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CLOSED INCOMPRESSIBLE SURFACES IN ALTERNATING KNOT AND LINK COMPLEMENTS

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§1. INTRODUCTION

Let $L \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a link which is alternating with respect to the projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$. We assume L has been isotoped to eliminate trivial crossings as in Fig. 1(a), and, more generally, crossings which decompose L as a (possibly trivial) connected sum $L = L' \# L''$ as in Fig. 1(b).

Our first result says that L can be split or non-prime link only in the obvious ways:

Theorem 1. (a) *If $\pi(L)$ is connected, then L is non-split, i.e. $S^3 - L$ is irreducible.*

(b) If L is non-split, then L is prime if for each disc $D \subset \mathbb{R}^2$ (the projection plane) with ∂D meeting $\pi(L)$ transversely in just two non-double points, $\pi(L) \cap D$ is an embedded arc.

The next special property we prove for alternating links is:

Theorem 2 (The Meridian Lemma). *If L is a non-split prime alternating link, and if $S \subset S^3 - L$ is a closed incompressible surface, then S contains a circle which is isotopic in $S^3 - L$ to a meridian of L.*

Corollary 1. *A non-split prime alternating link is simple. That is, every incompressible torus in $S^3 - L$ is peripheral (isotopic to the boundary of a tubular neighborhood of a component of L).*

Corollary 2. *If L is a non-split prime alternating link which is not a torus link, then $S^3 - L$ has a complete hyperbolic structure (of finite volume).*

After Theorem 2, it is natural in the study of closed incompressible surfaces $S \subset S^3 - L$ (for L alternating) to consider the operation of meridian surgery, indicated in Fig. 2. Such meridian surgery always preserves incompressibility, as one can easily verify. After finitely many meridian surgeries, S becomes "pairwise incompressible," in the following sense:

Definition. Let $S \subset S^3 - L$ be a properly embedded surface. Then S is called pairwise incompressible if for each disc $D \subset S^3$ meeting L transversely in one point, with $D \cap S = \partial D$, there is a disc $D' \subset S \cup L$ meeting L transversely in one point, with $\partial D' = \partial D$.

With this definition, Theorem 2 can be rephrased to say that there is no closed incompressible, pairwise incompressible surface in $S^3 - L$.

(a) (b)

Fig. 1.

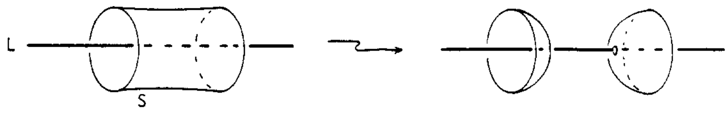

Fig. 2.

The study of closed incompressible surfaces $S \subset S^3 - L$ thus breaks into two parts: (1) analyzing the incompressible, pairwise incompressible surfaces in $S^3 - L$ having all their boundary components meridians of L, and (2) understanding when the "peripheral tubing" operation, inverse to meridian surgery, preserves incompressibility. Part (2) is dealt with in [2]. Concerning (1) we have:

THEOREM 3. *Let L be a non-split prime alternating link and suppose $S \subset S^3 - L$ is an incompressible, pairwise incompressible surface having $n > 0$ boundary components, all of which are meridians of L (hence n must be even). Then*:

(a) *If $n = 2$, S is an annulus, necessarily peripheral since L is prime.*

(b) *If $n = 4$ or 6, S has genus zero.*

(c) *For fixed n, there are only finitely many such surfaces S, up to isotopy.*

In the course of proving (b), it will be seen that all incompressible 4-punctured spheres $S \subset S^3 - L$, with ∂S consisting of meridians, can be recognized fairly easily in the projection $\pi(L)$.

By (a) of Theorem 3, to make a closed incompressible surface $S \subset S^3 - L$ of genus ≥ 2 pairwise incompressible at least two meridian surgeries are necessary. So the following theorem is relevant for alternating knots.

THEOREM 4. *Let $K \subset S^3$ be any knot, and let $S \subset S^3 - K$ be a closed incompressible surface such that there exist disjoint discs D_1, $D_2 \subset S^3$ satisfying*:

(i) *D_i intersects K transversely in one point*

(ii) *$D_i \cap S = \partial D_i$*

(iii) *∂D_1 is not isotopic to ∂D_2 on S.*

Then S remains incompressible in any closed manifold obtained by a non-trivial Dehn surgery on K.

Most of the results in this paper are restatements and amplifications of results that were contained in my thesis [1]. It was pointed out to me by Allen Hatcher that there is a gap in the proof of Theorem 4.1 in [1]. As of this writing this error has not been rectified.

I wish to thank Allen Hatcher for the considerable contribution he has made to the writing of this paper.

§2. PROOFS

Let $S^2 = \mathbb{R}^2 \cup \{\infty\}$. We position L so that it lies on S^2 except near crossings of L, where L lies on a "bubble" as shown in Fig. 3.

Let $S_+^2(S_-^2)$ be S^2 with each disc of S^2 inside a bubble replaced by the upper (lower) hemisphere of that bubble. Let B_+^3 be the ball in S^3 bounded by S_+^2 and lying above S_+^2, and let B_-^3 be the ball below S_-^2 with $\partial B_-^3 = S_-^2$. (We will use the notation $S_\pm^3$ to mean S_+^2 or S_-^2 and similarly for other symbols with subscript $\pm$.)

Let $S \subset S^3 - L$ be a surface whose boundary curves are all meridians of L which do not intersect the bubbles. We may isotope S to meet each ball bounded by a bubble in

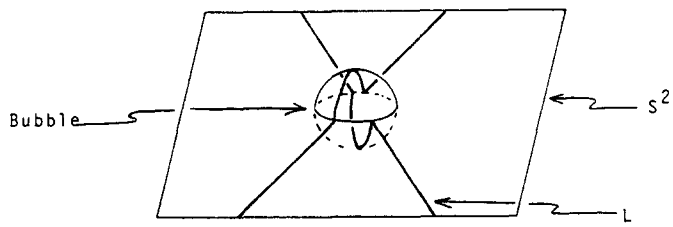

Fig. 3.

saddle-shaped discs, as in Fig. 4, by first isotoping S to be transverse to the polar axis of each bubble, then push S outward from that axis. We may suppose S meets S_+^2 and S_-^2 transversely. To each component C of $S \cap S_\pm^2$ can be associated a cyclic word $w_\pm(C)$ in the letters P (= puncture) and S (= saddle), which records, in order, the intersections of C with L and with the bubbles, respectively. (Strictly, $w_\pm(C)$ depends on an orientation for C.) $w_\pm(C)$ must have even length, and the number of boundary components of S equals the total number of P's in all the $w_\pm(C)$'s (or in all the $w_-(C)$'s).

We consider one of the following situations.

(1) $\pi(L)$ is connected and S is a 2-sphere not bounding a ball in $S^3 - L$.

(2) $S^3 - L$ is irreducible, and S is an annulus giving a non-trivial connected sum decomposition $L = L_1 \# L_2$.

(3) $S^3 - L$ is irreducible, L is prime, and S is incompressible and pairwise incompressible.

Lemma 1. *In each of the cases* (1–3), *the surface S can be replaced by another surface S′ of the same type* (*isotopic to S in case* (3)) *such that*:

(i) *no word* $w_\pm(C)$ *associated to S′ is empty*

(ii) *no loop of* $S' \cap S_\pm^2$ *meets a bubble in more than one arc.*

To prove (i), let C be an innermost loop of $S \cap S_+^2$ with $w_+(C)$ empty, with C' a nearby isotopic circle on $S \cap \mathring{B}_+^3$. C' bounds a disc $D \subset \mathring{B}_+^3$ with $D \cap S = \partial D = C'$. In case (1), surgering S along D yields two 2-spheres, at least one of which does not bound a ball in $S^3 - L$, and we replace S by this 2-sphere. In cases (2) and (3), C' also bounds a disc D' on S, and the 2-sphere $D \cup D'$ bounds a ball in $S^3 - L$, so we isotope S by isotoping D' to D, rel C'. This isotopy eliminates any loops of $S \cap S_+^2$ in D'; in particular, it eliminates C if $C \subset D'$. In all three cases (1–3) we call the new surface S again, and then perform the same procedure on a loop of $S \cap S_+^2 - C$ which is innermost among the trivial loops of $S \cap S_+^2 - C$; and so on, when all loops of $S \cap S_+^2$ have been considered, we operate in the same way on the loops of $S \cap S_-^2$. After this has been done, (i) holds. Also, every loop of $S \cap S_\pm^2$ bounds a disc in $B_\pm^2$.

For (ii), suppose some component C of $S \cap S_+^2$ ($S \cap S_-^2$ is treated similarly) meets the

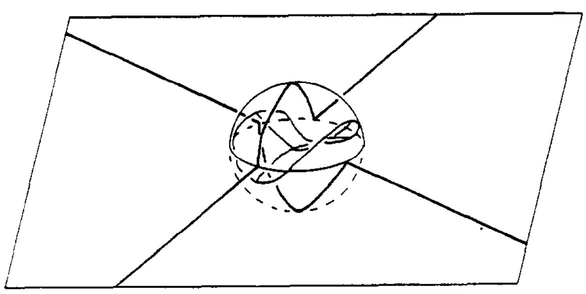

Fig. 4.

upper hemisphere H of a bubble in two or more arcs. Let $D \subset S_+^2$ be a disc bounded by C, chosen so that $D \cap H$ contains a rectangle R whose boundary consists of two arcs of C and two arcs of ∂H. Replacing C if necessary by another component of $S \cap S_-^2$ inside D, we may assume $S \cap \text{int}(R) = \emptyset$. We now have two possibilities, according to whether R meets L or not; see Fig. 5. C bounds a disc $D \subset S \cap B_+^3$. Let $B \subset D$ be a band joining the two arcs of $C \cap R$. If $R \cap L \neq \emptyset$ (Fig. 5a), these two arcs belong to the same saddle σ, so $B \cup \sigma$ contains a circle isotopic in $S^3 - L$ to a meridian of L. This is impossible in case (1). In case (2), S can be meridionally surgered along the core circle of $B \cup \sigma$ into two annuli, each with fewer saddles than S, and at least one of these two annuli must be of type (2). In case (3), since S is pairwise incompressible, it can be isotoped to eliminate the saddle σ.

If $R \cap L = \emptyset$, let $D' \subset B_+^3$ be a disc with $\partial D'$ consisting of an arc of B and an arc of R. In all cases (1)–(3) we may assume $D' \cap S \subset \partial D'$. Then we may use D' to isotope S so as to eliminate the two saddles of S containing the two arcs of $R \cap B$.□

If S satisfies (i) and (ii) of Lemma 1, we say S is in *standard position*. Notice that we have not used the hypothesis that L is alternating. If L is alternating, it is easy to see that $S \cap S_\pm^2$ has the following alternating property:

If B_1 and B_2 are two bubbles crossed in succession by a loop C of $S \cap S_\pm^2$, then:

(*) (i) If the two arcs of $L \cap S_\pm^2$ in B_1 and B_2 lie on opposite sides of C, then C crosses L (at punctures) an even number of times between crossing B_1 and B_2 (Fig. 6a).

(ii) If the two arcs of $L \cap S_\pm^2$ in B_1 and B_2 lie on the same side of C, then C crosses L and an odd number of times between crossing B_1 and B_2 (Fig. 6b).

Proof of Theorem 2. If this were false, $S \subset S^3 - L$ would be a closed incompressible, pairwise incompressible surface, which would be assumed to be standard position. Let C be an innermost circle of $S \cap S_+^2$, bounding a disc $D \subset S_+^2$ with $D \cap S = \partial D = C$. Since S is closed, $w_+(C) = S^i$ with $i \geq 2$. In two successive bubbles which C crosses, the arcs of L must lie on opposite sides of C, by the alternating property (*). One of these two arcs of L must then lie inside D, so we are in the situation. of Fig. 5(a), where C crosses the

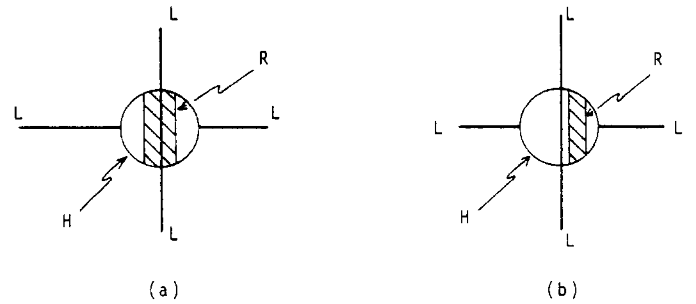

Fig. 5.

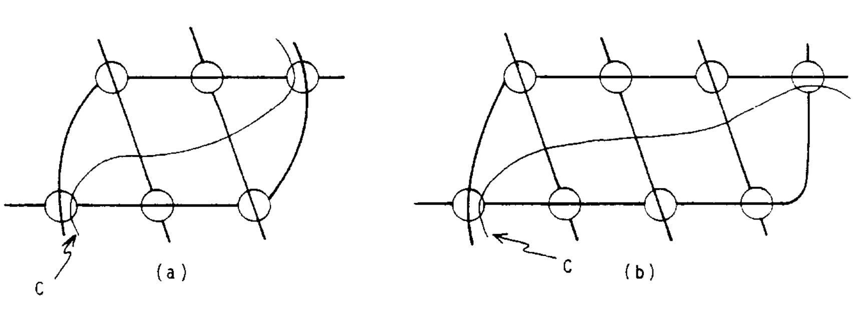

Fig. 6.

same bubble twice, contradicting (ii) of Lemma 1. Thus we have $S \cap S_+{}^2 = \emptyset$, hence also $S \cap S_-{}^2 = \emptyset$, which is impossible if S so incompressible.□

Proof of Theorem 1*a*. Let $S \subset C^3 - L$ be a 2-sphere not bounding a 3-ball in $S^3 - L$. We may assume S satisfies (i) and (ii) of Lemma 1. Then the same innermost circle argument applies as in the proof of Theorem 2 preceding.□

Proof of Corollary 1. Let $T \subset S^3 - L$ be an incompressible torus. By Theorem 2, T can be meridionally surgered to an annulus $A \subset S^3 - L$. Since L is prime, A is peripheral, hence also T.□

Proof of Corollary 2. This is a matter of checking that the hypotheses of Thurston's "Monster Theorem" [4] apply; namely, L must be non-trivial, non-split, simple, and $S^3 - L$ must contain no essential annuli. From [3], the latter condition means that L must not be a cable link. But the only simple cable links are torus links.□

LEMMA 2. *If L is alternating and $S \subset S^3 - L$ is a surface in standard position in either case* (2) *or* (3), *then no word $w_\pm(C)$ has the form $w_\pm(C) = P^iS^j$ with $j > 0$. Further, each word $w_\pm(C)$ contains at least two P's.*

Proof. Suppose $w_+(C) = P^iS^j, j > 0$, for some loop C of $S \cap S_+{}^2$. Let $\sigma_1, \ldots, \sigma_j$ be the saddles which C meets. Passing from $S_+{}^2$ to $S_-{}^2$, these saddles have the effect of surgering $S \cap S_+{}^2 j$ times along C. If $j = 1$, as in Fig. 7(a), C becomes in $S \cap S_-{}^2$ a loop C' which violates (ii) of Lemma 1. If $j > 1$, we look at either disc $D \subset S^2$ that $\pi(C)$ bounds. By (*), we have at least one σ_k with $\pi(\sigma_k) \subset D$. Then $\pi(S \cap S_-{}^2) \cap \text{int}(D)$ must contain some arcs with endpoints on ∂D. Let α be such an arc which is edgemost in D. The two possible configurations for α, depending on whether the endpoints of α lie in the same or different bubbles, are shown in Fig. 7(b) and (c). In either case, α gives a loop C' of $S \cap S^2_-$ which violates (ii) of Lemma 1.□

Proof of Theorem 1*b*. Let S be an annulus splitting L as a non-trivial connected sum. By Lemma 1, we can put S in standard position. Since S is an annulus, there are just two

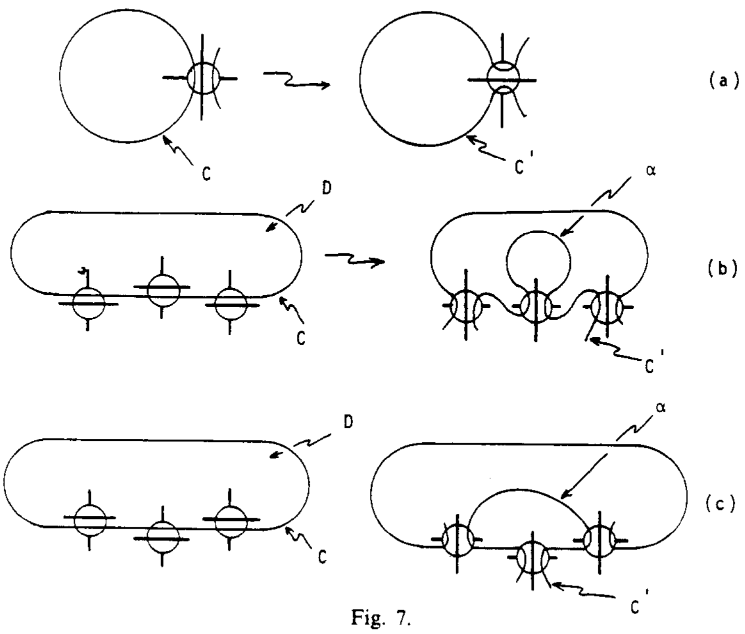

Fig. 7.

P's in all the words $w_+(C)$. By Lemma 2, $S \cap S_+^2$ then consists of a single circle C. If C crossed any bubbles, then a saddle which C met would also meet a loop C' of $S \cap S_+^2$, with $C' \neq C$ by (ii) of Lemma 1, a contradiction. So $w_-(C) = P^2$, and the result immediately follows.□

Proof of Theorem 3*a*. The same argument which proves Theorem 1b also proves this.□

LEMMA 3. *Let $S \subset S^3 - L$ satisfy*:

(i) *No word $w_+(C)$ associated to a loop C of $S \cap S_+^2$ is empty.*

(ii) *No loop of $S \cap S_+^2$ crosses the same bubble more than once.*

Then if S has n (meridian) boundary components, each word $w_+(C)$ has at most $n - 2$ S's.

Proof. Let C be an innermost loop of $S \cap S_+^2$ such that $w_+(C) \neq P^i$, and write $w_+(C) = P^{l_1}SP^{l_2}S \cdots P^{l_m}S$ with $l_i \geq 0$. By the alternating property (*), each l_i must be odd (in particular, $l_i > 0$), since C is innermost. Now let us change the system of loops of $S \cap S_+^2$ to a new system of loops in S_+^2 by the following steps (see Fig. 8):

(1) Delete C from $S \cap S_+^2$.

(2) At each saddle σ_i $(1 \leq i \leq m)$ which C meets, let C_i be the loop of $S^2 \cap S_+^2$ sharing a saddle with C at σ_i. Then deform C_i off the bubble containing σ_i, to obtain a curve C_i' which crosses L.

The resulting system of loops in S_+^2 is $S' \cap S_+^2$ for some surface $S' \subset S^3 - L$ which also satisfies (i) and (ii), clearly. The number of boundary components of S' is at most n, since we have added m boundary components in step 2 while removing $\sum_i l_i \geq m$ boundary components in step 1.

Now we prove the Lemma. Let C_0 be a given circle of $S \cap S_+^2$. We apply the operation $S \mapsto S'$ of the preceding paragraph repeatedly, using loops $C \neq C_0$, until there are no S's left in any words $w_+(C)$. In the end, C_0 has changed to a loop C'_0 with $w_+(C_0') = P^i$, $i \leq n$. At each replacement $S \mapsto S'$, the number of letters in $w_+(C_0)$ does not change. Initially, $w_+(C_0)$ contains at least two P's, by Lemma 2. So the number of S's in the original $w_+(C_0)$ is at most $i - 2 \leq n - 2$.□

Proof of Theorem 3*b*. When $n = 4$, there are at most two loops in $S \cap S_+^2$ since each $w_+(C)$ contains at least two P's, by Lemma 2. If $S \cap S_+^2$ has only one loop, its word must be P^4 (by the argument which proved Theorem 1b), making S a 4-punctured sphere. If $S \cap S_+^2$ has two loops C_1 and C_2, then $w_+(C_1)$ and $w_+(C_2)$ each contain two P's. If $w_+(C_1)$ and $w_+(C_2)$ are not both P^2 (which would make S the disjoint union of two annuli), then by Lemma 2, both $w_+(C_1)$ and $w_+(C_2)$ contain at least two S's. By Lemma 3, both words contain at most two S's. Thus $w_+(C_1) = w_+(C_2) = PSPS$. In this case $S \cup S_-^2$ must also have two loops C_1' and C_2', with the same word $PSPS$. Now we can compute the Euler characteristic of the surface $\hat{S}$ formed by capping with disks the boundary circles of S. It is $\chi(\hat{S}) = n_- - n_S + n_+$, where $n_\pm$ is the number of components of $S \cap S_\pm^2$ and n_S is the

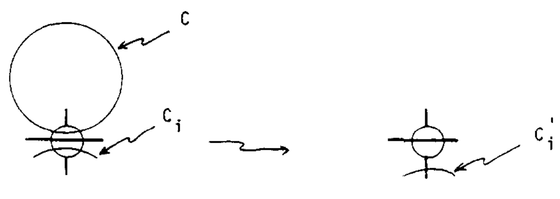

Fig. 8.

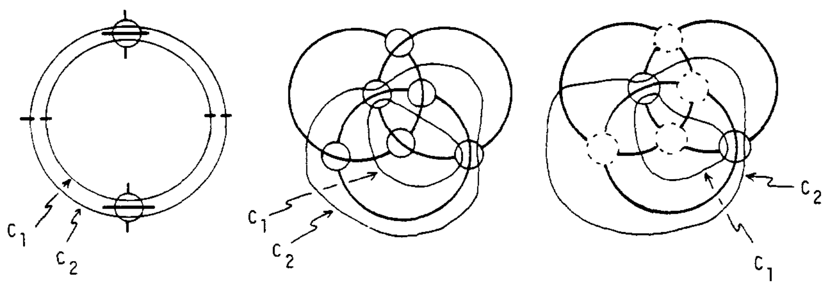

Fig. 9.

number of saddles of S, which is half the total number of S's in all words $w_+(C)$. In the case at hand, $\chi(\hat{S}) = 2 - 2 + 2 = 2$, so S has genus zero. Figure 9 below shows this pattern $S \cap S_+^2$. The simplest link for which this pattern arises is the Borromean rings; here S is a sphere which separates two components of L from each other, and is punctured 4 times by the third component of L. The general case of this configuration $w_+(C_1) = w_+(C_2) = PSPS$ is obtained from the Borromean rings case by replacing the four crossings of the Borromean rings which do not involve saddles of S by arbitrary 2-strand tangles.

When $n = 6$, the analysis is similar to the case $n = 4$, but there are at most three loops in $S \cap S_\pm^2$. We leave details to the reader.□

Proof of Theorem 3c. Each word $w_+(C)$ contains at least two P's by Lemma 2, so the number of loops in $S \cap S_+^2$ is at most $n/2$. Since each word $w_+(C)$ contains at most n P's and (by Lemma 3) at most $n - 2$ S's, the number of possible words $w_+(C)$ is bounded. Each word $w_+(C)$ can be realized by only finitely many loops C in S_+^2. So there are only finitely many possibilities for $S \cap S_+^2$, with fixed n, hence only finitely many possibilities for S (up to isotopy).□

Proof of Theorem 4. If a non-trivial Dehn surgery on K produced a 3-manifold in which S were compressible, then a compressing disc for S would yield a disc-with-punctures $D \subset S^3 - K$ with $D \cap S = \partial_0 D$, the "outer" boundary circle of D, and the "inner" boundary circles $\partial D - \partial_0 D$ non-meridian loops on (the boundary of a tubular neighborhood of) K. We make D transverse to $D_1 \cup D_2$ and then simplify the intersections of D with D_1 and D_2 in the usual way, until all components of $D \cap (D_1 \cup D_2)$ are arcs running from $\partial_0 D$ to $\partial D - \partial_0 D$. If the circles of $\partial D - \partial_0 D$ wrap around D q (>0) times longitudinally, then emanating from each circle of $\partial D - \partial_0 D$ there are q arcs of $D \cap D_1$ alternating with q arcs of $D \cap D_2$. There must be a rectangle $R \subset D$ with $R \cap [\partial D \cup (D \cap D_1) \cup (D \cap D_2)] = \partial R$, the four arcs of ∂R lying successively on $\partial_0 D$, $D \cap D_1$, $\partial D - \partial_0 D$, and $D \cap D_2$, as in Fig. 10(a). But then there is a compressing disc D'

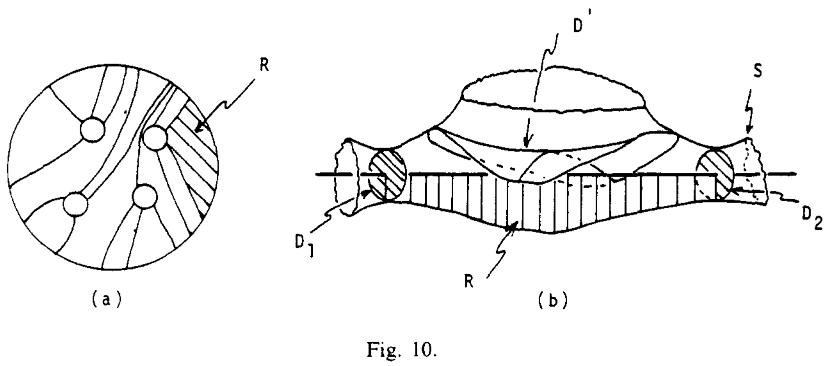

Fig. 10.

for S in $S^3 - K$, lying in a neighborhood of $R \cup D_1 \cup D_2$, as shown in Fig. 10(b). The boundary $\partial D'$ of this compressing disc is non-trivial in S since ∂D_1 and ∂D_2 are not parallel in S.□

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