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On the realization of Riemannian symmetric spaces in Lie groups ☆

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Abstract

In this paper we give a realization of some symmetric space G/K as a closed submanifold P of G. We also give several equivalent representations of the submanifold P. Some properties of the set $gK \cap P$ are also discussed, where gK is a coset space in G.

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1. Introduction

Suppose G is a connected Lie group, K a closed subgroup of G. Then G/K is a homogeneous space. When G, being considered as a principal K-bundle, is trivial, there is a global section S of this bundle. In this case, there is a natural isomorphism $G/K \cong S$. This happens when G is semisimple and K is a maximal compact subgroup of G, thanks to the Cartan decomposition. If G is not a trivial K-bundle, we seldom consider G/K as a submanifold of G. In this paper, we show that when G/K has the structure of Riemannian symmetric space where K is the closed subgroup of G consist of the fixed points

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of the involution of G defining the symmetric space, we can embed G/K in G as a closed submanifold in a good manner.

More precisely, let G be a connected Lie group, σ an involution of G. Let $K = \{g \in G \mid \sigma(g) = g\}$, then K is a closed subgroup of G. We suppose there exists a G-invariant Riemannian structure on G/K. Then G/K becomes a Riemannian (globally) symmetric space (see Helgason [2]). In this case, we call the triple (G, σ, K) a *Riemannian symmetric triple*. The differential $d\sigma$ of σ gives an involution of $\mathfrak{g}$, the Lie algebra of G. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the eigenspaces of $d\sigma$ in $\mathfrak{g}$ with eigenvalues 1 and -1, respectively. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. They satisfy the relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Note that $\mathfrak{k}$ is the Lie algebra of K. Note also that there exists a G-invariant Riemannian structure on G/K if and only if $\mathrm{Ad}(K)|_{\mathfrak{p}} = \{\mathrm{Ad}(k)|_{\mathfrak{p}} \in GL(\mathfrak{p}) \mid k \in K\}$ is compact.

In Section 2, we will prove that $P = \exp(\mathfrak{p})$ is a closed submanifold of G, and there is a natural isomorphism $G/K \cong P$. That is, G/K can be embedded as a closed submanifold of G. The most hard part of the proof is the closedness of P. We will deduce it by proving the fact that P coincides with the connected component R_0 containing e of the set $R = \{g \in G \mid \sigma(g) = g^{-1}\}$. In fact, we will give several equivalent representations of P, namely $P = Q = R_0 = R_0' = R^2$. This is our main Theorem 2.5 in Section 2. Then, as a corollary, we get the natural embedding of G/K in G. In the case that G is semisimple, the relation $P = R_0$ was mentioned in Hermann [3, Chapter 6], but the author did not give a proof there. It is interesting to notice that we can prove each connected component of R is a closed submanifold of G, but different components may have different dimensions.

Even if P is a closed submanifold of G, it is not a global section of the principal K-bundle $G \to G/K$ in general. But since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, P is a local section around $[e] = K$. Then it is naturally to ask that how far is P from being a global section. This will be discussed in Section 3. We will prove, among other things, that $gK \cap P \neq \emptyset$ for each coset space gK, and almost all coset space gK intersects P transversally.

2. Realization of symmetric spaces in Lie groups

We always suppose (G, σ, K) is a Riemannian symmetric triple as we have defined in Section 1 from now on. We construct the sets

$$P = \exp(\mathfrak{p}),$$

$$Q = \{g\sigma(g)^{-1} \mid g \in G\},$$

$$R = \{g \in G \mid \sigma(g) = g^{-1}\}.$$

Let R_0, R_0' be the connected component and the path component of R containing the identity e, respectively, and let $R^2 = \{g^2 \mid g \in R\}$. Let K_0 be the identity component of K.

Lemma 2.1. *The map $\Phi : P \times K_0 \to G$, $\Phi(p, k) = pk$ is surjective.*

Proof. $\forall g \in G$, we prove that there are $k \in K_0$ and $X \in \mathfrak{p}$ such that $g = e^X k$. Note that G/K_0 is also a symmetric space, which is a covering space of G/K. We denote $x_0 = [e] \in G/K_0$, and let $x_1 = gx_0$. Let $\gamma(t)$ be a geodesic in G/K_0 such that $\gamma(0) = x_0$ and $\gamma(1) =$

x_1. Then $\gamma(t)$ is of the form $\gamma(t) = e^{tX}x_0$ for some $X \in \mathfrak{p}$ (see Helgason [2, Chapter IV, Section 3)]). So $x_1 = e^X x_0$. Let $k = e^{-X}g$, we have $kx_0 = e^{-X}(gx_0) = e^{-X}x_1 = x_0$. So $k \in K_0$, and $g = e^X k$. □

Lemma 2.2. *For all $p, p' \in P$, we have $pp'p \in P$.*

Proof. Suppose $p' = e^X$, where $X \in \mathfrak{p}$. Let $p_1 = e^{X/2} \in P$, then $p' = p_1^2$. By Lemma 2.1, $pp_1 = p_2 k$ for some $p_2 \in P, k \in K_0$. Then we have $p^{-1}p_1^{-1} = \sigma(pp_1) = \sigma(p_2 k) = p_2^{-1}k$, this implies $p_1 p = k^{-1}p_2$. So $pp'p = (pp_1)(p_1 p) = (p_2 k)(k^{-1}p_2) = p_2^2 \in P$. □

A neighborhood U of 0 in $\mathfrak{g}$ is *symmetric* if $X \in U$ implies $-X \in U$.

Lemma 2.3. *Suppose U is a symmetric neighborhood of 0 in $\mathfrak{g}$ with $d\sigma(U) = U$ such that $\exp|_U : U \to \exp(U)$ is a diffeomorphism. If $g \in \exp(U)$ satisfies $\sigma(g) = g^{-1}$, then $g \in P$.*

Proof. Suppose $g = e^X$, $X \in U$. Applying the equation $\exp \circ d\sigma = \sigma \circ \exp$ to X, we have $\exp(d\sigma(X)) = \sigma(\exp(X)) = \exp(-X)$. Since U is symmetric and $d\sigma(U) = U$, $d\sigma(X), -X \in U$. But exp is injective on U, so we have $d\sigma(X) = -X$. This implies $X \in \mathfrak{p}$, so $g = e^X \in P$. □

Lemma 2.4. *Suppose U is a symmetric neighborhood of 0 in $\mathfrak{g}$ with $d\sigma(U) = U$ such that $\exp|_U$ is a diffeomorphism onto its image, and suppose $p \in P$. If $g \in p\exp(U)p$ satisfies $\sigma(g) = g^{-1}$. Then $g \in P$.*

Proof. Since $g \in p\exp(U)p$, $p^{-1}gp^{-1} \in \exp(U)$. But $\sigma(p^{-1}gp^{-1}) = pg^{-1}p = (p^{-1}gp^{-1})^{-1}$. By Lemma 2.3, $p^{-1}gp^{-1} \in P$. Then by Lemma 2.2, $g \in P$. □

Now we are prepared to formulate our main Theorem in this section.

Theorem 2.5. *Suppose (G, σ, K) is a Riemannian symmetric triple, and let the subsets P, Q, R, R_0, R_0', R^2 of G be as defined above. Then $P = Q = R_0 = R_0' = R^2$.*

Proof. We prove $P \subset R^2 \subset Q \subset R_0' \subset P$ and $R_0 = R_0'$.

(i) "$P \subset R^2$". Suppose $g \in P$, then $g = e^X$ for some $X \in \mathfrak{p}$. But $\sigma(e^{X/2}) = e^{d\sigma(X/2)} = e^{-X/2}$, so $e^{X/2} \in R$, and then $g = (e^{X/2})^2 \in R^2$.

(ii) "$R^2 \subset Q$". Suppose $g \in R^2$, then $g = h^2$, $\sigma(h) = h^{-1}$. Now $h\sigma(h)^{-1} = h^2 = g$, so $g \in Q$.

(iii) "$Q \subset R_0'$". For $g\sigma(g)^{-1} \in Q$, $\sigma(g\sigma(g)^{-1}) = \sigma(g)g^{-1} = (g\sigma(g)^{-1})^{-1}$, so $Q \subset R$. But Q is path connected and containing e, so $Q \subset R_0'$.

(iv) "$R_0' \subset P$". We first suppose that G is simply connected. Let $g : [0,1] \to R_0'$ be a continuous path in R_0' with $g(0) = e$, it suffices to prove $g(1) \in P$. Let $S = \{t \in [0,1] \mid g(t) \in P\}$. Since $g(0) \in P$, $0 \in S$, so $S \neq \emptyset$. We will prove that S is open and closed. Then by the connectedness of $[0,1]$, $S = [0,1]$, and then we will have $g(1) \in P$.

For the openness of S, suppose $t_0 \in S$, that is $g(t_0) = e^X$ for some $X \in \mathfrak{p}$. Let $p = e^{X/2}$, then $g(t_0) = p^2$. Let U be a symmetric neighborhood of 0 in $\mathfrak{g}$ with $d\sigma(U) = U$ such that

$\exp|_U$ is a diffeomorphism onto its image. Then $p\exp(U)p$ is a neighborhood of $g(t_0)$. So there is an open neighborhood (t_1,t_2) of t_0 such that $g(t)\in p\exp(U)p, \forall t\in(t_1,t_2)$. By Lemma 2.4, $g(t)\in P$, that is $t\in S$, $\forall t\in(t_1,t_2)$. This proves the openness.

To prove that S is closed, we endow a left invariant Riemannian structure on G. This induces a left invariant metric $d(\cdot,\cdot)$ on G. Since G is simply connected, there is an $\mathrm{Ad}(G)$ invariant, $d\sigma$-invariant symmetric neighborhood V of 0 in $\mathfrak{g}$ such that $\exp|_V$ is a diffeomorphism onto its image (see Varadarajan [5, Theorem 2.14.6]). Then $g\exp(V)g^{-1}=\exp(V), \forall g\in G$. Let $r>0$ such that $B_r(e)=\{g\in G \mid d(e,g)<r\}\subset\exp(V)$. Suppose $\{t_n\}_{n\in\mathbb{N}}\subset S$ is a sequence such that $\lim_{n\to\infty}t_n=t_0$, we prove $t_0\in S$. Choose $N\in\mathbb{N}$ such that $d(g(t_N),g(t_0))<r$. Since $g(t_N)\in P$, $g(t_N)=p^2$ for some $p\in P$. So $g(t_0)\in g(t_N)B_r(e)=p^2B_r(e)\subset p^2\exp(V)=p^2(p^{-1}\exp(V)p)=p\exp(V)p$. By Lemma 2.4, $g(t_0)\in P$. Hence S is closed. This conclude the proof when G is simply connected.

For general G, let $\widetilde{G}$ be its universal covering group with covering map $\pi:\widetilde{G}\to G$. Let the corresponding involution of $\widetilde{G}$ is $\tilde{\sigma}$, and let $\widetilde{R}'_0$, $\widetilde{P}$ be the corresponding subsets of $\widetilde{G}$. We claim that $R'_0\subset\pi(\widetilde{R}'_0)$. In fact, suppose $g\in R'_0$. Then there is a continuous path $g(t)$ $(t\in[0,1])$ in R'_0 such that $g(0)=e$ and $g(1)=g$. Let $\tilde{g}(t)$ be a lift of $g(t)$ to $\widetilde{G}$ such that $\tilde{g}(0)=e$. Then $\pi(\tilde{g}(t)\tilde{\sigma}(\tilde{g}(t)))=g(t)\sigma(g(t))=e$, that is $\tilde{g}(t)\tilde{\sigma}(\tilde{g}(t))\in\ker(\pi)$. But $\ker(\pi)$ is discrete and $\tilde{g}(0)\tilde{\sigma}(\tilde{g}(0))=e$, so $\tilde{g}(t)\tilde{\sigma}(\tilde{g}(t))=e$, that is $\tilde{g}(t)\in\widetilde{R}'_0$. In particular, $\tilde{g}(1)\in\widetilde{R}'_0$. But $g=g(1)=\pi(\tilde{g}(1))$, so $g\in\pi(\widetilde{R}'_0)$. Hence we have $R'_0\subset\pi(\widetilde{R}'_0)\subset\pi(\widetilde{P})=P$. Then (iv) is proved.

(v) "$R_0=R'_0$". It is well known that $R'_0\subset R_0$. To prove the converse, we let $V=\{X\in\mathfrak{g} \mid |\mathrm{Im}(\lambda)|<\pi$ for each eigenvalue λ of $\mathrm{ad}(X)\}$. Then V is an $\mathrm{Ad}(G)$ invariant, $d\sigma$-invariant symmetric neighborhood of $0\in\mathfrak{g}$. By Varadarajan [5, Theorem 2.14.6], there is a discrete additive subgroup Γ of $\mathfrak{g}$ such that for $X,X'\in V$, $e^X=e^{X'}$ if and only if $X-X'\in\Gamma$. Choose a neighborhood $U\subset V$ of $0\in\mathfrak{g}$ and $r_0>0$ such that $\exp|_U$ is a diffeomorphism onto its image and $\exp(U)=B_{r_0}(e)$. (G being endowed a left invariant Riemannian structure.) For a continuous function $\rho:G\to(0,r_0)$, let $N_\rho=\bigcup_{g\in R'_0}B_{\rho(g)}(g)$. Then N_ρ is an open neighborhood of R'_0. It is easy to prove that $R'_0\subset N_{\rho/2}\subset\overline{N_{\rho/2}}\subset N_\rho$. We will prove that for sufficient small $\rho:G\to(0,r_0)$, $N_\rho\cap R=R'_0$. So R'_0 is open in R. $N_\rho\cap R=R'_0$ implies $\overline{N_{\rho/2}}\cap R=R'_0$, so R'_0 is closed in R. This means that R'_0 is in fact a connected component of R. So we will have $R_0=R'_0$.

Now we prove $N_\rho\cap R=R'_0$ for sufficient small ρ. Since $\Gamma\subset\mathfrak{g}$ is discrete, there is $\varepsilon>0$ such that $B_\varepsilon(0)\cap\Gamma=\{0\}$. Let $K_1\subset K_2\subset\cdots\subset K_n\subset\cdots$ be a sequence of compact subsets of G such that $\bigcup_{n=1}^\infty K_n=G$. Let $C_n=\sup_{g\in K_n}\|d\sigma+\mathrm{Ad}(g)\|+1$. Choose $r_n\in(0,r_0)$ such that $B_{r_n}(e)\subset\exp(B_{\varepsilon/C_n}(0))$. We claim that if the function $\rho:G\to(0,r_0)$ satisfies $\rho(g)<r_n$, for $g\in K_n, \forall n\in\mathbb{N}$, then $N_\rho\cap R=R'_0$. In fact, for $g'\in N_\rho\cap R$, by the definition of N_ρ, there exists $g\in R'_0$ such that $g'\in B_{\rho(g)}(g)$. Suppose $g\in K_n$. Let $g'=gh$. By the left invariance of the Riemannian structure, $h\in B_{\rho(g)}(e)\subset B_{r_n}(e)$. Suppose $h=e^X$, where $X\in U$. By $B_{r_n}(e)\subset\exp(B_{\varepsilon/C_n}(0))$, we know that $|X|<\varepsilon/C_n$. Since $g,g'\in R$, $g^{-1}\sigma(h)=\sigma(gh)=\sigma(g')=g'^{-1}=h^{-1}g^{-1}$, that is $\sigma(h)=gh^{-1}g^{-1}$. But $h=e^X$, this implies $\exp(d\sigma(X))=\exp(-\mathrm{Ad}(g)X)$. Since $d\sigma(X), -\mathrm{Ad}(g)X\in V$, $d\sigma(X)-(-\mathrm{Ad}(g)X)=(d\sigma+\mathrm{Ad}(g))X\in\Gamma$. But $|(d\sigma+\mathrm{Ad}(g))X|\leqslant\|d\sigma+\mathrm{Ad}(g)\|\cdot|X|<C_n\cdot\varepsilon/C_n=\varepsilon$. So $(d\sigma+\mathrm{Ad}(g))X\in B_\varepsilon(0)\cap\Gamma=\{0\}$, that is, $d\sigma(X)=-\mathrm{Ad}(g)X$. Now

let $\gamma(t) = ge^{tX}$, then $\gamma(0) = g$, $\gamma(1) = g'$. But $\sigma(\gamma(t)) = \sigma(ge^{tX}) = g^{-1}\exp(td\sigma(X)) = g^{-1}\exp(-t\mathrm{Ad}(g)X) = g^{-1}ge^{-tX}g^{-1} = (ge^{tX})^{-1} = \gamma(t)^{-1}$. So $\gamma(t) \in R$. This proves $g' = \gamma(1) \in R'_0$. (v) is proved. □

Remark 2.1. In general, $P \subsetneqq R$, even in some very simple cases. For example, let $G = SL(2n, \mathbb{R})$, $\sigma(g) = (g^t)^{-1}$. Then $\mathrm{diag}(-1, \ldots, -1)$ is in R, but not in P. It is obvious by the above theorem that $P = R$ if and only if R is connected (or path connected).

Remark 2.2. Using the same method as in the proof of (v) of the above theorem, we can prove that for each path component R'_i of R, there are open subsets U_i, V_i of G such that $R'_i \subset V_i \subset \overline{V_i} \subset U_i$ and $U_i \cap R = R'_i$. Thus R'_i is open and closed in R, and then R'_i is in fact a connected component of R. The proof of (v) of the above theorem also shows that R'_0 is a closed submanifold of G of dimension $\dim(\ker(d\sigma + \mathrm{Ad}(g)))$, $g \in R'_0$. Similarly, each R'_i is a closed submanifold of G of dimension $\dim(\ker(d\sigma + \mathrm{Ad}(g)))$, $g \in R'_i$. So each connected component of R is a closed submanifold of G. But different components of R can have different dimensions. For example, let $G = SO(5)$ and $\sigma(g) = sgs$, where $s = \mathrm{diag}(1, -1, -1, -1, -1)$. Then $\dim R'_0 = \dim(\ker(d\sigma + \mathrm{Ad}(e))) = 4$. But $g_0 = \mathrm{diag}(-1, -1, -1, -1, 1) \in R$, and the connected component of R containing g_0 has dimension $\dim(\ker(d\sigma + \mathrm{Ad}(g_0))) = 6$. We leave the detail of the proofs of these conclusions to the reader.

We define the *twisted conjugate action* of G on G by $\tau_g(h) = gh\sigma(g)^{-1}$. Then $Q = P$ is the orbit of this action containing the identity e. The next conclusion says that the symmetric space G/K can be embedded in G as a closed submanifold, which is just the set $P \subset G$.

Corollary 2.6. *P is a closed submanifold of G. The map $\varphi : G/K \to P$ defined by $\varphi([g]) = g\sigma(g)^{-1}$ is a diffeomorphism. Under the actions of G by left multiplication on G/K and by the twisted conjugate action on P, the isomorphism φ is equivariant.*

Proof. The twisted conjugate action τ is smooth, so Q, as an orbit of this action, is an immersion submanifold of G. As a connected component of R, R_0 is closed in R. But R is closed in G, so R_0 is closed in G. By Theorem 2.5, $P = Q = R_0$ is a closed submanifold of G. Notice that the isotropic subgroup of the action τ associated with the identity e is just K, so $\varphi : G/K \to P$ is a diffeomorphism. It is obviously equivariant. □

Remark 2.3. Corollary 2.6 says that the symmetric space G/K can be realized in G as a closed submanifold, which is just P. But we should point out that for any subgroup K' of G satisfying $K_0 \subset K' \subset K$, G/K' is also a symmetric space. In general, G/K' cannot be embedded in G as closed submanifold. The submanifold $P \subset G$ is isomorphic at most one of such G/K', and this happens if and only if $K' = K$.

Remark 2.4. If G is compact, the closedness of P can be implied from the fact that $P = R^2$, which is much easier to be obtained than $P = R_0$. In fact, to prove that P is a closed submanifold when G is compact, we need only to show that $P = R^2 = Q$. $P \subset R^2 \subset Q$

have been showed in the proof of Theorem 2.5, (i) and (ii). $Q \subset P$ can be proved by the following simple argument. Let $g = h\sigma(h)^{-1} \in Q$. Since h can be expressed as $h = pk$, where $p \in P, k \in K$, $g = (pk)(\sigma(pk))^{-1} = (pk)(p^{-1}k)^{-1} = p^2 \in P$.

3. How far is P from being a section

We regard G as a principal K-bundle with base space G/K. We have proved that $P = \exp(\mathfrak{p})$ is a closed submanifold of G. It is obvious that the tangent space T_eP of P at e is $\mathfrak{p}$. But $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. So P is a local section of the bundle $G \to G/K$ around $[e]$. If G is semisimple (or more generally, connected reductive, in the sense of Knapp [4]) and σ is the global Cartan involution of G, then $K = K_0$, and by the Cartan decomposition, the map $\Phi : P \times K \to G$ defined in Lemma 2.1 is a diffeomorphism. So G is a trivial K-bundle, and P is a global section. This means that for each coset space gK, $gK \cap P$ consists of just one point. But in general G is not a trivial K-bundle. So we may ask the question: for a coset space gK, how many points in $gK \cap P$?

Theorem 3.1. *For each coset space gK_0, there is a homeomorphism between $gK_0 \cap P$ and $\Phi^{-1}(g)$. In particular,*

$$gK_0 \cap P \neq \emptyset.$$

Proof. Let $\pi_1 : P \times K_0 \to P$ be the projection to the first factor. We prove that $\pi_1|_{\Phi^{-1}(g)}$ is a homeomorphism between $\Phi^{-1}(g)$ and $gK_0 \cap P$. First, let $(p, k) \in \Phi^{-1}(g)$, then $g = pk$. So $\pi_1(p, k) = p = gk^{-1} \in gK_0 \cap P$. This proves $\pi_1(\Phi^{-1}(g)) \subset gK_0 \cap P$. Let $(p_1, k_1), (p_2, k_2) \in \Phi^{-1}(g)$ and $(p_1, k_1) \neq (p_2, k_2)$. Since $p_1k_1 = p_2k_2 = g$, $p_1 \neq p_2$. This shows $\pi_1|_{\Phi^{-1}(g)}$ is injective. Let $p \in gK_0 \cap P$, then there is some $k \in K_0$ such that $p = gk$. So $(p, k^{-1}) \in \Phi^{-1}(g)$, and $\pi_1(p, k^{-1}) = p$. This means that $\pi_1|_{\Phi^{-1}(g)} : \Phi^{-1}(g) \to gK_0 \cap P$ is surjective. Since π_1 is continuous and open, so is $\pi_1|_{\Phi^{-1}(g)}$. Hence $\pi_1|_{\Phi^{-1}(g)} : \Phi^{-1}(g) \to gK_0 \cap P$ is a homeomorphism. By Lemma 2.1, Φ is surjective. So $gK_0 \cap P \cong \Phi^{-1}(g) \neq \emptyset$. This proves the theorem. □

Corollary 3.2. *For each coset space gK,*

$$gK \cap P \neq \emptyset.$$

Similar to the map $\Phi : P \times K_0 \to G$, we can define the map $\Phi' : P \times K \to G$ by $\Phi(p, k) = pk$. It is easy to see that Φ' satisfies all the properties of Φ that we have mentioned above.

In the following we denote the left and right translations of $g \in G$ by L_g and R_g, respectively.

Lemma 3.3. *Let $g \in G$. Then g is a regular value of Φ' if and only if gK intersects P transversally.*

Proof. Suppose $(p, k) \in \Phi'^{-1}(g)$. Then

$$\begin{aligned}
&\operatorname{Im}(d\Phi')_{(p,k)} \\
&\quad = (d\Phi')_{(p,k)}\big(T_{(p,k)}\big(P \times \{k\}\big)\big) + (d\Phi')_{(p,k)}\big(T_{(p,k)}\big(\{p\} \times K\big)\big) \\
&\quad = T_g(Pk) + T_g(pK) \\
&\quad = (dR_k)_p(T_pP) + (dR_k)_p\big(T_p(gK)\big) \\
&\quad = (dR_k)_p\big(T_pP + T_p(gK)\big).
\end{aligned}$$

Since $(dR_k)_p$ is an isomorphism and $\pi_1(\Phi'^{-1}(g)) = gK \cap P$, we have

$$\begin{aligned}
&g \text{ is a regular value of } \Phi' \\
&\quad \Longleftrightarrow \operatorname{Im}(d\Phi')_{(p,k)} = T_gG, \quad \forall (p,k) \in \Phi'^{-1}(g) \\
&\quad \Longleftrightarrow T_pP + T_p(gK) = T_pG, \quad \forall (p,k) \in \Phi'^{-1}(g) \\
&\quad \Longleftrightarrow T_pP + T_p(gK) = T_pG, \quad \forall p \in gK \cap P \\
&\quad \Longleftrightarrow gK \text{ intersects } P \text{ transversally.} \qquad \square
\end{aligned}$$

Lemma 3.4. *The set of all regular values of Φ' is right K-invariant.*

Proof. For $g \in G$ and $k_1 \in K$, $(p,k) \in \Phi'^{-1}(g) \Longleftrightarrow (p, kk_1) \in \Phi'^{-1}(gk_1)$. Since

$$R_{k_1} \circ \Phi' = \Phi' \circ (id \times R_{k_1}),$$

for $(p,k) \in \Phi'^{-1}(g)$, we have

$$(dR_{k_1})_g \circ (d\Phi')_{(p,k)} = (d\Phi')_{(p,kk_1)} \circ (\mathrm{id} \times dR_{k_1})_{(p,k)}.$$

Since $(dR_{k_1})_g$ and $(\mathrm{id} \times dR_{k_1})_{(p,k)}$ are isomorphisms, $(d\Phi')_{(p,k)}$ is an isomorphism if and only if $(d\Phi')_{(p,kk_1)}$ is an isomorphism. So g is a regular value if and only if gk_1 is a regular value. This proves the lemma. $\square$

Theorem 3.5. *For almost all coset space gK in G/K, gK intersects P transversally.*

Proof. Let G_r be the set of all regular values of Φ'. By Sard's theorem, $G \setminus G_r$ is a set with measure zero. But Lemma 3.4 tells us that G_r is the union of some coset spaces gK. So by choosing local trivializations of the principal bundle $\pi : G \to G/K$ and using Fubini's Theorem, we know that $\pi(G \setminus G_r) = (G/K) \setminus \pi(G_r)$ has measure zero in G/K. By Lemma 3.3, $[g] = gK$ intersects P transversally, $\forall [g] \in \pi(G_r)$. This proves the theorem. $\square$

Corollary 3.6. *For almost all coset space gK in G/K, $gK \cap P$ is a discrete set. In particular, if K is compact, then $gK \cap P$ is a finite set for almost all coset space gK in G/K.*

Proof. Since $\dim(gK) + \dim P = \dim G$, gK intersects P transversally implies that $gK \cap P$ is a 0-dimensional submanifold of G, which is discrete. In particular, if K is compact, then so is gK. But $gK \cap P \subset gK$. So gK intersects P transversally implies $gK \cap P$ is a finite set. By Theorem 3.5, the corollary holds. $\square$

Denote $x_0 = [e] \in G/K$. We know that a curve $\gamma(t)(t \in \mathbb{R})$ in G/K with $\gamma(0) = x_0$ is a geodesic if and only if $\gamma(t) = e^{tX}x_0$ for some $X \in \mathfrak{p}$, and such X is unique.

Theorem 3.7. *Let $g \in G$. If for every two geodesics $\gamma_i(t) = e^{tX_i}x_0$ $(i = 1, 2)$ through the point $[g]$, where $X_i \in \mathfrak{p}$, we have $[X_1, X_2] = 0$. Then $\#(gK \cap P) \leqslant \#(K \cap P)$.*

Proof. Suppose $g = pk$, where $p \in P, k \in K$. We prove $L_{p^{-1}}(gK \cap P) \subset (K \cap P)$. Suppose $p' \in (gK \cap P)$. Since $p \in gK$, $L_{p^{-1}}(p') = p^{-1}p' \in K$. Let $p = e^X$, $p' = e^{X'}$, where $X, X' \in \mathfrak{p}$. Since $[p] = [p'] = [g]$, the two geodesics $\gamma_1(t) = e^{tX}x_0$ and $\gamma_2(t) = e^{tX'}x_0$ satisfy $\gamma_1(1) = \gamma_2(1) = [p]$. By the conditions of the theorem, $[X, X'] = 0$. So $L_{p^{-1}}(p') = e^{-X}e^{X'} = e^{X'-X} \in P$. This proves $L_{p^{-1}}(gK \cap P) \subset (K \cap P)$. But $L_{p^{-1}}$ is injective, this proves the theorem. □

By Lemma 2.1, all coset space gK is of the form pK for some $p \in P$. We conclude this section by an example in which we show what the set $pK \cap P$ is for each $p \in P$.

Example. Let $G = SU(2)$, $\sigma(g) = (g^t)^{-1}$. Then $K = SO(2)$.

$$R = \{g \in SU(2) \mid g^t = g\} = \left\{\begin{pmatrix} a+bi & ci \\ ci & a-bi \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 = 1\right\}.$$

So $R \cong S^2$ is connected, and then $P = R$. For $p \in P$, we show what the set $pK \cap P$ is. The element of pK has the form $pk, k \in K$. But

$$pk \in P \iff \sigma(pk) = (pk)^{-1} \iff p^{-1}k = k^{-1}p^{-1} \iff kp = pk^{-1}.$$

Let $k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $p = \begin{pmatrix} a+bi & ci \\ ci & a-bi \end{pmatrix}$. Then it is easy to show that $kp = pk^{-1} \iff a\sin\theta = 0$. So if $a \neq 0$, $pk \in P \iff \sin\theta = 0 \iff k = \pm I$. In this case $pK \cap P = \{\pm p\}$. In particular, $K \cap P = \{\pm I\}$. If $a = 0$, then $\forall k \in K, pk \in P$. This implies $pK \subset P$. So in this case, $pK \cap P = pK$. It should be noted that all $p = \begin{pmatrix} bi & ci \\ ci & -bi \end{pmatrix}$ correspond the same coset space pK, which is the antipodal point of $[e]$ in the symmetric space $SU(2)/SO(2) \cong S^2$. □

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