P_m$ where the test $\ell \in P_m$ is recursive in ℓ, m.
   (e) Now, starting with $m = 1$, each of these sets is tested for the existence of $\ell \leqslant m$ with $\ell \in P_m$. Whenever no such element is found, we let $j(i+m) = j(i)$ and check for the next value of m.
   (f) If, at some stage, there is $\ell \leqslant m$ with $\ell \in P_m$ then set $j(i+m) = \langle h(\ell), g(\ell)\rangle + 1$, increment i by m, set $k = \ell + 1$, and go to step (3.2).
(7) Now $B_{j(0)} \sqsubseteq B_{j(1)} \sqsubseteq \cdots$ and $\bigvee^{\uparrow}_{i\in\mathbb{N}} B_{j(i)} \sqsubseteq \xi_X(n)$.

We define $f(n)$ to be the derived index of the function j. It remains to verify that $C_n = \xi_X(n)$ if the latter is maximal. Maximality of $\xi_X(n)$ means that $\lim_{i\in\mathbb{N}} |||A_{\phi_{\eta_X(n)}(i)}||| = 0$, so we also have

$$\lim_{i\in\mathbb{N}} q_{g(i)} = 0 \tag{1}$$

as $q_{g(i)} < |||A_{\phi_{\eta_X(n)}(i)}|| + 1/2^i$ by construction (step (4)). Therefore, it suffices to show that for all $i, k \in \mathbb{N}$ there is $\ell \geqslant k$ such that $B_{\langle h(i),g(i)\rangle+1} \ll B_{\langle h(\ell),g(\ell)\rangle+1}$. This ensures that the sequence $(B_{j(i)})_{i\in\mathbb{N}}$ is not eventually constant and thus its limit is maximal by (1). As $q_{g(i)} > |||A_{\phi_{\eta_X(n)}(i)}||$, by (1) there is $\ell \geqslant i, k$ with $q_{g(i)} - |||A_{\phi_{\eta_X(n)}(i)}|| > q_{g(\ell)}$. As $\ell \geqslant i$, we have $x_{h(\ell)} \in A_{\phi_{\eta_X(n)}(\ell)} \subseteq A_{\phi_{\eta_X(n)}(i)}$. But also $x_{h(i)} \in A_{\phi_{\eta_X(n)}(i)}$, hence $\|x_{h(i)} - x_{h(\ell)}\| \leqslant |||A_{\phi_{\eta_X(n)}(i)}|| < q_{g(i)} - q_{g(\ell)}$. This implies $B_{\langle h(i),g(i)\rangle+1} \ll B_{\langle h(\ell),g(\ell)\rangle+1}$. □

Thus, the effective structure on X as a Banach space coincides with the constructed effective metric space structure. From now on, we will assume the more convenient latter form of presentation via a dense sequence $(x_n)_{n\in\mathbb{N}}$ in X and the basis consisting of $\perp$ and the $C(x_i, |q_j|)$ as defined above. The results concerning computability of points and sequences in metric spaces specialise as follows.

Corollary 15. (1) *An element $x \in X$ is computable iff it is an effective limit of a sequence $(x_{n_i})_{i\in\mathbb{N}}$, i.e. iff there is $f : \mathbb{N} \to \mathbb{N}$ recursive such that $\|x - x_{f(n)}\| \leqslant 1/2^n$ for all $n \in \mathbb{N}$. This equivalence is effective.*

(2) *A sequence $(y_n)_{n\in\mathbb{N}}$ is computable if and only if there is a recursive function $r : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $\|x_{r(n,k)} - y_n\| \leqslant 2^{-k}$ for all $n, k \in \mathbb{N}$. This equivalence is effective.*

3.3. Computable operators

As expected, we define those operators to be computable which have a computable extension.

Definition 16. A linear operator $f: X \to Y$ between effectively given Banach spaces is *computable* iff there is an extension $g: B(X) \to B(Y)$ (i.e. $g(\{x\}) = \{f(x)\}$ for all $x \in X$) which is computable in the sense of effective domain theory.

Recall that a linear operator $f: X \to Y$ between Banach spaces is *bounded* if there is $M \in \mathbb{R}$ such that $\|f(x)\| \leqslant M\|x\|$ for all $x \in X$. The smallest such bound M is the *norm* $\|f\|$ of f. By linearity, boundedness is equivalent to Lipschitz-continuity. Even more, continuity at the origin implies boundedness.

Theorem 17. *A linear operator is computable if and only if it maps computable sequences of points to computable sequences and is bounded with an explicitly given bound.*

Proof. As boundedness is equivalent to Lipschitz-continuity the (if) part is a direct consequence of Theorem 6. For the (only if) part, it is clear that computable sequences are mapped to computable sequences. To obtain a bound, suppose $g: B(X) \to B(X)$ is the computable function extending f. Then $M := \|\|g(C(\mathbf{0},1))\|\|$ is a bound for f. To see this, suppose $g(C(\mathbf{0},1)) = C(y,r)$ and $x \in X$ with $\|x\| = 1$. Then $C(\mathbf{0},1) \sqsubseteq C(x,0)$, hence $C(y,r) = g(C(\mathbf{0},1)) \sqsubseteq g(C(x,0)) = C(f(x),0)$, thus $\|y - f(x)\| + 0 \leqslant r$. But we also have $C(\mathbf{0},1) \sqsubseteq C(\mathbf{0},0)$ which implies $C(y,r) \sqsubseteq C(\mathbf{0},0)$, thus $\|y\| \leqslant r$. Therefore $\|f(x)\| \leqslant r + r = M$. Linearity implies that M is a bound. $\square$

Remark. (a) As in the case of Lipschitz functions on metric spaces (Theorem 6), it is essential to have a bound explicitly given; the mere knowledge of existence of a bound will not suffice to construct the extension to the domain of balls.

(b) Since every computable function is continuous and continuity and Lipschitz-continuity coincide for linear operators on Banach spaces, we do not need to assume in Theorem 17 that the function is Lipschitz-continuous. This is in contrast to the case of metric spaces (Theorem 6), where this assumption had to be made. In fact, there are continuous and computable operators on metric spaces which are not Lipschitz-continuous, e.g. the squaring function $x \mapsto x^2$ on the real line.

(c) One might ask the question why computable operators do not necessarily have a computable norm. Apart from the fact that there are bounded operators which appear to be computable but have non-computable norm,[2] this would not be analogous with the classification of computable functions on the reals. In that context, effective uniform continuity on computable intervals was crucial. Of course, every bounded operator is effectively uniformly continuous, the modulus given by some computable bound.

[2] Marian Pour-El, personal communication.

Demanding that the norm be computable would correspond to demanding the *optimal* modulus of continuity to be effective in the case of the real-valued functions. Certainly, this need not be the case.

3.4. Computability structure in the sense of Pour-El and Richards

We want to compare our notion of computability structure for Banach spaces with the one from [18]. The definition of Pour-El and Richards reads as follows:

Definition 18. Suppose $(X, \|\cdot\|)$ is a Banach space. A PR-computability *structure for* X is a nonempty set $\mathscr{S}(X)$ of sequences in X (referred to as *computable sequences*) such that the following three axioms are satisfied.
Linear forms: If $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in \mathscr{S}(X)$, and $(r_{nk})_{n,k\in\mathbb{N}}$ and $(s_{nk})_{n,k\in\mathbb{N}}$ are computable double sequences of real numbers and if $d:\mathbb{N}\to\mathbb{N}$ is recursive then $(w_n)_{n\in\mathbb{N}}$ is computable, where

$$w_n = \sum_{k=0}^{d(n)} (r_{nk}y_k + s_{nk}z_k).$$

Limits: If $(y_{nk})_{n,k\in\mathbb{N}}$ is a computable double sequence which effectively converges to $(z_n)_{n\in\mathbb{N}}$ as $k\to\infty$, then $(z_n)_{n\in\mathbb{N}} \in \mathscr{S}(X)$.
Norm: If $(y_n)_{n\in\mathbb{N}} \in \mathscr{S}(X)$ then $(\|y_n\|)_{n\in\mathbb{N}}$ is a computable sequence of real numbers.

(As usual, a double sequence $(y_{kn})_{k,n\in\mathbb{N}}$ is computable, if the sequence $(y_{\pi_1(n),\pi_2(n)})_{n\in\mathbb{N}}$ is computable.) The space is effectively separable, if there is a sequence $(e_n)_{n\in\mathbb{N}} \in \mathscr{S}(X)$ such that the linear span of $\{e_0, e_1, e_2, \ldots\}$, i.e. the set of all finite linear combinations of vectors from this set, is dense in X.

Theorem 19. *The computable sequences in an effectively given Banach space X constitute a PR-computability structure for X. The resulting space is effectively separable.*

Proof. Let us start with the (linear forms) axiom. Suppose $\oplus:\mathbb{N}^2\to\mathbb{N}$ is the recursive function defining addition on $B(X)$, i.e. $\xi_X(n)+\xi_X(m)=\xi_X(n\oplus m)$. Similarly, let $\odot:\mathbb{N}^2\to\mathbb{N}$ implement scalar multiplication. In the given situation, there are recursive functions $f_y, f_z:\mathbb{N}\to\mathbb{N}$ and $f_r, f_s:\mathbb{N}^2\to\mathbb{N}^2$ such that $y_n=\xi_X(f_y(n))$, $z_n=\xi_X(f_z(n))$, $r_{nk}=\xi_{\mathbb{R}}(f_r(n,k))$, and $s_{nk}=\xi_{\mathbb{R}}(f_s(n,k))$ for all $n,k\in\mathbb{N}$. To determine an index $h(n)$ for the element w_n, we use the following algorithm:
(1) Set $a=n_0$, where n_0 is an index for the origin $C(\mathbf{0},0)$.
(2) Set $k=0$.
(3) Set $b=a\oplus((f_r(n,k)\odot f_y(k)\oplus(f_s(n,k)\odot f_z(k)))$.
(Hence $\xi_X(b)=\xi_X(a)+r_{nk}y_k+s_{nk}z_k$.)
(4) Set $a=b$. Increment k.
(5) If $k>d(n)$ then set $h(n)=a$. Otherwise go to step (3).
It is clear that this gives the desired result $\xi_X(h(n))=w_n$. Furthermore, the algorithm is effective in n. Thus $h:\mathbb{N}\to\mathbb{N}$ is recursive and so the sequence $(w_n)_{n\in\mathbb{N}}$ is computable.

Now the (limits) axiom. If $(y_{nk})_{n,k\in\mathbb{N}}$ is a computable double sequence in X then, by Corollary 15(2), there is $r:\mathbb{N}^2\to\mathbb{N}$ such that

$$\|x_{r(n,k)} - y_{\pi_1(n),\pi_2(n)}\| \leqslant \frac{1}{2^k}$$

for all $n,k\in\mathbb{N}$. The double sequence $(y_{nk})_{n,k\in\mathbb{N}}$ converges to $(z_n)_{n\in\mathbb{N}}$, hence we also have

$$\|y_{ni} - z_n\| \leqslant \frac{1}{2^i}$$

for all $n,i\in\mathbb{N}$. Putting these together, we get

$$\|x_{r(\langle n,i\rangle,k)} - z_n\| \leqslant \|x_{r(\langle n,i\rangle,k)} - y_{ni}\| + \|y_{ni} - z_n\| \leqslant \frac{1}{2^k} + \frac{1}{2^i}$$

for all $n,k,i\in\mathbb{N}$. Thus setting $r'(n,k) := r(\langle n,k+1\rangle,k+1)$ yields

$$\|x_{r'(n,k)} - z_n\| \leqslant \frac{1}{2^k}$$

for all $n,k\in\mathbb{N}$, so the limit sequence $(z_n)_{n\in\mathbb{N}}$ is computable by Corollary 15(2).

The (norm) axiom is an immediate consequence of our axioms and the fact that computable functions preserve computability of sequences.

Finally, effective separability follows from Proposition 11. □

So every effectively given Banach space comes with a computability structure in the sense of Pour-El and Richards such that the space is effectively separable. Our aim is to show that the two notions coincide. We start with a Banach space $(X,\|\cdot\|)$ and a specified set of sequences $\mathscr{S}(X)$ satisfying the axioms to give a PR-computability structure on X. Furthermore, we assume $(e_n)_{n\in\mathbb{N}}$ to be a sequence with dense linear span. In order to be able to define an effective basis for the domain of balls $B(X)$, we need a dense sequence. The natural candidate is the *rational* linear span of $(e_n)_{n\in\mathbb{N}}$. Let us recall how to enumerate the set $\mathbb{N}^*$ of finite sequences over $\mathbb{N}$ (see for example [19, p. 71]). For a natural number $n>0$, let $\psi_n:\mathbb{N}\to\mathbb{N}^n$ denote the n-tupling function. (Defined inductively via $\langle\cdot,\cdot\rangle$.) Then $\psi:\mathbb{N}\to\mathbb{N}^*$ is defined by $\psi(0)=()$ and $\psi(\langle n,m\rangle+1) = \psi_m(n)$. Now we define $x_0 = \mathbf{0}$ and

$$x_{n+1} = \sum_{i=0}^{\pi_2(n)-1} q_{a_i} e_i,$$

where $\psi(n) = (a_0, a_1, \ldots, a_{\pi_2(n)-1})$.

Lemma 20. *There are total recursive functions $f:\mathbb{N}\to\mathbb{N}$ and $\bar{\oplus},\bar{\odot}:\mathbb{N}^2\to\mathbb{N}$ such that $x_{f(n)} = e_n$, $x_{n\bar{\oplus}m} = x_n + x_m$ and $x_{n\bar{\odot}m} = q_n\cdot x_m$.*

Clearly, $\{x_n \mid n\in\mathbb{N}\}$ is dense in X. We set $B_0 := \perp$ and

$$B_{\langle n,k\rangle+1} := C(x_n, |q_k|),$$

so the set $\{B_0, B_1, \ldots\}$ is a countable basis of $B(X)$. For the proof of the following theorem, we need the *Effective Density Lemma* from [18]:

Lemma 21 (Theorem 2-1 of [18]). *Suppose that $\mathscr{S}(X)$ is a PR-computability structure for the Banach space $(X, \|\cdot\|)$ and that $(e_n)_{n\in\mathbb{N}} \in \mathscr{S}(X)$ has dense linear span. Then a sequence $(y_n)_{n\in\mathbb{N}}$ is in $\mathscr{S}(X)$ if and only if there is a double sequence $(p_{nk})_{n,k\in\mathbb{N}}$ and recursive functions $d : \mathbb{N}^2 \to \mathbb{N}$ and $j : \mathbb{N}^3 \to \mathbb{N}$ such that*

$$p_{nk} = \sum_{i=0}^{d(n,k)} q_{j(n,k,i)} e_i$$

and such that $\|p_{nk} - y_n\| \leqslant 1/2^k$ for all $n, k \in \mathbb{N}$.

Theorem 22. *The above constructed basis for $B(X)$ makes $(X, \|\cdot\|)$ an effectively given Banach space. The computable sequences on this space are exactly the sequences from $\mathscr{S}(X)$.*

Proof. We have

$$C(x_n, q_k) \ll C(x_{n'}, q_{k'}) \Leftrightarrow \|x_n - x_{n'}\| + |q_{k'}| < |q_k|.$$

By the (norm) and (linear forms) axiom, this relation is r.e. in n, n', k, k'.

Computability of the operations and the norm is shown in the same manner as in the proof of Theorem 12. Instead of referring to computability w.r.t. the original basis for $B(X)$ we have to refer to the three axioms for sequences, of course. Let us give the details for addition: Using the recursive function $\bar{\oplus} : \mathbb{N}^2 \to \mathbb{N}$ from Lemma 20, we see that $C(x_i, |q_{i'}|) \ll C(x_n, |q_{n'}|) + C(x_m, |q_{m'}|)$ iff $C(x_i, |q_{i'}|) \ll C(x_{n\bar{\oplus} m}, |q_{n'}| + |q_{m'}|)$ iff $\|x_i - x_{n\bar{\oplus} m}\| + |q_{n'}| + |q_{m'}| < |q_{i'}|$ which is r.e. in all indices by the (norm) and (linear form) axioms. Hence addition is computable.

So it remains to verify that the resulting set $S'(X)$ of computable sequences coincides with the original $\mathscr{S}(X)$. Assume $(y_n)_{n\in\mathbb{N}} \in \mathscr{S}(X)$. By the Effective Density Lemma 21 it is the limit of a double sequence of recursive rational linear combinations of the e_n. This means that it is the limit of a recursive double sequence of the x_n. By Corollary 15, this implies $(y_n)_{n\in\mathbb{N}} \in \mathscr{S}'(X)$. Conversely, the recursive function f of Lemma 20 shows that $(e_n)_{n\in\mathbb{N}} \in \mathscr{S}'(X)$. We know from Theorem 19 that the set $\mathscr{S}'(X)$ satisfies the (linear forms) and (limits) axioms, so this implies $\mathscr{S}'(X) \subseteq \mathscr{S}(X)$. □

Vice versa we can start with an effective Banach space in the sense of the present paper, take the resulting set of computable sequences, considered as a computability structure in the sense of [18], and finally perform the above construction to get another effective structure for the domain of balls. Propositions 13 and 14 show that this structure is equivalent to the one we started with. This yields:

Theorem 23. *The domain-theoretic notion of computability structure for Banach spaces coincides with the notion of Pour-El and Richards from* [18].

Acknowledgement

We are grateful to the anonymous referees for various helpful remarks.

References

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, vol. 3, Clarendon Press, Boca Raton, FL, 1994, pp. 1–168.
[2] G. Alefeld, J. Herzberger, Introduction to Interval Arithmetic, Academic Press, New York, 1983.
[3] J. Blanck, Computability on topological spaces by effective domain representations, Ph.D. Thesis, Upsalla University, 1997.
[4] J. Blanck, Domain representability of metric spaces, Ann. Pure Appl. Logic 83 (1997) 225–247.
[5] G.S. Ceitin, Algorithmic operators in constructive metric spaces, Translations AMS 64 (1967) 1–80.
[6] A. Edalat, Domains for computation in mathematics, physics and exact real arithmetic, Bull. Symbolic Logic 3 (4) (1997) 401–452.
[7] A. Edalat, R. Heckmann, A computational model for metric spaces, Theoret. Comput. Sci. 193 (1–2) (1998) 53–73.
[8] A. Edalat, P.J. Potts, A new representation for exact real numbers, Electronic Notes in Theoretical Computer Science, vol. 6, 1997. URL: http://www.elsevier.nl/locate/entcs/volume6.html.
[9] A. Edalat, Ph. Sünderhauf, A domain theoretic approach to computability on the real line. Theoret. Comput. Sci. 210 (1) (1998) 73–98.
[10] Y.L. Ershov, Computable functionals of finite type, Algebra Logic 11 (4) (1972) 203–242.
[11] A. Grzegorczyk, Computable functionals, Fund. Math. 42 (1955) 168–202.
[12] J. Hauck, Berechenbare reelle Funktionenfolgen, Zeitschr. f. Math. Logik und Grundlagen d. Math. 19 (1973) 121–140.
[13] Chr. Kreitz, K. Weihrauch, Theory of representations, Theoret. Comput. Sci. 38 (1985) 35–53.
[14] D. Lacombe, Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles, C.R. Acad. Sci. Paris 240 (1955) 2478–2480, 241 (1955) 13–14, 241 (1955) 151–153.
[15] R.E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1966.
[16] Y.N. Moschovakis, Recursive metric spaces, Fund. Math. 55 (1964) 215–238.
[17] P.J. Potts, A. Edalat, Exact real computer arithmetic, March 1997, Department of Computing Technical Report DOC 97/9, Imperial College, available from http://theory.doc.ic.ac.uk/~pjp.
[18] M.B. Pour-El, J.I. Richards, Computability in Analysis and Physics, Springer, Berlin, 1989.
[19] H. Rogers, Theory of Recursive Functions and Effective Computability, McGraw-Hill, New York, 1967.
[20] D.S. Scott, Outline of a mathematical theory of computation, Proc. of 4th Annual Princeton Conf. on Information Sciences and Systems, 1970, pp. 169–176.
[21] V. Stolenberg-Hansen, J.V. Tucker, Complete local rings as domains, J. Symbolic Logic 53 (1988) 603–624.
[22] V. Stolenberg-Hansen, J.V. Tucker, Effective algebras, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, vol. 4, Clarendon Press, Boca Raton, FL, 1995, pp. 357–526.
[23] V. Stolenberg-Hansen, J.V. Tucker, Concrete models of computation for topological algebras, Technical Report 1997:34, Department of Mathematics, Upsalla University, 1997.
[24] K. Weihrauch, Computability, volume 9 of EATCS Monographs on Theoretical Computer Science, Springer, Berlin, 1987.
[25] K. Weihrauch, Computability on computable metric spaces, Theoret. Comput. Sci. 113 (1993) 191–210.
[26] K. Weihrauch, A foundation for computable analysis, In: D.S. Bridges, C.S. Calude, J. Gibbons, S. Reeves, I.H. Witten (Eds.), Combinatorics, Complexity and Logic, Discrete Mathematics and Theoretical Computer Science, Singapore, 1997, pp. 66–89, Proceedings of DMTCS'96, Springer, Berlin.