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Recurrence and the shadowing property ☆

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Abstract

In this paper, we study the relationship between the recurrent set and the shadowing property. We give a necessary condition for a homeomorphism restricted to its nonwandering set to have the shadowing property. Also, we consider a condition for an arbitrary homeomorphism to be a nonwandering homeomorphism. Finally, we consider homeomorphisms which cannot have the shadowing property.

Keywords: Shadowing property; Recurrent set; Nonwandering set; Expansive

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1. Introduction and definitions

As an attempt to approach some problems of stability theory in dynamical systems, homeomorphisms of metric spaces with the shadowing property (also called pseudo-orbit-tracing property) are studied together with the related concepts of various recurrent properties such as periodicity, recurrence and nonwanderingness.

We know the problem of which recurrent sets can accept the shadowing property of homeomorphisms. This problem was suggested by Morimoto [5], and Aoki [2] proved that if f is a homeomorphism with the shadowing property of a compact space, then its restriction to its nonwandering set also has the shadowing property. He proved for f the existence of the spectral decomposition of $\Omega(f)$ under the condition $\overline{\mathrm{Per}(f)} = \Omega(f)$. We shall prove that if f is a homeomorphism of a relatively compact space X and $f|_{\Omega(f)}$ has the shadowing property, then so does $f|_{\overline{C(f)}}$ (Theorem 2.2). As a corollary of this result

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we obtain that if f is an expansive homeomorphism of a compact metric space which has the shadowing property, then $\overline{C(f)}$ has a local product structure (this definition is from Shub's book [7]) (Theorem 2.7).

It is a good problem to find various conditions for when a homeomorphism is to be a nonwandering homeomorphism [3,4,7]. In [4], Hurley claimed that if f is topologically stable, X is connected and $\Omega(f)$ has interior, then $\Omega(f) = X$. Moreover, it was shown that if f is an expansive homeomorphism of a compact metric space X which has the shadowing property and $f|_{\Omega(f)}$ is topologically transitive, then $X = \Omega(f)$ [3]. We extend this result to the case of arbitrary homeomorphisms (Theorem 2.8). Finally, we consider some homeomorphisms which cannot have the shadowing property (Theorem 3.1).

Throughout this paper we will assume that X is a metric space with a metric d and $f: X \to X$ is a homeomorphism. For x in X, $O_f(x) = \{\ldots, f^{-1}(x), x, f(x), \ldots\}$ and $O_f^+(x) = \{x, f(x), f^2(x), \ldots\}$ is called the *f-orbit* and *positive f-orbit* of x, respectively. Let $C(f)$ and $\Omega(f)$ be the recurrent set and nonwandering set of f, respectively. Recall that

$$C(f) = \bigl\{x \in X\colon\ x \in \omega_f(x) \cap \alpha_f(x),\ \text{where } \omega_f(x) \text{ and } \alpha_f(x) \text{ denote the positive and negative limit set of } x \text{ for } f,\ \text{respectively}\bigr\},$$

$$\Omega(f) = \bigl\{x \in X\colon\ \text{for every neighborhood } U \text{ of } x \text{ and integer } n_0 > 0 \text{ there exists an integer } n \geqslant n_0 \text{ such that } f^n(U) \cap U \neq \emptyset\bigr\}.$$

A sequence of points $\{x_i\}_{i\in(a,b)}$ $(-\infty \leqslant a < b \leqslant \infty)$, is called *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. A pseudo-orbit $\{x_i\}$ is called *periodic* if $x_n = x_0$ for some integer n. A sequence $\{x_i\}_{i\in(a,b)}$ is called *ε-shadowed* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ holds for $i \in (a, b)$. We say that f has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-shadowed by some point $x \in X$.

A homeomorphism f of X is called *nonwandering* if $\Omega(f) = X$ and is called *expansive* if there exists a number $e > 0$ such that $d(f^n(x), f^n(y)) \leqslant e$ for all integer n implies $x = y$.

A homeomorphism f is called *minimal* if orbit of every point in X is dense in X and a closed subset M of X which is f-invariant is called a *minimal set* for f if $f|_M$ is minimal.

X is called *relatively compact* if every closure of bounded subset of X is compact. Let $B(x,\varepsilon)$ denote $\{y \in X\colon\ d(x,y) < \varepsilon\}$ and $\overline{M}$ denote the closure of $M \subset X$.

2. Recurrent set and the shadowing property

First, we give a necessary condition for a homeomorphism restricted to its nonwandering set to have the shadowing property. For the proof we need the following lemma.

Lemma 2.1. *Let f be a homeomorphism of a relatively compact space X. If f has the shadowing property, then for every $\varepsilon > 0$ there is $\delta > 0$ such that every periodic δ-pseudo-orbit of f can be ε-shadowed by some point in $C(f)$.*

Proof. Let $\varepsilon > 0$ be given, and let $\delta = \delta(\frac{1}{2}\varepsilon) > 0$ be the number with the property of the shadowing property for f. Let $\{x_i\} = \{x_0, x_1, \ldots, x_n = x_0\}$ be a periodic δ-pseudo-orbit of f. Since f has the shadowing property there is a point y in X such that

$$d\big(f^{ni+j}(y), x_j\big) < \tfrac{1}{2}\varepsilon, \tag{$*$}$$

for every integer i and j $(0 \leqslant j < n)$. In particular, $f^{ni}(y) \in B(x_0, \frac{1}{2}\varepsilon)$ for every integer i and hence we have

$$\overline{O_{f^n}(y)} \subset \overline{B\big(x_0, \tfrac{1}{2}\varepsilon\big)}.$$

Since $\overline{O_{f^n}(y)}$ is compact and invariant for the homeomorphism $f^n : X \to X$, it contains a minimal set M for f^n. Let $z_0 \in M$. Then we get

$$\overline{O_{f^n}(z_0)} = \omega_{f^n}(z_0) = \alpha_{f^n}(z_0) = M.$$

Therefore, we have

$$z_0 \in \omega_{f^n}(z_0) \cap \alpha_{f^n}(z_0) \subset \omega_f(z_0) \cap \alpha_f(z_0),$$

which implies $z_0 \in C(f)$. The lemma will be established if we can show that $\{x_i\}$ is ε-shadowed by z_0. We check this. When $z_0 \in O_{f^n}(y)$ we are done. So assume that $z_0 \in \omega_{f^n}(y)$ and $d(f^{j_0}(z_0), x_{j_0}) > \varepsilon$ for some integer j_0 $(0 \leqslant j_0 < n)$. By the continuity of f there is an open set G containing z_0 such that $f^{j_0}(G) \subset X \backslash B(x_{j_0}, \varepsilon)$. Since $z_0 \in \omega_{f^n}(y)$ it follows that there is a sequence $\{f^{nl_i}(y)\}$ which converges to z_0 for some $l_i \to +\infty$. Therefore, there is sufficiently large L in $\{l_i\}$ satisfying $f^{nL}(y) \in G$ and $d(f^{nL+j_0}(y), x_{j_0}) > \varepsilon$. But this contradicts $(*)$. This shows that $\{x_i\}$ is ε-shadowed by $z_0 \in C(f)$ as required. □

Theorem 2.2. *Let f be a homeomorphism of a relatively compact space X. If $f : \Omega(f) \to \Omega(f)$ has the shadowing property, then so does $f : \overline{C(f)} \to \overline{C(f)}$.*

Proof. Let $\varepsilon > 0$ be arbitrary and choose $\alpha = \alpha(\frac{1}{2}\varepsilon) > 0$ as in Lemma 2.1. Let $0 < \delta < \frac{1}{2}\alpha$ and $\{x_i\}_{i=-\infty}^{\infty}$ be any δ-pseudo-orbit in $\overline{C(f)}$ of f. We have to find a point in $\overline{C(f)}$ ε-shadowing the δ-pseudo-orbit $\{x_i\}$.

Take and fix a positive integer n. Here, we assert the existence of an α-pseudo-orbit of f which contains $\{x_i\}_{i=-n}^{n}$. To see this, for each integer l $(-n \leqslant l \leqslant n)$, let us choose a number β_l with $0 < \beta_l < \delta$, an integer $k(l)$ and a point $y_l \in C(f) \cap B(x_l, \beta_l)$ as follows:

For $l = n$:

$$d(x_l, y) < \beta_l \text{ implies } d\big(f(x_l), f(y)\big) < \delta,$$
$$d\big(f^{k(l)+1}(y_l), x_l\big) < \beta_l, \quad k(l) > 1.$$

For $-n < l < n$:

$$d(x_l, y) < \beta_l \text{ implies } d\big(f^2(x_l), f^2(y)\big) < \delta,$$
$$d\big(f^{k(l)+1}(y_l), x_l\big) < \beta_l, \quad k(l) > 2.$$

For $l = -n$:

$$d(x_l, y) < \beta_l \text{ implies } d\big(f^2(x_l), f^2(y)\big) < \delta,$$
$$d\big(f^{k(l)+1}(y_l), x_l\big) < \beta_l, \quad k(l) > 2.$$

Then we can define a sequence $\{a_i^n\}$ by putting

$$\begin{aligned}
&a_{i+j}^n = x_{-n+j}, && i = -n, \quad 0 \leqslant j \leqslant 2n,\\
&a_{i+j}^n = f^{j+1}(y_n), && i = n+1, \quad 0 \leqslant j \leqslant k(n) - 1,\\
&a_i^n = f(x_{n-1}), && i = n + k(n) + 1,\\
&a_{i+j}^n = f^{j+2}(y_{n-1}), && i = n + k(n) + 2, \quad 0 \leqslant j \leqslant k(n-1) - 2,\\
&a_i^n = f(x_{n-2}), && i = n + k(n) + k(n-1) + 1,\\
&\vdots && \vdots\\
&a_{i+j}^n = f^{j+2}(y_{-n+1}), && i = n + k(n) + k(n-1) + \cdots + k(-n+2) + 2,\\
& && \qquad 0 \leqslant j \leqslant k(-n+1) - 2,\\
&a_i^n = f(x_{-n}), && i = n + k(n) + k(n-1) + \cdots + k(-n+1) + 1,\\
&a_{i+j}^n = f^{j+2}(y_{-n}), && i = n + k(n) + \cdots + k(-n+2) + k(-n+1) + 2,\\
& && \qquad 0 \leqslant j \leqslant k(-n) - 2,\\
&a_i^n = x_{-n}, && i = n + k(n) + \cdots + k(-n) + 1.
\end{aligned}$$

Obviously, the sequence $\{a_i^n\}_{i=-n}^K$, where $K = n + 1 + \sum_{l=-n}^{l=n} k(l)$, defined as above is a periodic α-pseudo-orbit of f in $\Omega(f)$ which contains $\{x_i\}_{i=-n}^n$. Since $f|_{\Omega(f)}$ has the shadowing property, by the above lemma there is a $z_n \in C(f)$ with $d(f^i(z_n), a_i^n) < \frac{1}{2}\varepsilon$ for every integer i $(-n \leqslant i \leqslant K)$. In particular, we have

$$d\big(f^i(z_n), x_i\big) < \tfrac{1}{2}\varepsilon, \quad -n \leqslant i \leqslant n.$$

Let a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converge to z as $n_k \to \infty$. Then $z \in \overline{C(f)}$. Using the continuity of f it is easy to see that $d(f^i(z), x_i) \leqslant \frac{1}{2}\varepsilon < \varepsilon$ for all integer i. Thus $f : \overline{C(f)} \to \overline{C(f)}$ has the shadowing property and this completes the proof. □

It is well known that for an expansive homeomorphism f with the shadowing property, $\overline{\mathrm{Per}(f)} = \Omega(f)$ holds.

Corollary 2.3. *Let f be a homeomorphism of a compact space X which has the shadowing property. Then we have $\overline{C(f)} = \Omega(f)$.*

Proof. $\overline{C(f)} \subset \Omega(f)$ is clear. For every $\varepsilon > 0$, let $\delta = \delta(\varepsilon) > 0$ be as in Lemma 2.1. Then every $x \in \Omega(f)$ can be a member of a periodic δ-pseudo-orbit of f and which is ε-shadowed by a point in $C(f)$. Thus $x \in \overline{C(f)}$ since ε is arbitrary.

Now, we point out that from Lemma 2.1 and by using a method similar to the one used in the proof of the above theorem we can show that the result due to Aoki [2] also holds on relatively compact spaces. Here, we will omit the proof of this result.

Corollary 2.4. *If f is a homeomorphism of a relatively compact space X which has the shadowing property, then $f|_{\Omega(f)}$ also has the shadowing property.*

Remark 2.5. In general, the converse of the above theorem does not hold. For example, $X = \{(x,y) \in \mathbb{R}^2\colon\ x^2+y^2 \leqslant 1\}$. Define a continuous real flow $g: X \times \mathbb{R} \to X$ on X using an autonomous system of differential equations satisfying: the points $(0,1)$ and $(0,-1)$ are fixed for g. For $p = (x,y) \in S^1$, let $\omega_g(p) = (0,-1)$ and $\alpha_g(p) = (0,1)$ if $x < 0$ and $\omega_g(p) = (0,1)$ and $\alpha_g(p) = (0,-1)$ if $x > 0$. The phase portrait of g in $\{(x,y) \in X\colon\ x^2+y^2 < 1\}$ is the same as that of the flow defined by the differential equations (polar coordinate)

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r(1-r), \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1.$$

Consider a homeomorphism $f: X \to X$ defined by $f(x) = g(x,1)$. For this homeomorphism f, we have $\Omega(f) = S^1 \cup \{(0,0)\}$ and $\overline{C(f)} = \{(0,0),(0,1),(0,-1)\}$. Obviously, $f: \Omega(f) \to \Omega(f)$ does not have the shadowing property though $f: \overline{C(f)} \to \overline{C(f)}$ has the shadowing property.

For $x \in X$, let $W^s_\varepsilon(x)$, $W^u_\varepsilon(x)$ be the local stable set and local unstable set for x defined by

$$W^s_\varepsilon(x) = \{y \in X\colon\ d(f^i(x), f^i(y)) \leqslant \varepsilon \text{ for all integer } i \geqslant 0\},$$
$$W^u_\varepsilon(x) = \{y \in X\colon\ d(f^i(x), f^i(y)) \leqslant \varepsilon \text{ for all integer } i \leqslant 0\}.$$

For sufficiently small $\varepsilon, \delta > 0$ and $x, y \in X$ with $d(x,y) < \delta$, let us define $\{x,y\}^\delta_\varepsilon$ by

$$\{x,y\}^\delta_\varepsilon = \{z \in X\colon\ z \in W^s_\varepsilon(x) \cap W^u_\varepsilon(y)\}.$$

Note that if f has the shadowing property, then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\{x,y\}^\delta_\varepsilon$ is nonempty.

Lemma 2.6. *Let f be a homeomorphism of a relatively compact space X which has the shadowing property. Then, for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in \overline{C(f)}$, we have $\{x,y\}^\delta_\varepsilon \cap \overline{C(f)} \neq \emptyset$.*

Proof. Let $\varepsilon > 0$ be given and $\delta' = \delta'(\frac{1}{3}\varepsilon) > 0$ be a number with the property of the shadowing property for f. Let $0 < \delta < \frac{1}{2}\delta'$ and $d(x,y) < \delta$, $x, y \in \overline{C(f)}$.

For each integer $n > 0$, we can choose points

$$x_n \in B(x, \tfrac{1}{4}\delta) \cap C(f), \qquad y_n \in B(y, \tfrac{1}{4}\delta) \cap C(f)$$

and integers $k(n) > n$, $l(n) < -n$ satisfying that

$$d(f^i(x), f^i(x_n)) < \tfrac{1}{2}\varepsilon, \quad d(f^{-i}(y), f^{-i}(y_n)) < \tfrac{1}{2}\varepsilon, \qquad 0 \leqslant i \leqslant n, \tag{1}$$

and

$$d(f^{k(n)}(x_n), x_n) < \tfrac{1}{4}\delta, \quad d(f^{l(n)}(y_n), y_n) < \tfrac{1}{4}\delta. \tag{2}$$

(1) and (2) ensure the existence of the periodic δ-pseodo-orbit of f in $C(f)$ defined by

$$\{a_i\}_{i=l(n)}^{k(n)} = \{f^{l(n)}(y_n),\ f^{l(n)+1}(y_n), \ldots, f^{-1}(y_n),$$
$$x_n,\ f(x_n),\ f^2(x_n), \ldots, f^{k(n)-1}(x_n),\ f^{l(n)}(y_n)\}.$$

Applying Lemma 2.1, we can find $z_n \in C(f)$ $\varepsilon/3$-shadowing $\{a_i\}$. Further, a straightforward calculation yields

$$d(f^i(z_n),\ f^i(x)) < \varepsilon, \quad d(f^{-i}(z_n),\ f^{-i}(y)) < \varepsilon, \qquad 0 \leqslant i \leqslant n.$$

Let a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converge to z as $n_k \to \infty$. Then $z \in \overline{C(f)}$ and

$$d(f^i(z),\ f^i(x)) \leqslant \varepsilon, \qquad d(f^{-i}(z),\ f^{-i}(y)) \leqslant \varepsilon$$

for every nonnegative integer i. So, $z \in \{x, y\}_\varepsilon^\delta \cap \overline{C(f)}$ and this completes the proof. □

If $f|_{\overline{C(f)}}$ is expansive and has the shadowing property, then, for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in \overline{C(f)}$ the set $\{x, y\}_\varepsilon^\delta$ is a singleton. So the above result ensures the existence of the function

$$a_{\varepsilon,\delta} : U_\delta(\Delta, \overline{C(f)}) = \{(x, y) \in \overline{C(f)} \times \overline{C(f)}:\ d(x, y) < \delta\} \to \overline{C(f)},$$
$$(x, y) \rightsquigarrow \{x, y\}_\varepsilon^\delta.$$

Further, this function is continuous on compact spaces [6,7].

Hence, from the above lemma we have:

Theorem 2.7. *If f is an expansive homeomorphism of a compact space X which has the shadowing property, then $\overline{C(f)}$ has a local product structure.*

A homeomorphism f is *topologically transitive* if, for every pair of nonempty open set U and V, there is an integer n such that $f^n(U) \cap V$ is nonempty. Note that a homeomorphism f of a compact space X is topologically transitive if and only if there is a point z in X whose orbit is dense.

Theorem 2.8. *Let f be a homeomorphism of a compact space X which has the shadowing property and $f|_{\Omega(f)}$ be topologically trasitive, then $X = \Omega(f)$.*

Proof. Suppose that $X \neq \Omega(f)$. Since $\omega_f(x_0)$ and $\alpha_f(x_0)$ are contained in $\Omega(f)$ for $x_0 \in X \backslash \Omega(f)$, we have $\overline{O_f^+(x_0)} \cap \Omega(f) \neq \emptyset$ and $\overline{O_f^-(x_0)} \cap \Omega(f) \neq \emptyset$. Let

$\varepsilon_0 = d(x_0, \Omega(f))$ and $\delta = \delta(\varepsilon_0) > 0$ be the number with the property of the shadowing property for f. Then for sufficiently large n_0, $d(\Omega(f), f^{n_0}(x_0)) < \delta$ and $d(\Omega(f), f^{-n_0}(x_0)) < \delta$ and hence there are points $x_{n_0+1}, x_{-n_0-1} \in \Omega(f)$ with $d(f^{n_0}(x_0), x_{n_0+1}) < \delta$ and $d(f^{-n_0}(x_0), f(x_{-n_0-1})) < \delta$. Since $f|_{\Omega(f)}$ is topologically transitive there is δ-pseudo-orbit $\{x_{n_0+1}, z_1, \ldots, z_k, x_{-n_0-1}\}$ of f from x_{n_0+1} to x_{-n_0-1} in $\Omega(f)$. So we can obtain a periodic δ-pseudo-orbit of f

$$\{y_i\} = \{x_{-n_0-1}, f^{-n_0}(x_0), f^{-n_0+1}(x_0), \ldots, x_0, f(x_0), \ldots, \\ f^{n_0-1}(x_0), x_{n_0+1}, z_1, \ldots, z_k, x_{-n_0-1}\}.$$

Since f has the shadowing property there is a y in $C(f) \subset \Omega(f)$ ε_0-shadowing the pseudo-orbit $\{y_i\}$ with $d(y, x_0) < \varepsilon_0$. This means $d(x_0, \Omega(f)) < \varepsilon_0$ and this contradicts the fact that $d(x_0, \Omega(f)) = \varepsilon_0$. □

3. Homeomorphisms without the shadowing property

In this section we consider homeomorphisms which cannot have the shadowing property.

A homeomorphism $f : X \to X$ is called *positively (negatively) recurrent* if $x \in \omega_f(x)$ ($x \in \alpha_f(x)$) for each x in X. An ε-chain from x to y is a finite sequence $\{z_0 = x, z_1, \ldots, z_n = y\}$ such that $d(z_i, z_{i+1}) < \varepsilon$ for each $0 \leqslant i < n$.

Theorem 3.1. *Let f be a homeomorphism of a compact connected metric space X which is not minimal. If X is not one point, and if f is positively or negatively recurrent, then f does not have the shadowing property.*

Proof. Suppose that f has the shadowing property. Since f is not minimal there is $y_0 \in X$ with $\overline{O_f(y_0)} \neq X$. Let $x_0 \in X \backslash \overline{O_f(y_0)}$ and $d(x_0, \overline{O_f(y_0)}) = 3\varepsilon$. Let $\delta = \delta(\varepsilon)$ with $0 < \delta < \varepsilon$ be a number with the property of the shadowing property for f. By connectedness of X there is a $\frac{1}{2}\delta$-chain $\{z_0 = x_0, z_1, z_2, \ldots, z_{n+1} = y_0\}$ from x_0 to y_0.

First, assume that f is positively recurrent. Then, for each z_i, $z_i \in \omega_f(z_i)$. So there is positive integer $k(i)$ such that $d(f^{k(i)+1}(z_i), z_i) < \frac{1}{2}\delta$ for each $0 \leqslant i \leqslant n+1$. Thus we can construct a δ-pseudo-orbit by putting

$$\{a_i\}_{i=0}^{\infty} = \{z_0, f(z_0), \ldots, f^{k(0)}(z_0), z_1, f(z_1), \ldots, f^{k(1)}(z_1), \\ z_2, f(z_2), \ldots, f^{k(2)}(z_2), z_3, f(z_3), \ldots, z_n, \\ f(z_n), \ldots, f^{k(n)}(z_n), z_{n+1}, f(y_0), f^2(y_0), f^3(y_0), \ldots\}.$$

Since f has the shadowing property there is a in X such that $d(f^i(a), a_i) < \varepsilon$ for all $i \geqslant 0$. Let $K = k(0) + k(1) + \cdots + k(n) + n$. Then

$$d(a, a_0) = d(a, x_0) < \varepsilon \quad \text{and} \\ d(f^{K+i}(a), f^i(y_0)) < \varepsilon, \quad i \geqslant 0.$$

So we have

$$O_f^+\big(f^K(a)\big) \subset B\big(O_f^+(y_0),\varepsilon\big) \subset B\big(\overline{O_f^+(y_0)},\varepsilon\big) \subset B\big(\overline{O_f(y_0)},\varepsilon\big)$$

and hence

$$\omega_f(a) = \omega_f\big(f^K(a)\big) \subset \overline{O_f^+\big(f^K(a)\big)} \subset \overline{B\big(\overline{O_f(y_0)},\varepsilon\big)}.$$

However, by the positive recurrence of $f, a \in \omega_f(a)$ and therefore $d(\overline{O_f(y_0)}, a) \leqslant \varepsilon$. So we have

$$d\big(x_0, \overline{O_f(y_0)}\big) \leqslant d(x_0, a) + d\big(a, \overline{O_f^+(y_0)}\big) \leqslant 2\varepsilon.$$

Thus contradicting that $3\varepsilon = d(x_0, \overline{O_f(y_0)})$.

Next, assume that f is negatively recurrent. Then for each point z_i in the $\frac{1}{2}\delta$-chain $\{z_0, z_1, \ldots, z_{n+1}\}$ from x_0 to y_0, there is negative integer $c(i)$ such that $d(f^{c(i)}(z_i), z_i) < \frac{1}{2}\delta$. Let us define a sequence $\{b_i\}$ by putting

$$\begin{aligned}\{b_i\}_{i=-\infty}^0 = \big\{ &\ldots, f^{-2}(y_0), f^{-1}(y_0), f^{c(n+1)}(z_{n+1}), f^{c(n+1)+1}(z_{n+1}), \ldots, \\ &f^{-1}(z_{n+1}), f^{c(n)}(z_n), f^{c(n)+1}(z_n), \ldots, f^{-1}(z_n), f^{c(n-1)}(z_{n-1}), \\ &f^{c(n-1)+1}(z_{n-1}), \ldots, f^{-1}(z_2), f^{c(1)}(z_1), f^{c(1)+1}(z_1), \ldots, \\ &f^{-1}(z_1), z_0\big\}.\end{aligned}$$

Obviously, $\{b_i\}$ is a δ-pseudo-orbit. So there is a point b in X with $d(f^i(b), b_i) < \varepsilon$ for each $i \leqslant 0$. In particular, we have

$$\begin{aligned}&d(b, b_0) = d(b, x_0) < \varepsilon \quad \text{and}\\ &d\big(f^{L+i}(b), f^i(y_0)\big) < \varepsilon, \quad i \leqslant -1,\end{aligned}$$

where $L = c(1) + c(2) + \cdots + c(n+1)$. Similarly, we can obtain the following fact that

$$b \in \alpha_f(b) \subset \overline{B\big(\overline{O_f(y_0)},\varepsilon\big)}.$$

That is $d(\overline{O_f(y_0)}, b) \leqslant \varepsilon$, so that

$$d\big(x_0, \overline{O_f(y_0)}\big) \leqslant d(x_0, b) + d\big(b, \overline{O_f(y_0)}\big) \leqslant 2\varepsilon.$$

Hence we can derive the same contradiction, and so completes the proof. □

Now in view of [1], our theorem yields the following.

Corollary 3.2. *Let f be a homeomorphism of a compact connected space X which is not one point. If all orbit closures are minimal sets, then f does not have the shadowing property.*

Corollary 3.3. *Let f be a homeomorphism of a compact connected space X which is not one point. If $X = C(f)$, then f does not have the shadowing property.*

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