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On fractional tempered stable motion[☆]

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Abstract

Fractional tempered stable motion (fTSm) is defined and studied. FTSm has the same covariance structure as fractional Brownian motion, while having tails heavier than Gaussian ones but lighter than (non-Gaussian) stable ones. Moreover, in short time it is close to fractional stable Lévy motion, while it is approximately fractional Brownian motion in long time. A series representation of fTSm is derived and used for simulation and to study some of its sample paths properties.

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1. Introduction

Fractional Brownian motion (fBm) and its various extensions are not only rich mathematical objects but also have been extensively used in applications to model asset price dynamics, data traffic in telecommunication networks, daily hydrological series, and turbulence, to mention but a few topics. We recall that standard fBm $\{B_t^H : t \in \mathbb{R}\}$, $H \in (0, 1]$, is a centered Gaussian

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process with continuous paths and with the following covariance structure:

$$\mathrm{Cov}\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad t, s \in \mathbb{R}. \tag{1}$$

Modeling drawbacks of fBm and of some of its extensions have also been discussed in the literature. For example, although Gaussianity provides analytical tractability, its light tails are often inadequate for modeling higher variability observed in various natural phenomena. In contrast, non-Gaussian stable generalizations have infinite second moment, but also a lack of a closed form expression for the corresponding density, resulting in significant analytical difficulties. Moreover, the self-similar and stationary increments properties of fBm are sometimes viewed as unrealistic in practice.

In order to remove these drawbacks, we introduce and study *fractional tempered stable motion (fTSm)*, which has the following properties:

(i) Its marginals have tails heavier than Gaussian ones but lighter than (non-Gaussian) stable ones (Proposition 2.5).
(ii) It has the same second-order structure as fBm (Proposition 3.1).
(iii) In long time it behaves like fBm, while in short time it is more akin to fractional stable motion (Theorem 6.4).

We present a series representation for fTSm (Proposition 4.1), which is potentially useful for simulation and which is also used to study sample paths properties. For some ranges of the defining parameters, fTSm has a.s. Hölder continuous sample paths (Proposition 5.2), is not a semimartingale (Proposition 5.3), or its paths become nowhere bounded (Proposition 5.1).

In the next section, and motivated by the case of fBm, fTSm will be defined as a process of stochastic integrals, i.e., $\{\int_{\mathcal{S}} f(t,s)\mathrm{d}X_s^{TS} : t \in \mathcal{T}\}$, where $f : \mathcal{T} \times \mathcal{S} \to \mathbb{R}$ is a deterministic function, and where $\{X_t^{TS} : t \geq 0\}$ is a tempered stable Lévy process as described by Rosiński [21]. Various integral representations of fBm have been studied in the literature. The moving average representation and the harmonizable representation are the most commonly used. These have also been extended to non-Gaussian stable marginals. (See Chapter 7 of Samorodnitsky and Taqqu [23].) A lesser known representation of fBm, which will be of importance to us, involves a Volterra kernel and is due to Decreusefond and Üstünel [8]. Let us further put our work in perspective and describe its analogies to and differences from existing works (see also Remark 6.7). Recently, Benassi, Cohen and Istas [3] have defined the moving average fractional Lévy motion (MAFLM) as the stochastic integral of a moving average kernel with respect to a centered Lévy process with finite moments of arbitrary order. These processes have stationary increments, the same covariance structure as fBm and, asymptotically, their finite dimensional distributions converge to those of fBm. Moreover, when the Lévy integrator is a truncated stable random measure, the *increments* of MAFLM have finite dimensional distributions converging, in small time, to those of a fractional moving average stable motion. The Poissonized telecom process studied by Cohen and Taqqu [6] also possesses asymptotic properties similar to those of MAFLMs. Benassi et al. [2], as well as Gailagas and Kaj [10], proved, for various classes of processes, results which are somewhat opposite to ours: local convergence to fBm and global stable Lévy behaviors.

Let us close this section by introducing some notation and definitions which will be used throughout the text. $\mathbb{R}^d$ is the d-dimensional Euclidean space with the norm $\|\cdot\|$. $\mathbb{R}_0^d := \mathbb{R}^d \backslash \{0\}$, and $\mathcal{B}(\mathbb{R}_0^d)$ is the Borel σ-field of $\mathbb{R}_0^d$. $(\Omega, \mathcal{F}, \mathbb{P})$ is our underlying probability space. $\mathcal{L}(Y)$ is the law of the random vector Y, while "$\stackrel{\mathcal{L}}{=}$" denotes equality in law, or equality of the finite dimensional distributions when stochastic processes are considered. Similarly, "$\xrightarrow{\mathcal{L}}$" is used for convergence in law, or of the finite dimensional distributions. $C([0,\infty),\mathbb{R})$ is the space of continuous functions from $[0,\infty)$ to $\mathbb{R}$ endowed with the uniform metric $\sup_{t\in K\subset[0,\infty)}|x(t)-y(t)|$, for each compact set $K\subset[0,\infty)$. A sequence of stochastic processes $\{X_t^n : t\geq 0\}_{n\in\mathbb{N}}$ in $C([0,\infty),\mathbb{R})$ is said to be *tight* if for each compact set $K\subset[0,\infty)$ and each $\epsilon>0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{t,s\in K,|t-s|\leq\delta}|X_t^n - X_s^n|>\epsilon\right)=0.$$

A sequence of stochastic processes $\{X_t^n : t\geq 0\}_{n\in\mathbb{N}}$ is said to converge *uniformly on compacts in probability (ucp)* to a stochastic process $\{X_t : t\geq 0\}$ if for each compact set $K\subset[0,\infty)$ and each $\epsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in K}|X_t^n - X_t|>\epsilon\right)=0.$$

This last convergence will be denoted by "$X^n \xrightarrow{ucp} X$". Finally, $\ln^+ a=\max(0,\ln a)$.

2. Definition of fractional tempered stable motion

We begin by briefly reviewing tempered stable distributions and processes (Rosiński [21]) which are building blocks for the fTSm defined below. Let μ be an infinitely divisible probability measure, without a Gaussian component, on $\mathbb{R}^d$ (see Sato [22] for a good introduction to infinitely divisible laws and Lévy processes). Then, μ is called tempered stable if its Lévy measure has the form

$$\nu(B)=\int_{\mathbb{R}_0^d}\int_0^\infty \mathbf{1}_B(sx)s^{-\alpha-1}\mathrm{e}^{-s}\mathrm{d}s\rho(\mathrm{d}x),\quad B\in\mathcal{B}(\mathbb{R}_0^d),$$

where $\alpha\in(0,2)$ and where ρ, *the inner measure*, is such that

$$\int_{\mathbb{R}_0^d}\|x\|^\alpha\rho(\mathrm{d}x)<+\infty. \tag{2}$$

The two parameters α and ρ uniquely identify the Lévy measure of tempered stable distributions. Under the additional condition

$$\begin{cases}\displaystyle\int_{\mathbb{R}_0^d}\|x\|\rho(\mathrm{d}x)<+\infty, & \text{if } \alpha\in(0,1),\\ \displaystyle\int_{\mathbb{R}_0^d}\|x\|(1+\ln^+\|x\|)\rho(\mathrm{d}x)<+\infty, & \text{if } \alpha=1,\end{cases} \tag{3}$$

the characteristic function of μ has a closed form expression given by

$$\widehat{\mu}(y)=\exp\left[\mathrm{i}\langle y,b\rangle+\int_{\mathbb{R}_0^d}\phi_\alpha(\langle y,x\rangle)\rho(\mathrm{d}x)\right], \tag{4}$$

for some $b\in\mathbb{R}^d$ and where, for $s\in\mathbb{R}$,

$$\phi_\alpha(s)=\begin{cases}\Gamma(-\alpha)((1-\mathrm{i}s)^\alpha-1+\mathrm{i}\alpha s), & \text{if } \alpha\in(0,1)\cup(1,2),\\ (1-\mathrm{i}s)\ln(1-\mathrm{i}s)+\mathrm{i}s, & \text{if } \alpha=1.\end{cases} \tag{5}$$

Below, we write $\mu \sim TS(\alpha, \rho; b)$ if $\widehat{\mu}$ is given by (4) and denote by $\{X_t^{TS} : t \geq 0\}$ a real valued tempered stable Lévy process (and so $X_0^{TS} = 0$ a.s.) such that $X_1^{TS} \sim TS(\alpha, \rho; b)$. Note that the condition (3) implies that $\mathbb{E}[|X_t^{TS}|] < +\infty$, $t \geq 0$. Thus, setting $b = 0$ gives $\mathbb{E}[X_t^{TS}] = 0$, $t \geq 0$, and then $\{X_t^{TS} : t \geq 0\}$ is a martingale with respect to its natural filtration.

From now on, we always assume that for any $t \geq 0$,

$$\mathbb{E}[X_t^{TS}] = 0,$$

and further that

$$\int_{\mathbb{R}_0} |x|^2 \rho(\mathrm{d}x) < +\infty, \tag{6}$$

so that for any $t \geq 0$, $\mathbb{E}[(X_t^{TS})^2] < +\infty$.

Next, recall that a Volterra kernel is a function $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $K(t, s) = 0$ for $s > t$. In the present paper, we will use the Volterra kernel $K_{H,\alpha} : [0, \infty) \times [0, \infty) \to [0, \infty)$, given by

$$K_{H,\alpha}(t, s) := c_{H,\alpha} \left[\left(\frac{t}{s}\right)^{H-1/\alpha} (t-s)^{H-1/\alpha} - \left(H - \frac{1}{\alpha}\right) s^{1/\alpha - H} \int_s^t u^{H-1/\alpha-1} (u-s)^{H-1/\alpha} \mathrm{d}u \right] \mathbf{1}_{[0,t]}(s), \tag{7}$$

where $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$, $\alpha \in (0, 2)$, and

$$c_{H,\alpha} = \left(\frac{G(1-2G)\Gamma(1/2 - G)}{\Gamma(2-2G)\Gamma(G+1/2)} \right)^{1/2}, \tag{8}$$

with (throughout) $G := H - 1/\alpha + 1/2$. Clearly, $K_{1/\alpha,\alpha}(t, s) = \mathbf{1}_{[0,t]}(s)$. Note that when $H \in (1/\alpha, 1/\alpha + 1/2)$, we also have

$$K_{H,\alpha}(t, s) = c_{H,\alpha}(H - 1/\alpha) s^{1/\alpha - H} \int_s^t (u-s)^{H-1/\alpha-1} u^{H-1/\alpha} \mathrm{d}u \, \mathbf{1}_{[0,t]}(s). \tag{9}$$

Despite its complex structure, there are two main advantages to using this kernel as an integrand:

(i) It is defined only on $[0, t]$, $t > 0$, while the domain of definition of the moving average kernel $(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}$ is the whole real line. This is meaningful, in particular, in sample paths simulations, since it is rather impractical to generate a stochastic process as an integrator on the whole real line.

(ii) It can be written as a Riemann–Liouville fractional integral, whose inverse function has a closed form expression which is also a Riemann–Liouville fractional derivative. This is important for the prediction of the sample paths of fTSm, a problem of consequence in financial modeling. This will be presented in a subsequent paper.

Below, we derive a necessary and sufficient condition on H so that for each $t > 0$ and $p \geq 2$, the kernel is in $L^p([0, t])$.

Lemma 2.1. *Let $t > 0$, let $\alpha \in (0, 2)$, and let $p \geq 2$. Then, $K_{H,\alpha}(t, \cdot) \in L^p([0, t])$ if and only if $H \in (1/\alpha - 1/p, 1/\alpha + 1/p)$. In particular, $K_{H,\alpha}(t, \cdot) \in L^2([0, t])$. Moreover, when*

$K_{H,\alpha}(t,\cdot) \in L^p([0,t])$,

$$\int_0^t K_{H,\alpha}(t,s)^p \mathrm{d}s = C_{H,\alpha,p}\, t^{p(H-1/\alpha)+1}, \tag{10}$$

where

$$C_{H,\alpha,p} = c_{H,\alpha}^p \int_0^1 v^{p\left(\frac{1}{\alpha}-H\right)} \left[(1-v)^{H-\frac{1}{\alpha}} - \left(H-\frac{1}{\alpha}\right) \int_v^1 w^{H-\frac{1}{\alpha}-1}(w-v)^{H-\frac{1}{\alpha}} \mathrm{d}w \right]^p \mathrm{d}v, \tag{11}$$

and where $c_{H,\alpha}$ is given by (8).

Proof. The case $H = 1/\alpha$ is trivial since then $K_{H,\alpha}(t,s) = \mathbf{1}_{[0,t]}(s)$. If $H > 1/\alpha$, then $K_{H,\alpha}(t,s) \geq 0$, $K_{H,\alpha}(t,s)$ is decreasing in s, and as $s \downarrow 0$, $K_{H,\alpha}(t,s) \sim C' s^{1/\alpha - H}$, for some constant C'. Hence, $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$ if and only if $p(1/\alpha - H) > -1$, i.e., $H < 1/\alpha + 1/p$. When $H < 1/\alpha$, $K_{H,\alpha}(\cdot,s)$ explodes at $s = 0$ and at $s = t$. In fact, as $s \downarrow 0$, $K_{H,\alpha}(t,s) \sim C'' s^{H-1/\alpha}$, and as $s \uparrow t$, $K_{H,\alpha}(t,s) \sim C'''(t-s)^{H-1/\alpha}$, for some constants C'' and C'''. Thus, $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$ if and only if $p(H-1/\alpha) > -1$, i.e., $H > 1/\alpha - 1/p$, which proves the first claim. The last claim follows from an elementary computation. □

Remark 2.2. Above, we only considered the case $p \geq 2$ because of the moment condition (6) and since $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$ in (7). However, it is easily seen that the above results remain true for arbitrary $p > 0$, provided the kernel is defined for arbitrary H. In particular, we have $K_{H,\alpha}(t,\cdot) \in L^\alpha([0,t])$ since $(0, 2/\alpha) \supset (1/\alpha - 1/2, 1/\alpha + 1/2)$ and, moreover, $\int_0^t K_{H,1}(t,s)(1 + \ln^+ K_{H,1}(t,s))\mathrm{d}s < +\infty$.

Let us state two other known properties of $K_{H,\alpha}$. The proof of (ii) below can be found in, e.g., Decreusefond and Üstünel [8], or Nualart [15], while (i) is immediate.

Lemma 2.3. (i) *For each $h > 0$,*

$$K_{H,\alpha}(ht,s) = h^{H-1/\alpha} K_{H,\alpha}(t, s/h).$$

(ii) *For $t, s > 0$,*

$$\int_0^{t\wedge s} K_{H,\alpha}(t,u) K_{H,\alpha}(s,u) \mathrm{d}u = \frac{1}{2}(t^{2G} + s^{2G} - |t-s|^{2G}),$$

where $G = H - 1/\alpha + 1/2$.

We are now in a position to define fTSm.

Definition 2.4. Fractional tempered stable motion $\{L_t^H : t \geq 0\}$ in $\mathbb{R}$ is given by

$$L_t^H := \int_0^t K_{H,\alpha}(t,s) \mathrm{d}X_s^{TS}, \quad t \geq 0, \tag{12}$$

where the integral is defined in the $L^2(\Omega, \mathcal{F}, \mathbb{P})$-sense.

The integral above is well defined by the moment condition (6), Lemma 2.3(ii) and with the help of the Wiener–Itô isometry. (See also the proof of Proposition 3.1 below.) For convenience, we will henceforth write $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ when $\{L_t^H : t \geq 0\}$ is defined as in (12). Since $K_{1/\alpha,\alpha}(t, s) = \mathbf{1}_{[0,t]}(s)$, we note that $L_\cdot^{1/\alpha} = X_\cdot^{TS}$, which is thus a Lévy process.

The following is an important result on the marginals of fTSm.

Proposition 2.5. *The finite dimensional distributions of fTSm are tempered stable with finite second moment.*

Proof. Let $k \in \mathbb{N}$. It suffices to show that for any reals $\{a_i\}_{i=1}^k$ and any nonnegative nondecreasing reals $\{t_i\}_{i=1}^k$, the random variable $\sum_{i=1}^k a_i L_{t_i}^H$ is tempered stable. First, observe that

$$\sum_{i=1}^k a_i L_{t_i}^H = \int_0^{t_k} \left(\sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right) \mathrm{d}X_s^{TS}.$$

Then, by Proposition 35 of Rocha-Arteaga and Sato [17], we get

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{\mathrm{i}y\sum_{i=1}^k a_i L_{t_i}^H}] &= \exp\left[\int_0^{t_k} \int_{\mathbb{R}_0} \phi_\alpha \left(yx \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right) \rho(\mathrm{d}x)\mathrm{d}s \right] \\ &= \exp\left[\int_{\mathbb{R}_0} \phi_\alpha(yx)\eta(\mathrm{d}x) \right],\end{aligned}$$

where ϕ_α is given by (5) and where $\eta = M \circ J$ with $M(\mathrm{d}x, \mathrm{d}s) = \rho(\mathrm{d}x)\mathrm{d}s$ and

$$J(B) = \left\{ (x, s) \in \mathbb{R}_0 \times [0, t_k] : x \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \in B \right\}, \quad B \in \mathcal{B}(\mathbb{R}_0).$$

The measure η is well defined as an inner measure with finite second moment since for each i, $K_{H,\alpha}(t_i, \cdot) \in L^2([0, t_i])$ and

$$\int_{\mathbb{R}_0} |x|^2 \eta(\mathrm{d}x) = \int_{\mathbb{R}_0} |x|^2 \rho(\mathrm{d}x) \int_0^{t_k} \left(\sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right)^2 \mathrm{d}s < +\infty,$$

which concludes the proof. □

It is worth noting the one-dimensional marginal result as a corollary.

Corollary 2.6. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ and let ϕ_α be given by (5). For each $t > 0$,*

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}yL_t^H}] = \exp\left[\int_0^t \int_{\mathbb{R}_0} \phi_\alpha(yxK_{H,\alpha}(t, s))\rho(\mathrm{d}x)\mathrm{d}s \right], \tag{13}$$

and thus

$$L_t^H \sim TS(\alpha, \eta_t; 0) \tag{14}$$

where $\eta_t = M \circ J_t$ with $M(\mathrm{d}x, \mathrm{d}s) = \rho(\mathrm{d}x)\mathrm{d}s$ and $J_t(B) = \{(x, s) \in \mathbb{R}_0 \times [0, t] : xK_{H,\alpha}(t, s) \in B\}$, $B \in \mathcal{B}(\mathbb{R}_0)$.

3. Covariance structure and long-range dependence

Let us first describe the covariance structure of fTSm.

Proposition 3.1. *Let* $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$*. Then,* $\mathbb{E}[L_t^H] = 0$*, and*

$$\mathrm{Cov}(L_t^H, L_s^H) = \frac{1}{2}\left(t^{2G} + s^{2G} - |t-s|^{2G}\right)\mathbb{E}[(X_1^{TS})^2], \quad s, t > 0. \tag{15}$$

Proof. Recall that $\{X_t^{TS} : t \geq 0\}$ is a square-integrable centered martingale. The first claim is thus trivial. For the second claim, observe that for $s \in [0, t]$,

$$\begin{aligned}\mathrm{Cov}(L_t^H, L_s^H) &= \mathbb{E}[L_t^H L_s^H] \\ &= \mathbb{E}\left[\int_0^t K_{H,\alpha}(t,u)\mathrm{d}X_u^{TS} \int_0^s K_{H,\alpha}(s,u)\mathrm{d}X_u^{TS}\right] \\ &= \mathbb{E}\left[\int_0^{s\wedge t} K_{H,\alpha}(t,u)\mathrm{d}X_u^{TS} \int_0^{s\wedge t} K_{H,\alpha}(s,u)\mathrm{d}X_u^{TS}\right] \\ &= \mathbb{E}[(X_1^{TS})^2]\int_0^{s\wedge t} K_{H,\alpha}(t,u)K_{H,\alpha}(s,u)\mathrm{d}u,\end{aligned}$$

where the last equality holds by the Wiener–Itô isometry. Lemma 2.3(ii) then gives the result. □

Let us state some immediate consequences of the previous result.

Corollary 3.2. *Let* $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$*. For each* $t > 0$ *and each* $h > 0$*,*

$$\mathbb{E}[(L_{ht}^H)^2] = h^{2G}\mathbb{E}[(L_t^H)^2], \tag{16}$$

and for $s, t > 0$*,*

$$\mathbb{E}[(L_t^H - L_s^H)^2] = \mathbb{E}[(L_{|t-s|}^H)^2] = |t-s|^{2G}\mathbb{E}[(X_1^{TS})^2]. \tag{17}$$

The property (16) is sometimes called *second-order self-similarity*. Moreover, (17) says that fTSm has *second-order stationary increments*, which clearly implies its continuity in probability.

Remark 3.3. The self-similarity property and the stationarity property of the increments hold true only in the second-order sense, and *not* in the strict sense, i.e., for finite dimensional distributions. The strict self-similarity cannot hold unless the background driving Lévy process is strictly self-similar, which is not the case of tempered stable processes. Concerning stationarity, $\int_0^2 K_{H,\alpha}(2,s)\mathrm{d}X_s^{TS} - \int_0^1 K_{H,\alpha}(1,s)\mathrm{d}X_s^{TS}$ is not identical in law to $\int_0^1 K_{H,\alpha}(1,s)\mathrm{d}X_s^{TS}$, since with the notation of the proof of Proposition 2.5, we can find a set $B \in \mathcal{B}(\mathbb{R}_0)$ such that $\{(x,s) \in \mathbb{R}_0 \times [0,2] : x(K_{H,\alpha}(2,s) - K_{H,\alpha}(1,s)) \in B\} \neq \{(x,s) \in \mathbb{R}_0 \times [0,1] : xK_{H,\alpha}(1,s) \in B\}$. Note that the moving average fractional Lévy motions defined in [3] always have strictly stationary increments due to the nature of the moving average kernel.

We are now in a position to discuss the long-range dependence of fTSm. The definition of long-range dependence is often ambiguous and varies among authors. In the present paper, we will follow Samorodnitsky and Taqqu [23] and say that the increments of a second-order stochastic process $\{X_t : t \geq 0\}$ exhibit *long-range dependence* if for each $h > 0$,

$$\sum_{n=1}^{\infty} |\mathrm{Cov}(X_h - X_0, X_{nh} - X_{(n-1)h})| = +\infty,$$

and *short-range dependence*, if for each $h > 0$,

$$\sum_{n=1}^{\infty} |\mathrm{Cov}(X_h - X_0, X_{nh} - X_{(n-1)h})| < +\infty.$$

Proposition 3.4. *The increments of fTSm exhibit long-range dependence when $H \in (1/\alpha, 1/\alpha + 1/2)$, and short-range dependence when $H \in (1/\alpha - 1/2, 1/\alpha]$.*

Proof. By Proposition 3.1, we have for each $h > 0$,

$$\begin{aligned}\mathrm{Cov}(L_h^H, L_{t+h}^H - L_t^H) &= \frac{1}{2} t^{2G}((1 + h/t)^{2G} - 2 + (1 - h/t)^{2G}) \\ &\sim \frac{1}{2} t^{2(G-1)} G(2G - 1)h^2,\end{aligned}$$

as $t \to \infty$. The claim then holds since $2(G - 1) > -1$ for $H \in (1/\alpha, 1/\alpha + 1/2)$, while $2(G - 1) \le -1$ for $H \in (1/\alpha - 1/2, 1/\alpha]$. □

We next consider the existence of higher moments of fTSm.

Proposition 3.5. *Let $\{L_t^H : t \ge 0\} \sim fTSm(H, \alpha, \rho)$. Then, for each $p > 2$ and each $t > 0$, $\mathbb{E}[|L_t^H|^p] < +\infty$ if and only if $H \in (1/\alpha - 1/p, 1/\alpha + 1/p)$ and $\int_{|x|>1} |x|^p \rho(\mathrm{d}x) < +\infty$.*

Proof. By Corollary 2.6, for each $t \ge 0$, $\mathcal{L}(L_t^H)$ is tempered stable. By Proposition 2.3(iii) of Rosiński [21], $\mathbb{E}[|L_t^H|^p] < +\infty$ if and only if $\int_{|x|>1} |x|^p \eta_t(\mathrm{d}x) < +\infty$, where η_t is the inner measure of L_t^H given as in Corollary 2.6, that is, if and only if

$$\iint_{|x|K_{H,\alpha}(t,s)>1} (|x|K_{H,\alpha}(t, s))^p \rho(\mathrm{d}x)\mathrm{d}s < +\infty.$$

The integral above can be decomposed into two terms;

$$\int_0^t K_{H,\alpha}(t, s)^p \int_{\frac{1}{K_{H,\alpha}(t,s)} < |x| \le \frac{1}{K_{H,\alpha}(t,s)} \vee 1} |x|^p \rho(\mathrm{d}x)\mathrm{d}s,$$

and

$$\int_0^t K_{H,\alpha}(t, s)^p \int_{|x|> \frac{1}{K_{H,\alpha}(t,s)} \vee 1} |x|^p \rho(\mathrm{d}x)\mathrm{d}s.$$

The first term is equivalent to $\int_0^t K_{H,\alpha}(t, s)^p \mathrm{d}s$ by the moment condition (6) on ρ, while the second term is clearly equivalent to $\int_0^t K_{H,\alpha}(t, s)^p \mathrm{d}s \int_{|x|>1} |x|^p \rho(\mathrm{d}x)$. Then, Lemma 2.1 concludes the proof. □

Remark 3.6. It is shown in Proposition 2.3(iv) of [21] that a tempered stable distribution has finite exponential moments if and only if its inner measure has compact support. The tempered stable marginals of fTSm cannot have exponential moment since, with the notation of the preceding proposition, for any $\epsilon > 0$,

$$\eta_t(\{x \in \mathbb{R}_0 : |x| > \epsilon\}) = \int_{|x|K_{H,\alpha}(t,s)>\epsilon} \rho(\mathrm{d}x)\mathrm{d}s > 0,$$

by the unboundedness of the kernel $K_{H,\alpha}$.

4. Series representation

In this section, we derive a series representation for fTSm which is, in fact, inherited from the one for tempered stable processes obtained by Rosiński [21]. This representation can also be used for simulation (see Section 7). We will further make use of its structure to derive some sample paths properties in Section 5.

Let $\{\Gamma_i\}_{i\geq 1}$ be arrival times of a standard Poisson process, let $\{E_i\}_{i\geq 1}$ be a sequence of iid exponential random variables with parameter 1, let $\{U_i\}_{i\geq 1}$ be a sequence of iid uniform random variables on [0, 1], let $\{V_i\}_{i\geq 1}$ be a sequence of iid random variables in $\mathbb{R}_0$ with common distribution

$$\frac{|x|^{\alpha}\rho(\mathrm{d}x)}{m(\rho)^{\alpha}},$$

and let $\{T_i\}_{i\geq 1}$ be a sequence of iid uniform random variables on $[0, T]$. Let also $m(\rho)^{\alpha}$, k', and z_T be constants respectively given by $m(\rho)^{\alpha} = \int_{\mathbb{R}_0} |x|^{\alpha}\rho(\mathrm{d}x)$, $k' = m(\rho)^{-\alpha}\int_{\mathbb{R}_0} x|x|^{\alpha-1}\rho(\mathrm{d}x)$, and

$$z_T = \begin{cases} m(\rho)(\alpha/T)^{-1/\alpha}\zeta(1/\alpha)k'T^{-1} + |\Gamma(1-\alpha)|\int_{\mathbb{R}_0} x\rho(\mathrm{d}x), & \text{if } \alpha \neq 1, \\ (\ln(m(\rho)T) + 2\gamma)\int_{\mathbb{R}_0} x\rho(\mathrm{d}x) - \int_{\mathbb{R}_0} x\ln|x|\rho(\mathrm{d}x), & \text{if } \alpha = 1, \end{cases}$$

where ζ denotes the Riemann zeta function and γ $(= 0.5772\ldots)$ is the Euler constant. Then, Theorem 5.4 of [21] tells us that

$$\sum_{i=1}^{\infty}\left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}\mathbf{1}(T_i \leq t) - m(\rho)\left(\frac{\alpha\,\mathrm{i}}{T}\right)^{-1/\alpha} k'\frac{t}{T}\right] + z_T\, t, \tag{18}$$

converges a.s. uniformly in $t \in [0, T]$ to a tempered stable process with $TS(\alpha, \rho; 0)$. This series representation can easily be extended to fTSm as follows.

Proposition 4.1. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ and let $T > 0$. Then,*

$$\{L_t^H : t \in [0, T]\} \overset{\mathcal{L}}{=} \left\{\sum_{i=1}^{\infty}\left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}K_{H,\alpha}(t, T_i) - m(\rho)\left(\frac{\alpha\,\mathrm{i}}{T}\right)^{-1/\alpha} k' C_{H,\alpha,1}\frac{t^{H-1/\alpha+1}}{T}\right] + z_T C_{H,\alpha,1}t^{H-1/\alpha+1} : t \in [0, T]\right\}, \tag{19}$$

where $C_{H,\alpha,1}$ is the constant defined in (11). If ρ is symmetric, then

$$\{L_t^H : t \in [0, T]\} \overset{\mathcal{L}}{=} \left\{\sum_{i=1}^{\infty}\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|\right) \times \frac{V_i}{|V_i|}K_{H,\alpha}(t, T_i) : t \in [0, T]\right\}.$$

Moreover, if $H \in [1/\alpha, 1/\alpha + 1/2)$*, the series converges almost surely uniformly in* t *to* $fTSm(H, \alpha, \rho)$.

Proof. We will only consider the asymmetric case. By arguments as in Theorem 5.4 of Rosiński [21], we get

$$\sum_{i=1}^{\infty}\left[m(\rho)\left(\frac{\alpha\,\mathrm{i}}{T}\right)^{-1/\alpha}k'C_{H,\alpha,1}\frac{t^{H-1/\alpha+1}}{T}-c_i(T)\mathbb{E}[K_{H,\alpha}(t,T_1)]\right]$$
$$= z_T C_{H,\alpha,1}t^{H-1/\alpha+1},$$

uniformly in t, where

$$c_i(T) := \int_{i-1}^{i}\mathbb{E}\left[\left(m(\rho)\left(\frac{\alpha r}{T}\right)^{-1/\alpha}\wedge E_1U_1^{1/\alpha}|V_1|\right)\frac{V_1}{|V_1|}\right]\mathrm{d}r.$$

Hence, the right hand side of (19) can be rewritten as

$$Z_t := \sum_{i=1}^{\infty}\left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\wedge E_iU_i^{1/\alpha}|V_i|\right)\right.$$
$$\left.\times\frac{V_i}{|V_i|}K_{H,\alpha}(t,T_i)-c_i(T)\mathbb{E}[K_{H,\alpha}(t,T_1)]\right]. \tag{20}$$

Next, we need to analyze the finite dimensional distributions of $\{Z_t : t \in [0, T]\}$. Let $k \in \mathbb{N}$, let $\{a_j\}_{j=1}^k$ be a real sequence, and let $\{t_j\}_{j=1}^k$ be nondecreasing reals taking values in $[0, T]$. We will show that the random variable $\sum_{j=1}^k a_j Z_{t_j}$ has the same law as $\sum_{j=1}^k a_j L_{t_j}^H$. In view of Proposition 2.5, we have

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}y\sum_{j=1}^k a_jL_{t_j}^H}\right]=\exp\left[\int_0^T\int_{\mathbb{R}_0}\phi_\alpha\left(yx\sum_{j=1}^k a_jK_{H,\alpha}(t_j,s)\right)\rho(\mathrm{d}x)\mathrm{d}s\right],$$

where ϕ_α is given by (5). Observe also that

$$\sum_{j=1}^k a_jZ_{t_j} = \sum_{i=1}^{\infty}\left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\wedge E_iU_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}\sum_{j=1}^k a_jK_{H,\alpha}(t_j,T_i)\right.$$
$$\left.-c_i(T)\mathbb{E}\left[\sum_{j=1}^k a_jK_{H,\alpha}(t_j,T_1)\right]\right].$$

This series representation is induced by the Lévy measure

$$\nu(B)=\int_0^T\int_{\mathbb{R}_0}\int_0^\infty\int_0^1\int_0^\infty\mathbf{1}_B\left(H(r/T,u,s,x)\sum_{j=1}^k a_jK_{H,\alpha}(t_j,v)\right)$$
$$\times\,\mathrm{d}r\mathrm{d}u\mathrm{e}^{-s}\mathrm{d}s\rho_1(\mathrm{d}x)\frac{\mathrm{d}v}{T}$$
$$=\int_0^T\int_{\mathbb{R}_0}\int_0^\infty\mathbf{1}_B\left(sx\sum_{j=1}^k a_jK_{H,\alpha}(t_j,v)\right)s^{-\alpha-1}\mathrm{e}^{-s}\mathrm{d}s\rho(\mathrm{d}x)\mathrm{d}v,$$

where $H(r, u, s, x) = (m(\rho)(\alpha r)^{-1/\alpha} \wedge s u^{1/\alpha}|x|)x/|x|$ and $\rho_1(\mathrm{d}x) = m(\rho)^{-\alpha}|x|^{\alpha}\rho(\mathrm{d}x)$. In fact, the measure ν is well defined as a Lévy measure since $K_{H,\alpha}(t_j, \cdot) \in L^2([0, t_j])$, for each j. Therefore, by Theorem 4.1(B) of Rosiński [20],

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}y\sum_{j=1}^{k} a_j Z_{t_j}}\right] = \exp\left[\int_{\mathbb{R}_0} (\mathrm{i}\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz)\nu(\mathrm{d}z)\right]$$
$$= \exp\left[\int_0^T \int_{\mathbb{R}_0} \phi_\alpha\left(yx\sum_{j=1}^{k} a_j K_{H,\alpha}(t_j, s)\right)\rho(\mathrm{d}x)\mathrm{d}s\right],$$

which proves the equality of all the finite dimensional distributions.

Next, let $H \in [1/\alpha, 1/\alpha + 1/2)$. Define, for $t \in [0, T]$ and $s \in [0, \infty)$,

$$Z_{t,s} := \sum_{\{i:\Gamma_i \le s\}} \left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}K_{H,\alpha}(t, T_i)\right.$$
$$\left. - c_i(T)\mathbb{E}[K_{H,\alpha}(t, T_1)]\right].$$

Notice that $\{Z_{t,s} : t \in [0, T],\ s \in [0, \infty)\}$ has independent increments in s (*not* in t, of course). Arguments as in the proof of Theorem 5.1 of [20] give the a.s. convergence of the series uniformly on $[0, T]$. □

Remark 4.2. Extending the decomposition method of Asmussen and Rosiński [1], Lacaux [13] derives a series representation for real harmonizable multifractional Lévy motions by decomposing the (infinite) Lévy measure into two parts: one approximates the component around the origin by a Gaussian law while the other component is simply compound Poisson. Our result is rather direct and exact thanks to the explicit representation (18) of tempered stable processes. Actually, the series representation above can easily be extended to the multifractional case. Simply replacing the parameter H by an Hölderian function $H(\cdot)$ from $[0, T]$ to $(1/\alpha - 1/2, 1/\alpha + 1/2)$ leads to

$$\{L_t^{H(t)} : t \in [0, T]\} \stackrel{\mathcal{L}}{=} \left\{\sum_{i=1}^{\infty}\left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}K_{H(t),\alpha}(t, T_i)\right.\right.$$
$$\left. - m(\rho)\left(\frac{\alpha\, \mathrm{i}}{T}\right)^{-1/\alpha} k' C_{H(t),\alpha,1}\frac{t^{H(t)-1/\alpha+1}}{T}\right]$$
$$\left. + z_T C_{H(t),\alpha,1} t^{H(t)-1/\alpha+1} : t \in [0, T]\right\}.$$

5. Sample paths properties

In this section, we investigate sample paths properties of fTSm. Let us begin with the case $H \in (1/\alpha - 1/2, 1/\alpha)$.

Proposition 5.1. *When $H \in (1/\alpha - 1/2, 1/\alpha)$, fTSm is a.s. unbounded on every interval of positive length.*

Proof. Let $T > 0$. For each $t \in (0, T]$, $\lim_{s \uparrow t} K_{H,\alpha}(t, s) = +\infty$. Hence, in the series representation given in Proposition 4.1, $K_{H,\alpha}(T_i, T_i) = +\infty$, for all $i \in \mathbb{N}$, and so none of the summands are well defined. This implies that $\sup_{t \in [0,T]} |L_t^H| = +\infty$ a.s. □

Unfortunately, the above sample paths property makes fTSm with short-range dependence of little practical use. We notice that for any $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$, $K_{H,\alpha}(t, 0) = +\infty$, but this turns out to be irrelevant for the unboundedness of the sample paths since $T_i \neq 0$ a.s., $i \in \mathbb{N}$.

FTSm with long-range dependence has better sample paths properties. In particular, it has a Hölder continuous version with exponent $\gamma \in (0, H - 1/\alpha)$.

Proposition 5.2. *If $H \in (1/\alpha, 1/\alpha + 1/2)$, there exists a continuous modification of fTSm, which is a.s. locally Hölder continuous with exponent $\gamma < H - 1/\alpha$.*

Proof. By Corollary 3.2, we have $\mathbb{E}[|L_t^H - L_s^H|^2] = |t - s|^{2G}\mathbb{E}[(X_1^{TS})^2]$. If $H > 1/\alpha$, then $2G > 1$, and thus the Kolmogorov–Čentsov Theorem (see, for example, Theorem 3.23 of Kallenberg [11]) applies directly, giving the result. □

We will henceforth always assume that when $H \in (1/\alpha, 1/\alpha + 1/2)$, we are using such a Hölder continuous version of fTSm.

Proposition 5.3. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ with $H \in (1/\alpha, 1/\alpha + 1/2)$.*

(i)

$$\lim_{N \to \infty} \mathbb{E}\left[\sum_{n=0}^{N-1} \left| L^H_{\frac{n+1}{N}T} - L^H_{\frac{n}{N}T} \right|^2\right] = 0.$$

(ii) *FTSm is a.s. of infinite variation on every interval of positive length.*

(iii) *FTSm is not semimartingale.*

Proof. (i) Immediate from

$$\mathbb{E}\left[\sum_{n=0}^{N-1} \left| L^H_{\frac{n+1}{N}T} - L^H_{\frac{n}{N}T} \right|^2\right] = (N/T)^{2(1/\alpha - H)}.$$

(ii) Let $T > 0$. For each $s \in [0, T]$, we have

$$\limsup_{t_1, t_2 \downarrow s} \frac{|K_{H,\alpha}(t_1, s) - K_{H,\alpha}(t_2, s)|}{|t_1 - t_2|} = +\infty,$$

which implies that for each $s \in [0, T]$, $K_{H,\alpha}(\cdot, s)$ is of infinite variation. By Theorem 4 of Rosiński [18] and a symmetrization argument given there, fTSm is of infinite variation with positive probability. FTSm is self-decomposable and hence by Corollary 3 of Rosiński [19], it obeys a zero–one law. This gives the result.

(iii) The convergence in (i) implies convergence in probability and, together with (ii), the claim follows as in Lin [14]. □

Remark 5.4. In view of (iii) above, stochastic integration with respect to fTSm cannot be defined in the classical semimartingale sense. However, a slight modification of the kernel $K_{H,\alpha}$, such as in Proposition 2.5 of Carmona et al. [4], induces a corresponding semimartingale, which can be arbitrarily close to fTSm. Recall that for $H \in (1/\alpha, 1/\alpha + 1/2)$, $K_{H,\alpha}$ admits a simpler expression given by (9). Then, observe that

$$\frac{\partial}{\partial t} K_{H,\alpha}(t,s) = c_{H,\alpha}(H - 1/\alpha)(t-s)^{H-1/\alpha-1} \left(\frac{t}{s}\right)^{H-1/\alpha} \mathbf{1}_{[0,t)}(s),$$

and that

$$K_{H,\alpha}(t,s) = \int_s^t \frac{\partial}{\partial u} K_{H,\alpha}(u,s) \mathrm{d}u \, \mathbf{1}_{[0,t)}(s).$$

Therefore,

$$L_t^H = \int_0^t K_{H,\alpha}(t,s) \mathrm{d}X_s^{TS} = \int_0^t \left(\int_s^t \frac{\partial}{\partial u} K_{H,\alpha}(u,s) \mathrm{d}u \right) \mathrm{d}X_s^{TS}. \tag{21}$$

If the two integrals could be interchanged, then fTSm would be of finite variation, i.e., it would be a semimartingale; we have seen that this is not the case. On the other hand, the integrability condition of the stochastic Fubini theorem (Theorem 46 of Protter [16]) can be achieved by slightly modifying the kernel $K_{H,\alpha}$. Set

$$K_{H,\alpha}^n(t,s) := c_{H,\alpha}(H - 1/\alpha) s^{1/\alpha - H} \int_s^t \left(u + \frac{1}{n} - s \right)^{H-1/\alpha-1} \times u^{H-1/\alpha} \mathrm{d}u \, \mathbf{1}_{[0,t]}(s), \quad n \in \mathbb{N}, \tag{22}$$

and so

$$\frac{\partial}{\partial t} K_{H,\alpha}^n(t,s) = c_{H,\alpha}(H - 1/\alpha) \left(t + \frac{1}{n} - s \right)^{H-1/\alpha-1} \left(\frac{t}{s}\right)^{H-1/\alpha} \mathbf{1}_{[0,t]}(s).$$

The integrability condition is then satisfied; for every $u \in [0, t]$,

$$\begin{aligned}\int_0^u \left(\frac{\partial}{\partial u} K_{H,\alpha}^n(u,s) \right)^2 \mathrm{d}s &= c_{H,\alpha}^2 (H - 1/\alpha)^2 \int_0^u \left(u + \frac{1}{n} - s \right)^{2(H-1/\alpha-1)} \\ &\quad \times \left(\frac{u}{s}\right)^{2(H-1/\alpha)} \mathrm{d}s \\ &\le c_{H,\alpha}^2 (H - 1/\alpha)^2 (1 - 2(H - 1/\alpha))^{-1} \\ &\quad \times n^{-2(H-1/\alpha-1)} u < +\infty,\end{aligned}$$

and thus the stochastic Fubini theorem applies. Therefore,

$$\int_0^t K_{H,\alpha}^n(t,s) \mathrm{d}X_s^{TS} = \int_0^t \left(\int_0^u \frac{\partial}{\partial u} K_{H,\alpha}^n(u,s) \mathrm{d}X_s^{TS} \right) \mathrm{d}u,$$

which is clearly of finite variation. It is also of interest to further modify the above to get an infinite variation semimartingale. This can be done as follows. For $\epsilon > 0$, set $K_{H,\alpha}^{n,\epsilon}(t,s) := K_{H,\alpha}^n(t,s) + \epsilon$. Since $\frac{\partial}{\partial t} K_{H,\alpha}^{n,\epsilon}(t,s) = \frac{\partial}{\partial t} K_{H,\alpha}^n(t,s)$, the stochastic Fubini theorem applies again and thus

$$\begin{aligned}\int_0^t K_{H,\alpha}^{n,\epsilon}(t,s) \mathrm{d}X_s^{TS} &= \int_0^t (\epsilon + K_{H,\alpha}^n(t,s)) \mathrm{d}X_s^{TS} \\ &= \epsilon X_t^{TS} + \int_0^t \left(\int_0^u \frac{\partial}{\partial u} K_{H,\alpha}^n(u,s) \mathrm{d}X_s^{TS} \right) \mathrm{d}u,\end{aligned}$$

which exactly follows the definition of the canonical decomposition of semimartingales, i.e. a martingale plus a finite variation process.

6. Short and long time behavior

In this section, we obtain both *the short and the long time behaviors* of fTSm. These are in fact inherited from the corresponding behaviors of the background driving tempered stable process as described in Rosiński [21]. In short time, fTSm is asymptotically fractional stable motion (to be defined below), while in long time it is approximately fBm.

Let us begin by briefly reviewing the corresponding behaviors of tempered stable processes as obtained in Theorem 3.1 of [21]. Below, "$\stackrel{d}{\to}$" denotes the weak convergence of stochastic processes in the space $D([0,\infty),\mathbb{R})$ of càdlàg functions from $[0,\infty)$ into $\mathbb{R}$ equipped with the Skorohod topology.

(i) *Short time behavior*: Let $\{X_t^{TS} : t \geq 0\} \sim TS(\alpha, \rho; 0)$ and let

$$b_{h,\alpha} = \begin{cases} h\Gamma(1-\alpha)\int_{\mathbb{R}_0} x\rho(\mathrm{d}x), & \text{if } \alpha \in (0,1), \\ -(1+\ln h)\int_{\mathbb{R}_0} x\rho(\mathrm{d}x), & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha \in (1,2). \end{cases} \tag{23}$$

Then,

$$\left\{h^{-1/\alpha}\left(X_{ht}^{TS} + b_{h,\alpha}t\right) : t \geq 0\right\} \stackrel{d}{\to} \{X_t^{\alpha} : t \geq 0\}, \quad \text{as } h \to 0,$$

where $\{X_t^{\alpha} : t \geq 0\}$ is an α-stable Lévy process in $\mathbb{R}$ such that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}yX_t^{\alpha}}] = \exp\left[t\int_{\mathbb{R}_0} \varphi_{\alpha}(yx)\rho(\mathrm{d}x)\right],$$

where

$$\varphi_{\alpha}(s) = \begin{cases} -\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}|s|^{\alpha}\left(1 - \mathrm{i}\tan\frac{\pi\alpha}{2}\mathrm{sgn}(s)\right), & \text{if } \alpha \in (0,1)\cup(1,2), \\ -\left(\frac{\pi}{2}|s| + \mathrm{i}s\ln|s|\right) + \mathrm{i}s, & \text{if } \alpha = 1. \end{cases} \tag{24}$$

(ii) *Long time behavior*: Let $\{X_t^{TS} : t \geq 0\} \sim TS(\alpha, \rho; 0)$. Then,

$$\{h^{-1/2}X_{ht}^{TS} : t \geq 0\} \stackrel{d}{\to} \{cB_t : t \geq 0\}, \quad \text{as } h \to \infty,$$

where $\{B_t : t \geq 0\}$ is a standard (centered) Brownian motion and

$$c^2 = \Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2\rho(\mathrm{d}x). \tag{25}$$

We will say that the limiting stable process $\{X_t^{\alpha} : t \geq 0\}$ given above is *associated with* the tempered stable Lévy process $\{X_t^{TS} : t \geq 0\}$.

Let us now define *fractional stable motion (fSm)*, which turns out to be a short time limit of fTSm. Below, the integral in (26) is well defined in probability since for $\alpha \neq 1$, $K_{H,\alpha}(t,\cdot) \in L^\alpha([0,t])$, $t > 0$, while for $\alpha = 1$, $K_{H,1}(t,\cdot)(1 + \ln^+ K_{H,1}(t,\cdot)) \in L^1([0,t])$, $t > 0$ (see Remark 2.2).

Definition 6.1. Let $\{L_t^H : t \geq 0\} \sim fTSm(H,\alpha,\rho)$. Fractional stable motion (fSm) $\{L_t^{H,\alpha} : t \geq 0\}$ associated with fTSm $\{L_t^H : t \geq 0\}$ is given via

$$L_t^{H,\alpha} := \int_0^t K_{H,\alpha}(t,s)\mathrm{d}X_s^\alpha, \quad t \geq 0, \tag{26}$$

where $\{X_t^\alpha : t \geq 0\}$ is associated with the tempered stable Lévy process $\{X_t^{TS} : t \geq 0\}$.

Let us derive some basic properties of fSm.

Lemma 6.2. *Let* $\{L_t^{H,\alpha} : t \geq 0\}$ *be fSm associated with* $\{L_t^H : t \geq 0\} \sim fTSm(H,\alpha,\rho)$. *Moreover, in* (ii)–(v) *below, assume that* ρ *is symmetric when* $\alpha = 1$.

(i) *The finite dimensional distributions of fSm are stable.*

(ii) *For each* $t > 0$, *the characteristic function of* $L_t^{H,\alpha}$ *is given by*

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}yL_t^{H,\alpha}}] = \exp\left[C_{H,\alpha,\alpha}t^{\alpha H}\int_{\mathbb{R}_0}\widetilde{\varphi}_\alpha(yx)\rho(\mathrm{d}x)\right], \tag{27}$$

where $C_{H,\alpha,\alpha}$ *is given by* (11) *and where*

$$\widetilde{\varphi}_\alpha(s) = \begin{cases} \varphi_\alpha(s), & \text{if } \alpha \in (0,1)\cup(1,2), \\ -\dfrac{\pi}{2}|s|, & \text{if } \alpha = 1, \end{cases}$$

with φ_α *is defined by* (24).

(iii) *FSm is self-similar, i.e., for any* $h > 0$, $\{h^{-H}L_{ht}^{H,\alpha} : t \geq 0\} \stackrel{\mathcal{L}}{=} \{L_t^{H,\alpha} : t \geq 0\}$.

(iv) *FSm has (strictly) stationary increments, i.e., for any* $t > s > 0$, $L_t^{H,\alpha} - L_s^{H,\alpha} \stackrel{\mathcal{L}}{=} L_{t-s}^{H,\alpha}$.

(v) *For* $H \in (1/\alpha, 1/\alpha + 1/2)$, *fSm has a continuous version.*

(vi) *For* $H \in (1/\alpha - 1/2, 1/\alpha)$, *fSm is unbounded on every interval of positive length.*

(vii) *For* $H = 1/\alpha$, *fSm is an* α*-stable (Lévy) process.*

(viii) *With the notation of Section* 4, *if* $\alpha \in (1,2)$ *with asymmetric* ρ, *then*

$$\begin{aligned}\{L_t^{H,\alpha} : t \in [0,T]\} \stackrel{\mathcal{L}}{=} \Bigg\{\sum_{i=1}^\infty \Bigg[& m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\frac{V_i}{|V_i|}K_{H,\alpha}(t,T_i) - m(\rho)\left(\frac{\alpha\mathrm{i}}{T}\right)^{-1/\alpha} \\ & \times k' C_{H,\alpha,1}\frac{t^{H-1/\alpha+1}}{T}\Bigg] \\ & + m(\rho)\left(\frac{\alpha}{T}\right)^{-1/\alpha}\zeta(1/\alpha)k'C_{H,\alpha,1}\frac{t^{H-1/\alpha+1}}{T} : t \in [0,T]\Bigg\},\end{aligned}$$

and if $\alpha = 1$, *then*

$$\{L_t^{H,1} : t \in [0,T]\} \stackrel{\mathcal{L}}{=} \Bigg\{\sum_{i=1}^\infty \Bigg[m(\rho)\left(\frac{\Gamma_i}{T}\right)^{-1}\frac{V_i}{|V_i|}K_{H,1}(t,T_i)$$

$$-m(\rho)\left(\frac{i}{T}\right)^{-1} k' C_{H,1,1} \frac{t^H}{T}\Bigg]$$

$$\left. + m(\rho)(\ln(m(\rho)T) + 2\gamma)k' C_{H,1,1} t^H : t \in [0, T]\right\},$$

while if $\alpha \in (0, 1)$*, or if* $\alpha \in [1, 2)$ *with* ρ *symmetric, then*

$$\{L_t^{H,\alpha} : t \in [0, T]\} \stackrel{\mathcal{L}}{=} \left\{\sum_{i=1}^{\infty} m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i) : t \in [0, T]\right\}.$$

Moreover, if $H \in [1/\alpha, 1/\alpha + 1/2)$*,* $\alpha \in (0, 2)$*, the above series converges almost surely uniformly in* $t \in [0, T]$ *to a version of* $\{L_t^{H,\alpha} : t \in [0, T]\}$.

Proof. Since $K_{H,\alpha}(t, \cdot) \in L^\alpha([0, t])$ for $\alpha \neq 1$ and $K_{H,1}(t, \cdot)(1 + \ln^+(K_{H,1}(t, \cdot))) \in L^1([0, t])$, (i) follows directly from Section 3.2 of Samorodnitsky and Taqqu [23]. Moreover, for $\alpha \neq 1$,

$$\begin{aligned}\mathbb{E}\left[e^{iyL_t^{H,\alpha}}\right] &= \exp\Bigg[-\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}\int_0^t \int_{\mathbb{R}_0} |K_{H,\alpha}(t, s)yx|^\alpha \\ &\quad\times \left(1 - i\tan\frac{\pi\alpha}{2}\operatorname{sgn}(K_{H,\alpha}(t, s)yx)\right)\rho(dx)ds\Bigg] \\ &= \exp\Bigg[-\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}\int_0^t K_{H,\alpha}(t, s)^\alpha ds \\ &\quad\times \int_{\mathbb{R}_0} |yx|^\alpha \left(1 - i\tan\frac{\pi\alpha}{2}\operatorname{sgn}(yx)\right)\rho(dx)\Bigg],\end{aligned}$$

and for $\alpha = 1$ with symmetric ρ,

$$\begin{aligned}\mathbb{E}\left[e^{iyL_t^{H,1}}\right] &= \exp\left[\int_0^t \int_{\mathbb{R}_0} \left(-\frac{\pi}{2}|K_{H,1}(t, s)yx|\right)\rho(dx)ds\right] \\ &= \exp\left[-\frac{\pi}{2}\int_0^t |K_{H,1}(t, s)|ds \int_{\mathbb{R}_0} |yx|\rho(dx)\right],\end{aligned}$$

which show (ii). (iii) and (iv) follow from the self-similarity and the stationary increments properties of $\{X_t^\alpha : t \geq 0\}$ and with the help of Lemma 2.3. For (v), let us first assume that ρ is symmetric when $\alpha = 1$. By (iii) and (iv), $\mathbb{E}[|L_t^{H,\alpha} - L_s^{H,\alpha}|^p] = |t - s|^{pH}\mathbb{E}[|L_1^{H,\alpha}|^p]$, $0 < p < \alpha$. Hence, when $H \in (1/\alpha, 1/\alpha + 1/2)$, the continuity of fSm follows from the Kolmogorov–Čentsov Theorem, which shows (v). Next, let $H \in (1/\alpha - 1/2, 1/\alpha)$. For each $T > 0$, $\sup_{t\in[0,T]} |K_{H,\alpha}(t, s)| = +\infty$, $s \in [0, T]$. The nowhere boundedness follows from (i) and Theorem 4 of Rosiński [18] as well as the zero–one law for stable processes and a symmetrization argument given there. This shows (vi). (vi) is immediate from $K_{1/\alpha,\alpha}(t, s) = \mathbf{1}_{[0,t]}(s)$. Finally, (viii) follows from arguments as in Proposition 5.5 of Rosiński [21] and from Proposition 4.1. □

We will henceforth always assume that when $H \in (1/\alpha, 1/\alpha+1/2)$*, we are using a continuous version of fSm.*

Remark 6.3. Many stable extensions of fBm have been studied; for example, linear fractional stable motion, log-fractional stable motion, harmonizable fractional stable motion (see, e.g., Samorodnitsky and Taqqu [23]). These various extensions are not necessarily identical in law (although their finite dimensional distributions are all stable) since their stable marginals are determined by the kernels in the stochastic integral representations. Indeed, fSm defined above is still different from any of them.

We are now in a position to present the main result of this section (below, the case $H = 1/\alpha$ is omitted since it corresponds to the results of [21]).

Theorem 6.4. *Let* $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ *with* $H \neq 1/\alpha$.

(i) *Short time behavior: Let* $b_{h,\alpha}$ *be the constant* (23) *and let* $k_t = \int_0^t K_{H,\alpha}(t, s)\mathrm{d}s$. *Then,*

$$\{h^{-H}(L_{ht}^H + h^{H-1/\alpha} b_{h,\alpha} k_t) : t \geq 0\} \xrightarrow{\mathcal{L}} \{L_t^{H,\alpha} : t \geq 0\} \quad \text{as } h \to 0,$$

where $\{L_t^{H,\alpha} : t \geq 0\}$ *is fSm associated with* $\{L_t^H : t \geq 0\}$.

(ii) *Long time behavior:*

$$\{h^{-G} L_{ht}^H : t \geq 0\} \xrightarrow{\mathcal{L}} \{c B_t^G : t \geq 0\} \quad \text{as } h \to \infty,$$

where $\{B_t^G : t \geq 0\}$ *is a standard fBm with parameter* $G = H - 1/2 + 1/\alpha$ *and where* c *is the constant given by* (25).

(iii) *When* $H \in (1/\alpha, 1/\alpha + 1/2)$, *with moreover* ρ *symmetric when* $\alpha = 1$, *the convergence in* (i) *and* (ii) *can be strengthened to the weak convergence in* $C([0, \infty), \mathbb{R})$.

Proof of (i) and (ii). (i) Observe that for each $t \geq 0$,

$$Y_t^h := h^{-H}(L_{ht}^H + h^{H-1/\alpha} b_{h,\alpha} k_t) = \int_0^t K_{H,\alpha}(t, s) h^{-1/\alpha} \mathrm{d}(X_{hs}^{TS} + b_{h,\alpha} s).$$

It thus suffices to show that for any reals $\{a_i\}_{i=1}^k$ and nonnegative nondecreasing reals $\{t_i\}_{i=1}^k$, $k \in \mathbb{N}$, the random variable $\sum_{i=1}^k a_i Y_{t_i}^h$ converges in law to $\sum_{i=1}^k a_i L_{t_i}^{H,\alpha}$, as $h \to 0$. Since

$$\sum_{i=1}^k a_i Y_{t_i}^h = \int_0^{t_k} \left(\sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right) h^{-1/\alpha} \mathrm{d}(X_{hs}^{TS} + b_{h,\alpha} s),$$

we have by Proposition 2.5 that

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}y \sum_{i=1}^k a_i Y_{t_i}^h}\right] = \exp\left[\int_0^{t_k} \int_{\mathbb{R}_0} h\psi_\alpha \left(yxh^{-1/\alpha} \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right) \rho(\mathrm{d}x)\mathrm{d}s \right], \tag{28}$$

where

$$\psi_\alpha(s) = \begin{cases} \Gamma(-\alpha)((1 - \mathrm{i}s)^\alpha - 1), & \text{if } \alpha \in (0, 1), \\ \frac{1}{2} \ln(1 + s^2) - s \tan^{-1} s, & \text{if } \alpha = 1, \\ \Gamma(-\alpha)((1 - \mathrm{i}s)^\alpha - 1 + \mathrm{i}\alpha s), & \text{if } \alpha \in (1, 2). \end{cases} \tag{29}$$

Note that ψ_1 is obtained by the symmetry of ρ. (See Proposition 2.8 of Rosiński [21].) We then want to show that as $h \to 0$, (28) tends to the characteristic function of the random variable $\sum_{i=1}^k a_i L_{t_i}^{H,\alpha}$.

The proof of Theorem 3.1(i) in [21] shows that for $\alpha \neq 1$,

$$\lim_{h\to 0} h\psi_\alpha(h^{-1/\alpha}s) = \varphi_\alpha(s),$$

where φ_α is given by (24) and

$$|h\psi_\alpha(h^{-1/\alpha}s)| \leq z_\alpha |s|^\alpha,$$

where z_α is some constant depending only on α ($\neq 1$). When $\alpha = 1$, we have

$$\lim_{h\to 0} h\psi_1(h^{-1}s) = -\frac{\pi}{2}|s|,$$

and the uniform boundedness (in $h > 0$) of $|h\psi_1(h^{-1}s)|$ holds true as

$$\begin{aligned}|h\psi_1(h^{-1}s)| &\leq |h \ln \sqrt{1+h^{-2}s^2}| + |s \tan^{-1}(h^{-1}s)| \\ &\leq |h\ln(1+h^{-1}|s|)| + \frac{\pi}{2}|s| \\ &\leq \left(1+\frac{\pi}{2}\right)|s|.\end{aligned}$$

Clearly, $\sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \in L^\alpha([0, t_k])$ since the $a_i K_{H,\alpha}(t_i, s)$ are in $L^\alpha([0, t_i])$. Together with the moment condition (2) on ρ, this indicates that the passage to the limit in (28) is justified. Hence,

$$\lim_{h\to 0} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}y\sum_{i=1}^k a_i Y_{t_i}^h}\right] = \exp\left[\int_0^{t_k}\int_{\mathbb{R}_0} \varphi_\alpha\left(yx\sum_{i=1}^k a_i K_{H,\alpha}(t_i, s)\right)\rho(\mathrm{d}x)\mathrm{d}s\right],$$

which is the characteristic function of $\sum_{i=1}^k a_i L_{t_i}^{H,\alpha}$.

(ii) We have that for each $h > 0$,

$$\mathrm{Cov}(h^{-G}L_{ht}^H, h^{-G}L_{hs}^H) = \frac{1}{2}\left(t^{2G}+s^{2G}-(t-s)^{2G}\right)\mathbb{E}[(X_1^{TS})^2], \quad s \in [0, t].$$

Hence, for the convergence of the finite dimensional distributions, we only need to show that the marginal law, at any time, of $\{h^{-G}L_{ht}^H : t \geq 0\}$ converges to a Gaussian law. Without loss of generality, let $t = 1$. By Lemma 2.3(i),

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{\mathrm{i}yh^{-G}L_h^H}] &= \exp\left[\int_0^h\int_{\mathbb{R}_0}\vartheta_\alpha(h^{-G}yxK_{H,\alpha}(h,s))\rho(\mathrm{d}x)\mathrm{d}s\right] \\ &= \exp\left[\int_0^1\int_{\mathbb{R}_0}\vartheta_\alpha(h^{-1/2}yxK_{H,\alpha}(1,s))\rho(\mathrm{d}x)\mathrm{d}s\right],\end{aligned}$$

where $\vartheta_\alpha(u) = \int_0^\infty(\mathrm{e}^{\mathrm{i}us}-1-\mathrm{i}us)s^{-\alpha-1}\mathrm{e}^{-s}\mathrm{d}s$. As in the proof of Theorem 3.1(ii) in [21], it follows that

$$|\vartheta_\alpha(yxK_{H,\alpha}(1,s))| \leq (yx)^2\Gamma(2-\alpha)\int_0^1 K_{H,\alpha}(1,s)^2\mathrm{d}s,$$

which justifies the passage to the limit below

$$\lim_{h\to\infty}\mathbb{E}[\mathrm{e}^{\mathrm{i}yh^{-G}L_h^H}] = \exp\left[-\frac{y^2}{2}\int_0^1 K_{H,\alpha}(1,s)^2\mathrm{d}s\,\Gamma(2-\alpha)\int_{\mathbb{R}_0}x^2\rho(\mathrm{d}x)\right].$$

This shows the convergence to a Gaussian law, which concludes the first part of the proof of the theorem. □

To prove (iii), let us present a technical lemma.

Lemma 6.5. *Let $\{X_t^n : t \geq 0\}_{n\in\mathbb{N}}$ and $\{Y_t^n : t \geq 0\}_{n\in\mathbb{N}}$ be sequences of stochastic processes in $C([0,\infty),\mathbb{R})$, let the sequence $\{X_t^n : t \geq 0\}_{n\in\mathbb{N}}$ be tight and for each $n \in \mathbb{N}$, let there exist a version $\widetilde{Y}^n$ of Y^n, defined on the same probability space as X^n, such that the sequence $\{\widetilde{Y}_t^n - X_t^n : t \geq 0\}$ converges ucp to zero. Then, the sequence $\{Y_t^n : t \geq 0\}_{n\in\mathbb{N}}$ is tight.*

Proof. For each compact set $K \subset [0,\infty)$ and for each $\delta > 0$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t,s\in K,|t-s|\leq\delta}|Y_t^n - Y_s^n| \wedge 1\right] \\
&\leq \mathbb{E}\left[\sup_{t,s\in K,|t-s|\leq\delta}|\widetilde{Y}_t^n - X_t^n| \wedge 1\right] + \mathbb{E}\left[\sup_{t,s\in K,|t-s|\leq\delta}|X_t^n - X_s^n| \wedge 1\right] \\
&\quad + \mathbb{E}\left[\sup_{t,s\in K,|t-s|\leq\delta}|X_s^n - \widetilde{Y}_s^n| \wedge 1\right] \\
&= 2\,\mathbb{E}\left[\sup_{t\in K}|\widetilde{Y}_t^n - X_t^n| \wedge 1\right] + \mathbb{E}\left[\sup_{t,s\in K,|t-s|\leq\delta}|X_t^n - X_s^n| \wedge 1\right]. \qquad (30)
\end{aligned}$$

The first term in (30) tends to zero as $n \to \infty$, since $\widetilde{Y}^n - X^n \xrightarrow{ucp} 0$. The second term also tends to zero as $n \to \infty$ and $\delta \to 0$ by the tightness of $\{X_t^n : t \geq 0\}_{n\in\mathbb{N}}$ in $C([0,\infty),\mathbb{R})$. The claimed result then follows. □

Proof of Theorem 6.4(iii). Using, for example, Lemma 16.2, Theorems 16.3 and 16.5 of Kallenberg [11], it suffices to show the tightness of the sequences $\{h^{-H}(L_{ht}^H + h^{H-1/\alpha}b_{h,\alpha}k_t) : t \geq 0\}$ (as $h \downarrow 0$) and $\{h^{-G}L_{ht}^H : t \geq 0\}$ (as $h \uparrow \infty$) in $C([0,\infty),\mathbb{R})$.

We begin with the short time behavior case. By Lemma 6.2(iii) and (iv), for each $p \in (0,\alpha)$,

$$\mathbb{E}[|h^{-H}L_{ht}^{H,\alpha} - h^{-H}L_{hs}^{H,\alpha}|^p] = (t-s)^{pH}\mathbb{E}[|L_1^{H,\alpha}|^p], \quad s \in [0,t]. \qquad (31)$$

By Corollary 16.9 of Kallenberg [11], the uniform boundedness in h seen in (31) implies the tightness of the sequence $\{h^{-H}L_{ht}^{H,\alpha} : t \geq 0\}$ in $C([0,\infty),\mathbb{R})$. Hence, by Lemma 6.5, it is enough to find $h^{-H}L_{h\cdot}^{H,\alpha}$ and $h^{-H}(L_{h\cdot}^H + h^{H-1/\alpha}b_{h,\alpha}k_\cdot)$ in $C([0,T],\mathbb{R})$, $T > 0$, defined on a common probability space and such that $h^{-H}L_{h\cdot}^{H,\alpha} - h^{-H}(L_{h\cdot}^H + h^{H-1/\alpha}b_{h,\alpha}k_\cdot) \xrightarrow{ucp} 0$. To this end, we make use of their series representations. First, let $\alpha \in (0,1)$, or let $\alpha \in [1,2)$ with a symmetric ρ. With the notation of Section 4, it follows from Proposition 4.1 and Lemma 6.2(vii) that the stochastic processes

$$h^{-H}\widetilde{L}_{h\cdot}^{H,\alpha} := h^{-H}\sum_{i=1}^{\infty} m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\frac{V_i}{|V_i|}K_{H,\alpha}(h\cdot, hT_i),$$

and

$$\begin{aligned}
h^{-H}(\widetilde{L}_{h\cdot}^H + h^{H-1/\alpha}b_{h,\alpha}k_\cdot) &:= h^{-H}\sum_{i=1}^{\infty}\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_iU_i^{1/\alpha}|V_i|\right) \\
&\quad\times\frac{V_i}{|V_i|}K_{H,\alpha}(h\cdot, hT_i),
\end{aligned}$$

respectively converge, almost surely uniformly on $[0, T]$, to versions of fSm $L^H_\cdot$ and of fTSm $h^{-H}(L^{H,\alpha}_{h\cdot} + h^{H-1/\alpha} b_{h,\alpha} k_\cdot)$, defined on a common probability space by using the common random sequences $\{\Gamma_i\}_{i\ge1}, \{T_i\}_{i\ge1}, \{V_i\}_{i\ge1}, \{E_i\}_{i\ge1}$ and $\{U_i\}_{i\ge1}$. Then, in view of Lemma 2.3(i),

$$h^{-H}\widetilde{L}^{H,\alpha}_{h\cdot} - h^{-H}(\widetilde{L}^{H}_{h\cdot} + h^{H-1/\alpha} b_{h,\alpha} k_\cdot) = \sum_{i=1}^{\infty}\left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} - h^{-1/\alpha}E_i U_i^{1/\alpha}|V_i|\right)\vee 0\right]\frac{V_i}{|V_i|}K_{H,\alpha}(\cdot, T_i), \tag{32}$$

which clearly converges ucp to zero as $h \to 0$. For $\alpha \in (1, 2)$ and ρ asymmetric, similar arguments with a careful treatment of the shift component yield (32). Hence, the sequence $\{h^{-H}(L^H_{ht} + h^{H-1/\alpha} b_{h,\alpha} k_t) : t \ge 0\}$ is tight in $C([0,\infty), \mathbb{R})$, which concludes the short time behavior case.

For the long time behavior case, Corollary 3.2 gives for $h > 0$,

$$\mathbb{E}[(h^{-G}L^H_{ht} - h^{-G}L^H_{hs})^2] = (t-s)^{2G}\mathbb{E}[(X_1^{TS})^2]. \tag{33}$$

Again by Corollary 16.9 of Kallenberg [11], the uniform boundedness in h seen in (33) implies the tightness of the sequence $\{h^{-G}L^H_{ht} : t \ge 0\}$ in $C([0,\infty), \mathbb{R})$, which completes the proof. □

Remark 6.6. The short time behavior result can also be seen from the series representation. For simplicity, consider the symmetric case. With the notation of Proposition 4.1, we have by Lemma 2.3(i)

$$\begin{aligned} h^{-H}L^H_{ht} &\stackrel{\mathcal{L}}{=} h^{-H}\sum_{i=1}^{\infty}\left(m(\rho)\left(\frac{\alpha\Gamma_i}{hT}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}K_{H,\alpha}(ht, hT_i) \\ &= \sum_{i=1}^{\infty}\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|h^{-1/\alpha}\right)\frac{V_i}{|V_i|}K_{H,\alpha}(t, T_i) \\ &\to \sum_{i=1}^{\infty} m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\frac{V_i}{|V_i|}K_{H,\alpha}(t, T_i) \quad \text{a.s., as } h \to 0. \end{aligned} \tag{34}$$

Remark 6.7. Let us further compare the results of this section to those obtained for MAFLM in [3]. As far as the long time Gaussian behavior is concerned, above, no specific structure of the Lévy measure is needed besides the square integrability condition. Our methods will thus also give convergence of the finite dimensional distributions of MAFLM. For an appropriate range of the parameter, convergence in $C([0, T], \mathbb{R})$ can further be proved by our methods since MAFLM possesses the same covariance structure as fTSm. This will thus extend the results of [3] to that stronger type of convergence (note that the scaling parameters for MAFLM and fTSm have different orders). To obtain the short time convergence of the finite dimensional distributions, all we need is the asymptotic stable structure of the Lévy measure near the origin. Indeed, in our case, the tempered stable Lévy measure certainly satisfies this, and so does the truncated stable one for MAFLM, discussed in Proposition 4.1 of [3]. So, again, our methods will give (with different scaling orders) the short time behavior of MAFLM while [3] is rather concerned with the short time behavior of the increments of MAFLM. As for convergence in $C([0, T], \mathbb{R})$, the situation is less clear and our approach is not applicable to MAFLM: the series representation of MAFLM is not available since its Lévy integrator is defined on the semi-infinite horizon.

We have seen in Lemma 6.2 that the marginals of fSm are α-stable and as a result their covariance does not exist. An alternative notion used in such situations is the one of *covariation*. The covariation of two jointly symmetric α-stable random variables X and Y, with $\alpha > 1$, is given by

$$\tau(X, Y) := \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha - \|X - Y\|_\alpha^\alpha, \tag{35}$$

where the norm $\|\cdot\|_\alpha$ gives the scale parameter, i.e. for $Z \sim S_\alpha(\sigma, 0, 0)$, $\|Z\|_\alpha = \sigma$. More generally, one can also define the *codifference* of any two jointly infinitely divisible random variables X and Y by

$$I(\theta_1, \theta_2; X, Y) := -\ln \mathbb{E}[e^{i(\theta_1 X + \theta_2 Y)}] + \ln \mathbb{E}[e^{i\theta_1 X}] + \ln \mathbb{E}[e^{i\theta_2 Y}], \tag{36}$$

for $\theta_1, \theta_2 \in \mathbb{R}$ (see [23] for more details on the covariation and the codifference and, in particular, the fact that (35) is a special case of (36)). Let $\{L_t^{H,\alpha} : t \geq 0\}$ be fSm associated with $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ where, moreover, ρ is assumed to be *symmetric*. Then,

$$I(1, -1; L_t^{H,\alpha}, L_s^{H,\alpha}) = C(t^{\alpha H} + s^{\alpha H} - (t - s)^{\alpha H}), \quad s \in [0, t], \tag{37}$$

for some constant C. In the Gaussian case, the codifference coincides with the covariance. For example, for a standard fBm $\{B_t^G : t \geq 0\}$,

$$\tau(B_t^G, B_s^G) = \frac{1}{2}(t^{2G} + s^{2G} - (t - s)^{2G}) = \mathrm{Cov}(B_t^G, B_s^G), \quad s \in [0, t],$$

and

$$\tau(B_{t+1}^G - B_t^G, B_1^G - B_0^G) = \mathrm{Cov}(B_{t+1}^G - B_t^G, B_1^G - B_0^G) \sim C t^{2(G-1)},$$

as $t \to \infty$.

Interestingly enough, as shown below, the codifference of the increments of fTSm has the same order of decay as the covariance of its increments as given in Proposition 3.4.

Proposition 6.8. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$. Then,*

$$I(\theta_1, \theta_2; L_{t+1}^H - L_t^H, L_1^H - L_0^H) \sim C(\theta_1, \theta_2) t^{2(G-1)},$$

as $t \to \infty$, where

$$C(\theta_1, \theta_2) = \frac{-i c_{H,\alpha} \theta_1 \pi}{\Gamma(\alpha) \sin(\pi\alpha)} \int_{[0,1]\times\mathbb{R}_0} ((1 - ix\theta_2 K_H(1, s))^{\alpha-1} - 1) x s^{1/\alpha - H} ds \rho(dx).$$

Proof. Observe that

$$\begin{aligned} &I(\theta_1, \theta_2; L_{t+1}^H - L_t^H, L_1^H - L_0^H) \\ &\quad = \Gamma(-\alpha) \int_{\mathbb{R}\times\mathbb{R}_0} (-(1 - ix(\theta_1(K_{H,\alpha}(t+1, s) - K_{H,\alpha}(t, s)) + \theta_2 K_{H,\alpha}(1, s)))^\alpha \\ &\qquad + (1 - ix\theta_1(K_{H,\alpha}(t+1, s) - K_{H,\alpha}(t, s)))^\alpha + (1 - ix\theta_2 K_{H,\alpha}(1, s))^\alpha - 1) ds \rho(dx), \end{aligned}$$

and that

$$K_{H,\alpha}(t+1, s) - K_{H,\alpha}(t, s) \sim c_{H,\alpha} s^{1/\alpha - H} t^{2(G-1)},$$

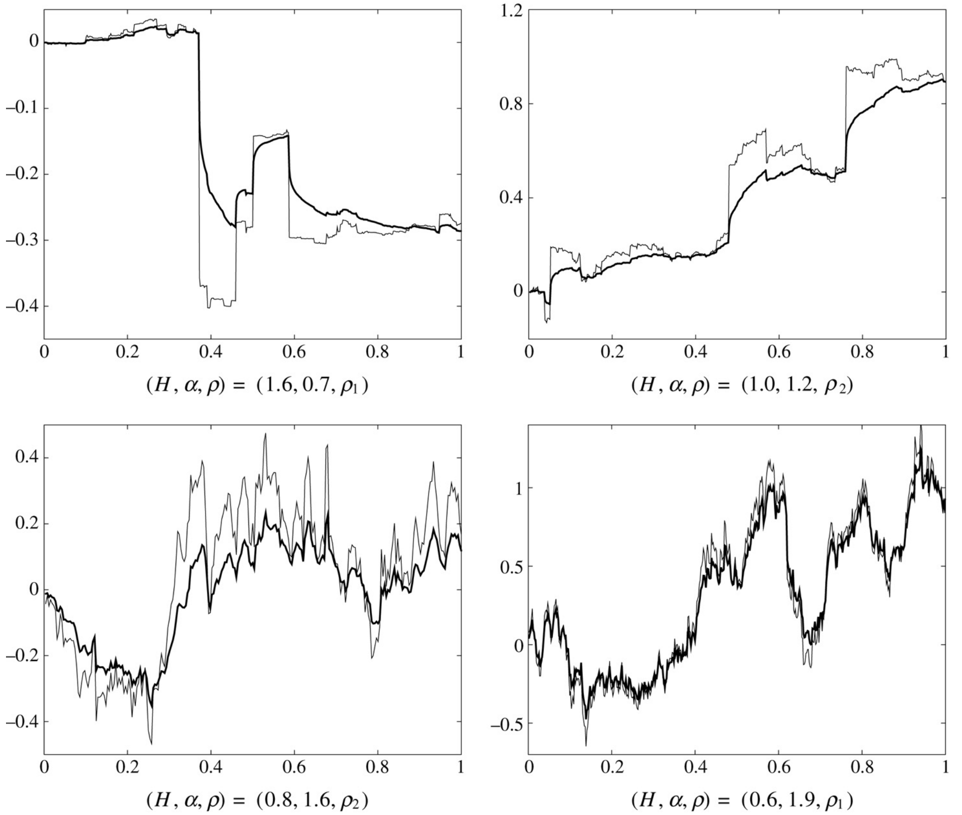

Fig. 1. Typical sample paths of fTSm (thick line) and of its background driving TS process (thin line) generated via the series representation.

as $t \to \infty$. Hence, for each $s > 0$,

$$\begin{aligned}
&-(1 - \mathrm{i}x(\theta_1(K_{H,\alpha}(t+1,s) - K_{H,\alpha}(t,s)) + \theta_2 K_{H,\alpha}(1,s)))^{\alpha} \\
&\quad + (1 - \mathrm{i}x\theta_1(K_{H,\alpha}(t+1,s) - K_{H,\alpha}(t,s)))^{\alpha} + (1 - \mathrm{i}x\theta_2 K_{H,\alpha}(1,s))^{\alpha} - 1 \\
&\sim \mathrm{i}\alpha x \theta_1 c_{H,\alpha} s^{1/\alpha - H}((1 - \mathrm{i}x\theta_2 K_{H,\alpha}(1,s))^{\alpha-1} - 1)t^{2(G-1)},
\end{aligned}$$

as $t \to \infty$. The result then holds since $\Gamma(-\alpha) = \frac{-\pi}{\alpha\Gamma(\alpha)\sin(\pi\alpha)}$. □

7. Concluding remarks

• Willinger et al. [25] assert that a numerical analysis of stock price time series indicates long-range dependence with marginal tails heavier than Gaussian but lighter than stable. Moreover, it is known that in shorter time, the asset price paths tend to lack higher moments, while they have a Gaussian behavior in long time. Indeed, fTSm achieves all these properties.

In Fig. 1, we give typical sample paths of fTSm and of its background driving tempered stable process, generated via the series representation presented in Proposition 4.1. We put the inner measures ρ_1 and ρ_2 as $\rho_1(\mathrm{d}x) = \delta_{-1.0}(\mathrm{d}x) + \delta_{1.0}(\mathrm{d}x)$ and $\rho_2(\mathrm{d}x) = 0.5^{-\alpha}\delta_{-0.5}(\mathrm{d}x) + \delta_{1.0}(\mathrm{d}x)$.

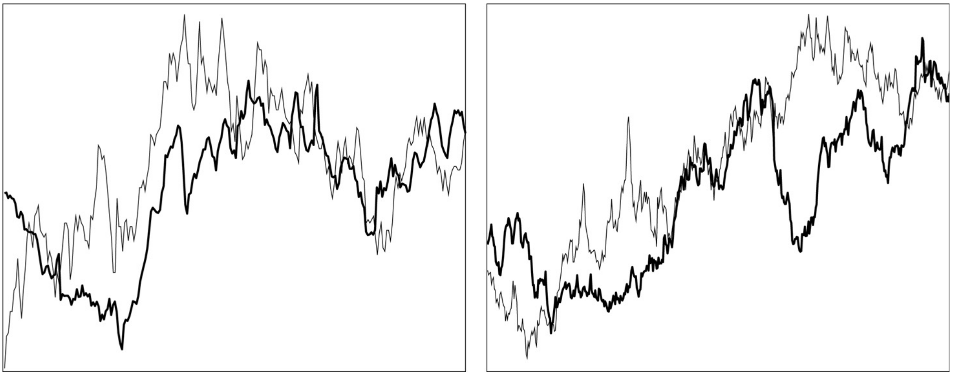

Fig. 2. fTSm $(H, \alpha, \rho) = (0.8, 1.6, \rho_2)$ (left thick) and $(0.6, 1.9, \rho_1)$ (right thick) with (scaled) daily time series of TOYOTA shares on the Tokyo Stock Exchange; 247 days (left thin) and 493 days (right thin) up to 11 February 2005.

(Tempered stable Lévy processes whose inner measure is discrete as above are studied in Carr et al. [7] with emphasis on financial application and called CGMY processes.) Observe that the sample paths of fTSm look like their background driving Lévy process as H is close to $1/\alpha$, while the dependency range gets longer and the paths of fTSm become less erratic as H becomes greater.

For the reader's convenience and for comparison, Fig. 2 shows daily time series of TOYOTA shares on the Tokyo Stock Exchange, together with fTSm as drawn in Fig. 1. It is observed that in short time the time series looks like fTSm with $(0.8, 1.6, \rho_2)$, while behaving in a Gaussian manner in longer time.

• Parameter estimation is important in practice. For the parameter H, the universal estimator introduced in Cohen and Istas [5] is directly applicable thanks to the local self-similarity property shown in Theorem 6.4. Our numerical experiments indicate that this estimator performs very well. On the other hand, it is unclear whether it is possible to estimate α directly from paths of fTSm. A reasonable way would be to estimate $G = H - 1/\alpha + 1/2$ by the well known aggregational variance method (see, for example, Taqqu et al. [24]), or by a simple but still very useful method introduced in Kettani and Gubner [12]. Both methods are based on the second-order self-similarity property. Again, our numerical experiments show that estimation via the method of [12] is fairly accurate. The inner measure ρ can also be estimated by specializing the results of Figueroa-Lopez and Houdré [9].

• To finish this study, let us mention that the long time behavior result provides yet another way to simulate the sample paths of fBm. For simplicity, consider a symmetric inner measure with a very simple structure, e.g. $\rho(\mathrm{d}x) = 2^{-1}(\delta_{-1}(\mathrm{d}x) + \delta_1(\mathrm{d}x))$, which reduces the random sequence $\{V_i\}_{i\ge1}$ to a sequence of iid Bernoulli random variables $\{\epsilon_i\}_{i\ge1}$. Observe that

$$h^{-G} L^H_{ht} \stackrel{\mathcal{L}}{=} \sum_{i=1}^{\infty} \left(m(\rho) \left(\frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} h^{1/\alpha - 1/2} \wedge E_i U_i^{1/\alpha} h^{-1/2} \right) \epsilon_i K_{H,\alpha}(t, T_i).$$

Clearly, for sufficiently large h, the above right hand side behaves like

$$h^{-1/2} \sum_{i=1}^{\infty} E_i U_i^{1/\alpha} \epsilon_i K_{H,\alpha}(t, T_i).$$

Theorem 6.4(ii) tells us that this stochastic process (on $[0, T]$) approximates fBm. In order for its second moments to be equal to those of a standard fBm, we set $h = \frac{2\alpha}{2+\alpha} N$ since then

$$\mathbb{E}\left[\left(h^{-1/2}\sum_{i=1}^{N} E_i U_i^{1/\alpha}\epsilon_i K_{H,\alpha}(t, T_i)\right)^2\right] = h^{-1} N \frac{2\alpha}{2+\alpha} t^{2G} = t^{2G}.$$

Therefore, for sufficiently large N, the stochastic process

$$\left\{\left(\frac{2\alpha}{2+\alpha}N\right)^{-1/2}\sum_{i=1}^{N} E_i U_i^{1/\alpha}\epsilon_i K_{H,\alpha}(t, T_i) : t \in [0, T]\right\}$$

can be used for the simulation of standard fBm.

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