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Products of characters and derived length ☆

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Abstract

Let G be a finite solvable group and $\chi \in \mathrm{Irr}(G)$ be a faithful character. We show that the derived length of G is bounded by a linear function of the number of distinct irreducible constituents of $\chi\overline{\chi}$. We also discuss other properties of the decomposition of $\chi\overline{\chi}$ into its irreducible constituents.

Keywords: Products of characters; Irreducible character; Derived length

1. Introduction

Let G be a finite group. Denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. Let 1_G be the principal character of G. Denote by $[\Theta, \Phi]$ the inner product of the characters Θ and Φ of G. Through this work, we will use the notation of [1].

Let $\chi \in \mathrm{Irr}(G)$. Define $\overline{\chi}(g)$ to be the complex conjugate $\overline{\chi(g)}$ of $\chi(g)$ for all $g \in G$. Then $\overline{\chi}$ is also an irreducible complex character of G. Since the product of characters is a character, $\chi\overline{\chi}$ is a character of G. So it can be expressed as an integral linear combination of irreducible characters. Now observe that

$$[\chi\overline{\chi}, 1_G] = [\chi, \chi] = 1,$$

where the last equality holds since $\chi \in \mathrm{Irr}(G)$. Assume now that $\chi(1) > 1$. Then the decomposition of the character $\chi\overline{\chi}$ into its distinct irreducible constituents 1_G, α_1,

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$\alpha_2, \ldots, \alpha_n$ has the form

$$\chi\overline{\chi} = 1_G + \sum_{i=1}^{n} a_i\alpha_i, \tag{1.1}$$

where $n > 0$ and $a_i > 0$ is the multiplicity of α_i.

Set $\eta(\chi) = n$, so that $\eta(\chi)$ is the number of distinct non-principal irreducible constituents of $\chi\overline{\chi}$. The number $\eta(\chi)$ carries information about the structure of the group. For example, if $\eta(\chi)$ is an odd number, then the order of the group has to be an even number. To see this, notice that $\chi\overline{\chi}$ is a real character. When $\eta(\chi)$ is odd, at least one of the irreducible characters α_i has to be real. Then G has a non-principal irreducible real character. So the order of G has to be even.

The purpose of this work is to give some answers to the following questions:

Question 1. Assume that we know $\eta(\chi)$ for some $\chi \in \mathrm{Irr}(G)$. What can we say about the structure of the group G and about the character χ?

Question 2. Knowing the set $\{a_i \mid i = 1, \ldots, \eta(\chi)\}$, what can we say about the group G?

Denote by $\mathrm{dl}(G)$ the derived length of the group G. The main results of this work regarding the first question are the following

Theorem A. *There exist constants C and D such that for any finite solvable group G and any irreducible character χ*

$$\mathrm{dl}\big(G/\mathrm{Ker}(\chi)\big) \leqslant C\eta(\chi) + D.$$

Theorem B. *Let G be a finite solvable group and $\chi \in \mathrm{Irr}(G)$. Then $\chi(1)$ has at most $\eta(\chi)$ distinct prime divisors.*

If, in addition, G is supersolvable and $\chi(1) > 1$, then $\chi(1)$ is a product of at most $\eta(\chi) - 1$ primes.

The main result of this work regarding the second question is

Theorem C. *Assume that G is a finite solvable group and $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$. Let $\{\alpha_i \in \mathrm{Irr}(G)^\# \mid i = 1, \ldots, n\}$ be the set of non-principal irreducible constituents of $\chi\overline{\chi}$. If $\mathrm{Ker}(\alpha_j)$ is maximal under inclusion among the subgroups $\mathrm{Ker}(\alpha_i)$ for $i = 1, \ldots, n$, of G, then $[\chi\overline{\chi}, \alpha_j] = 1$. Thus $1 \in \{[\chi\overline{\chi}, \alpha_i] \mid i = 1, \ldots, n\}$.*

Notation. Set $V^\# = V \setminus \{0\}$ and $\mathrm{Irr}(G)^\# = \mathrm{Irr}(G) \setminus \{1_G\}$.

2. Preliminaries

Definition 2.1. Let V be a finite $\mathbf{F}G$-module for some finite field $\mathbf{F}$. Then $m(G, V)$ is the number of distinct sizes of orbits of G on $V^\#$.

Lemma 2.2 (Keller). *There exist universal constants C_1 and C_2 such that for any finite solvable group G acting faithfully and irreducibly on a finite vector space V we have*

$$\mathrm{dl}(G) \leqslant C_1 \log\big(m(G,V)\big) + C_2.$$

Proof. See [2]. $\square$

Definition 2.3. We define the function

$$h(n) = C_1 \log(n) + C_2,$$

where C_1 and C_2 are as in Lemma 2.2.

3. The function $\eta(\chi)$

Given a finite group G and a character $\chi \in \mathrm{Irr}(G)$, we define $\eta(\chi)$ as the number of non-principal irreducible constituents of the product $\chi\overline{\chi}$. We give examples showing that there is no relation between induction of characters and η.

Example 3.1. If $\chi = \theta^G$ is induced from some $\theta \in \mathrm{Irr}(H)$, where $H \leqslant G$, then we need not have $\eta(\chi) \geqslant \eta(\theta)$.

Proof. Let E be an extra-special group of exponent p and order p^3 for some odd prime p. Let $a \in \mathrm{Aut}(E)$ be an element of prime order q that divides $p-1$. Assume that a acts fixed point free on E.

Set $G = \langle a\rangle E$. Let $\theta \in \mathrm{Irr}(E)$ be a non-linear character. Since a acts fixed point free, we have that $\theta^G = \chi \in \mathrm{Irr}(G)$.

Observe that G has q linear characters, namely the irreducible characters of G/E. Also G has $(p^2-1)/q$ irreducible characters of degree q, the characters that are induced from linear non-principal characters of E. And finally there are $(p-1)/q$ irreducible characters of degree pq. We conclude that G has $q + (p^2-1)/q + (p-1)/q$ distinct irreducible characters. Thus $\eta(\chi) \leqslant q - 1 + (p^2-1)/q + (p-1)/q$.

We can check that

$$q - 1 + \frac{p^2-1}{q} + \frac{p-1}{q} < p^2 - 1.$$

Observe that $\theta\overline{\theta} = (1_{\mathbf{Z}(E)})^E$. Thus $\eta(\theta) = p^2 - 1 > \eta(\chi)$. $\square$

Example 3.2. If $\chi = \theta^G$ is induced from some $\theta \in \mathrm{Irr}(H)$, where $H \leqslant G$, then we need not have $\eta(\chi) \leqslant \eta(\theta)$.

Proof. Let G be an extra-special group. Let $\chi \in \mathrm{Irr}(G)$ be a non-linear character. Let θ be a linear character of some subgroup H of G such that $\chi = \theta^G$. Then $\eta(\chi) > \eta(\theta) = 0$. $\square$

4. Proof of Theorem C

Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Consider the expression (1.1) for $\chi\overline{\chi}$. We will see in this section that if G is solvable, then $1 \in \{a_i\}$. That may not be true in general. For example, consider A_6, the alternating group on 6 letters, and $\chi_5 \in \mathrm{Irr}(A_6)$ with $\chi_5(1) = 10$. Using the notation of p. 289 of [1], we can check that

$$\chi_5\overline{\chi_5} = \chi_1 + 2\chi_2 + 2\chi_3 + 3\chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7.$$

Thus $\{a_i\} = \{2, 3\}$.

Lemma 4.1. *Let L and N be normal subgroups of G such that L/N is an abelian chief factor of G. Let $\theta \in \mathrm{Irr}(L)$ be a G-invariant character. Then the restriction θ_N is reducible if and only if*

$$\theta(g) = 0 \quad \textit{for all } g \in L \setminus N. \tag{4.2}$$

Also if θ_N is reducible, then

$$\theta\bar{\theta} = (1_N)^L + \Phi, \tag{4.3}$$

where Φ is either the zero function or a character of L, and $[\Phi_N, 1_N] = 0$.

Proof. Let $\varphi \in \mathrm{Irr}(N)$ be a character such that $[\varphi, \theta_N] \neq 0$. If θ_N is reducible, by Theorem 6.18 of [1] we have that either $\theta_N = e\varphi$, where $e^2 = |L : N|$, or $\theta = \varphi^L$. If $\theta_N = e\varphi$, where $e^2 = |L : N|$, by Exercise 6.3 of [1] we have that θ vanishes on $L \setminus N$. If $\theta = \varphi^L$, since N is a normal subgroup of L we have that $\theta(g) = 0$ for all $g \in L \setminus N$. Thus (4.2) holds.

Now assume that (4.2) holds. Then

$$\begin{aligned}[\theta_N, \theta_N] &= \frac{1}{|N|}\sum_{g\in N}\theta(g)\overline{\theta(g)} \\ &= \frac{1}{|N|}\sum_{g\in L}\theta(g)\overline{\theta(g)} \quad \text{by (4.2)} \\ &= \frac{1}{|N|}|L|[\theta,\theta] = \frac{|L|}{|N|},\end{aligned}$$

where the last equality holds since $\theta \in \mathrm{Irr}(L)$. Because $|L|/|N| > 1$, it follows that θ_N is a reducible character.

For any $\gamma \in \mathrm{Irr}(L/N)$ we have that

$$\begin{aligned}[\theta\bar{\theta}, \gamma] &= [\theta, \theta\gamma] \\ &= \frac{1}{|L|}\sum_{g\in L}\theta(g)\overline{\theta(g)\gamma(g)}\end{aligned}$$

$$= \frac{1}{|L|}\left[\sum_{g\in L\setminus N}\theta(g)\overline{\theta(g)\gamma(g)} + \sum_{g\in N}\theta(g)\overline{\theta(g)\gamma(g)}\right]$$

$$= \frac{1}{|L|}\left[\sum_{g\in L\setminus N}\theta(g)\overline{\theta(g)} + \sum_{g\in N}\theta(g)\overline{\theta(g)\gamma(g)}\right] \quad \text{by (4.2)}$$

$$= \frac{1}{|L|}\left[\sum_{g\in L\setminus N}\theta(g)\overline{\theta(g)} + \sum_{g\in N}\theta(g)\overline{\theta(g)}\right] \quad \text{since } \mathrm{Ker}(\gamma)\geqslant N \text{ and } \gamma(1)=1$$

$$= [\theta,\theta] = 1.$$

Thus (4.3) follows. □

Lemma 4.4. *Let G be a finite solvable group and $\chi\in\mathrm{Irr}(G)$. Let $\{\alpha_i \mid i=1,\dots,\eta(\chi)\}$ be the set of non-principal irreducible constituents of the product $\chi\overline{\chi}$. Let N be a normal subgroup of G. Then $\chi_N\in\mathrm{Irr}(N)$ if and only if $N\nleqslant\mathrm{Ker}(\alpha_i)$ for $i=1,\dots,\eta(\chi)$.*

Proof. Observe that

$$\begin{aligned}
[\chi_N,\chi_N] &= \left[\chi_N\overline{\chi}_N, 1_N\right] \\
&= \left[\left(1_G+\sum_{i=1}^{n}a_i\alpha_i\right)_N, 1_N\right] \quad \text{by (1.1)} \\
&= \left[1_N+\sum_{i=1}^{n}a_i(\alpha_i)_N, 1_N\right] \\
&= [1_N,1_N]+\sum_{i=1}^{n}a_i\left[(\alpha_i)_N,1_N\right] \\
&= 1+\sum_{i=1}^{n}a_i\left[(\alpha_i)_N,1_N\right].
\end{aligned}$$

Thus $[\chi_N,\chi_N]=1$ if and only if $\sum_{i=1}^{n}a_i[(\alpha_i)_N,1_N]=0$. Since $a_i>0$ for $i=1,\dots,n$, we have $[\chi_N,\chi_N]=1$ if and only if $[(\alpha_i)_N,1_N]=0$ for all i. Since $[(\alpha_i)_N,1_N]=0$ if and only if $N\nleqslant\mathrm{Ker}(\alpha_i)$, the result follows. □

Proof of Theorem C. Set $N=\mathrm{Ker}(\alpha_j)$. Let L be a normal subgroup of G such that L/N is a chief factor of G. Since $N=\mathrm{Ker}(\alpha_j)\nleqslant\mathrm{Ker}(\alpha_i)$ for $i=1,\dots,n$, we have $L\nleqslant\mathrm{Ker}(\alpha_i)$ for $i=1,\dots,n$. By Lemma 4.4 we have that $\chi_L\in\mathrm{Irr}(L)$. Set $\theta=\chi_L$. Since $N=\mathrm{Ker}(\alpha_j)$, we have that $[(\alpha_j)_N,1_N]=\alpha_j(1)$. Thus

$$[\chi_N,\chi_N]=\left[\left(\chi\overline{\chi}\right)_N,1_N\right]\geqslant 1+a_j\alpha_j(1)>1.$$

Therefore χ_N is reducible. By Lemma 4.1 we have that

$$\left(\chi\overline{\chi}\right)_L = \theta\bar{\theta} = 1_N^L + \Phi, \tag{4.5}$$

where Φ is either the zero function or a character of L and $[\Phi_N, 1_N] = 0$. Also, by (1.1) we have that

$$\left(\chi\overline{\chi}\right)_L = 1_L + \sum_{i=1}^{n} a_i(\alpha_i)_L.$$

Let $\gamma \in \mathrm{Irr}(L/\mathrm{Ker}(\alpha_j))$ be such that $[(\alpha_j)_L, \gamma] \neq 0$. Then

$$0 < a_j\left[(\alpha_j)_L, \gamma\right] = \left[(a_j\alpha_j)_L, \gamma\right] \leqslant \left[\left(\chi\overline{\chi}\right)_L, \gamma\right] = 1,$$

where the last equality follows from (4.5). Therefore $a_j = 1$.

Since there is some $j \in \{1, \dots, n\}$ such that $\mathrm{Ker}(\alpha_j)$ is maximal among the $\mathrm{Ker}(\alpha_i)$ for all i, the last part of Theorem C follows from that. □

5. Proof of Theorem B

Lemma 5.1. *Assume G is a finite group and $\chi \in \mathrm{Irr}(G)$ is a faithful character. Let $\{\alpha_i \in \mathrm{Irr}(G)^\# \mid i = 1, \dots, n\}$ be the set of non-principal irreducible constituents of $\chi\overline{\chi}$. Then*

$$\mathbf{Z}(G) = \bigcap_{i=1}^{n} \mathrm{Ker}(\alpha_i).$$

Proof. By Lemma 2.21 of [1],

$$\mathrm{Ker}\left(\chi\overline{\chi}\right) = \mathrm{Ker}(1_G)\bigcap_{i=1}^{n}\mathrm{Ker}(\alpha_i) = \bigcap_{i=1}^{n}\mathrm{Ker}(\alpha_i).$$

Since $(\chi\overline{\chi})(g) = \chi^2(1)$ if and only if $g \in \mathbf{Z}(\chi)$, it follows that $\mathrm{Ker}(\chi\overline{\chi}) = \mathbf{Z}(G)$, and the result follows. □

Definition 5.2. Let G be a group and L be a subgroup of G. We say that

$$(N, \theta) \leqslant (L, \phi)$$

if $N \leqslant L$, $\phi \in \mathrm{Irr}(L)$, $\theta \in \mathrm{Irr}(N)$ and $[\phi_N, \theta] \neq 0$. We say that

$$(N, \theta) < (L, \phi)$$

if $N < L$, $\phi \in \mathrm{Irr}(L)$, $\theta \in \mathrm{Irr}(N)$ and $[\phi_N, \theta] \neq 0$.

Let X be a family of normal subgroups of G with $G \in X$. We say that a chain

$$(N_0, \theta_0) > (N_1, \theta_1) > (N_2, \theta_2) > \cdots > (N_k, \theta_k),$$

where $N_0 = G$ and $\chi = \theta_0$, is an (X, χ)-*reducing chain* if $N_i \in X$ and $(\theta_i)_{N_{i+1}}$ is reducible for $i = 0, \ldots, k$.

We say that the above chain is a *maximal* (X, χ)-*reducing chain* if it is a (X, χ)-reducing chain with the following two properties:

(i) For any i with $0 < i \leqslant k$, the group N_i is a maximal subgroup in the set

$$\{M \in X \mid M \leqslant N_{i-1} \text{ and } (\theta_{i-1})_M \text{ is reducible}\}.$$

(ii) For any $M \in X$ such that $M < N_k$, the restriction $(\theta_k)_M$ is irreducible.

Remark. Given a family X of normal subgroups of G with $G \in X$ and given $\chi \in \mathrm{Irr}(G)$, there is always an (X, χ)-reducing chain, and a maximal (X, χ)-reducing chain. In fact (G, χ) is already an (X, χ)-reducing chain. We find a maximal reducing (X, χ) chain by induction. We start with $(N_0, \theta_0) = (G, \chi)$. If $(\theta_0)_M$ is irreducible for any $M \in X$, then (N_0, θ_0) is our maximal (X, χ)-reducing chain. Assume we have found (N_{i-1}, θ_{i-1}) for some integer $i \geqslant 1$. If the set

$$\{M \in X \mid M \leqslant N_{i-1} \text{ and } (\theta_{i-1})_M \text{ is reducible}\}$$

is non-empty, we choose N_i to be any maximal element in this set, and θ_i to be any character in $\mathrm{Irr}(N_i)$ such that $[(\theta_{i-1})_{N_i}, \theta_i] > 0$. Otherwise we stop our chain with $k = i - 1$.

Hypotheses 5.3. *Assume G is a finite solvable group and $\chi \in \mathrm{Irr}(G)$ is a faithful character. Set $n = \eta(\chi)$. Let $\{\alpha_i \in \mathrm{Irr}(G)^\# \mid i = 1, \ldots, n\}$ be the set of non-principal irreducible constituents of $\chi\overline{\chi}$. Set*

$$\Omega = \left\{ \bigcap_{i \in S} \mathrm{Ker}(\alpha_i) \mid S \subseteq \{1, 2, \ldots, n\} \right\}, \tag{5.4}$$

where $\bigcap_{i \in S} \mathrm{Ker}(\alpha_i)$ is taken to be G when S is empty.

Let

$$(G, \chi) = (N_0, \theta_0) > (N_1, \theta_1) > \cdots > (N_k, \theta_k)$$

be a maximal (Ω, χ)-reducing chain.

Lemma 5.5. *Assume Hypotheses* 5.3. *Then the maximal (Ω, χ)-reducing chain has the following properties*:

(a) *For any integer $i = 1, 2, \ldots, k$ and any normal subgroup M of G such that $N_i < M \leqslant N_{i-1}$ we have that*

$$(\theta_{i-1})_M \in \mathrm{Irr}(M). \tag{5.6}$$

(b) *N_k is abelian.*
(c) *$k \leqslant n$.*
(d) *If, in addition, G is supersolvable, then $k \leqslant n-1$.*

Proof. (a) If $M \in \Omega$, then $(\theta_{i-1})_M$ has to be irreducible. Otherwise N_i is not a maximal element in Ω such that $(\theta_{i-1})_{N_i}$ reduces, a contradiction with property (i) in Definition 5.2.

So we may assume that M is not an element of Ω. Let L be minimal among all elements $K \in \Omega$ such that $M \leqslant K \leqslant N_{i-1}$. By property (i) in Definition 5.2 we have that

$$\phi = (\theta_{i-1})_L \in \mathrm{Irr}(L).$$

Observe that

$$1 = [\phi, \phi] = \big[\phi\overline{\phi}, 1_L\big] \leqslant \big[(\phi\overline{\phi})_M, 1_M\big] = [\phi_M, \phi_M], \tag{5.7}$$

where equality holds if and only if $\phi_M \in \mathrm{Irr}(M)$.

Recall that $[\chi_{N_{i-1}}, \theta_{i-1}] \neq 0$. Thus $[\chi_L, \phi] \neq 0$. Let T be the stabilizer of ϕ in G and Y be a set of coset representatives of T in G. Thus if $g, h \in Y$ and $g \neq h$, we have that $\phi^g \neq \phi^h$ and therefore $[\phi^g, \phi^h] = 0$. By Clifford Theory we have that $\chi_L = e\sum_{g\in Y}\phi^g$ for some integer $e > 0$. Thus

$$\big[(\chi\overline{\chi})_L, 1_L\big] = [\chi_L, \chi_L] = \left[e\sum_{g\in Y}\phi^g, e\sum_{g\in Y}\phi^g\right] = e^2\sum_{g\in Y}\big[\phi^g, \phi^g\big]. \tag{5.8}$$

Since $\chi_M = (\chi_L)_M$, we have that

$$\big[(\chi\overline{\chi})_M, 1_M\big] = [\chi_M, \chi_M] = e^2\left[\sum_{g\in Y}(\phi^g)_M, \sum_{g\in Y}(\phi^g)_M\right]. \tag{5.9}$$

If $\phi_M \notin \mathrm{Irr}(M)$, then (5.7), (5.8) and (5.9) imply that

$$\big[(\chi\overline{\chi})_L, 1_L\big] < \big[(\chi\overline{\chi})_M, 1_M\big]. \tag{5.10}$$

By (1.1) and (5.10) there exists some α_j such that $\mathrm{Ker}(\alpha_j) \geqslant M$ but $\mathrm{Ker}(\alpha_j) \ngeqslant L$. Therefore $L \cap \mathrm{Ker}(\alpha_j)$ is a proper subset of L, contains M and lies in Ω. This contradicts our choice of L. Thus $(\theta_{i-1})_M = \phi_M \in \mathrm{Irr}(M)$.

(b) By Lemma 5.1 we have that $\mathbf{Z}(G) \subseteq M$ for any $M \in \Omega$. Thus $(\theta_k)_{\mathbf{Z}(G)}$ is irreducible by property (ii) in Definition 5.2. That implies that $\theta_k \in \mathrm{Irr}(N_k)$ is a linear character. Since N_k is normal in G and $[\chi_{N_k}, \theta_k] \neq 0$, all the irreducible components of χ_{N_k} are linear. By hypothesis $\chi \in \mathrm{Irr}(G)$ is a faithful character. Therefore N_k must be abelian.

(c) This follows from the definition of Ω and the fact that the set $\{\mathrm{Ker}(\alpha_j)\}$ has at most n elements.

(d) Suppose that $N_k = \mathbf{Z}(G)$. Let L/N_k be a chief factor of G with $L \leqslant N_{k-1}$. Since G is supersolvable, L/N_k is cyclic of prime order. Observe that L is abelian because it has a central subgroup N_k with a cyclic factor group L/N_k. So θ_k extends to L. By (a) we have that $(\theta_{k-1})_L \in \mathrm{Irr}(L)$. Thus $(\theta_{k-1})_{N_k} = \theta_k$. That can not be by Definition 5.2(i). We conclude that $N_k \neq \mathbf{Z}(G)$.

Since $N_k \neq \mathbf{Z}(G) = \bigcap_{i=1}^n \mathrm{Ker}(\alpha_i)$ and $\{\mathrm{Ker}(\alpha_i) \mid i = 1, 2, \ldots, n\}$ has at most n elements, we must have that $k \leqslant n - 1$. $\square$

Theorem B is an application of Lemma 5.5.

Proof of Theorem B. Working with the group $G/\mathrm{Ker}(\chi)$, by induction on the order of G we can assume that $\mathrm{Ker}(\chi) = 1$. Let

$$(G, \chi) = (N_0, \theta_0) > (N_1, \theta_1) > \cdots > (N_k, \theta_k)$$

be a maximal (Ω, χ)-reducing chain. For each $i = 1, 2, \ldots, k$, let L_i be a normal subgroup of G such that L_i/N_i is a chief factor of G and $L_i \leqslant N_{i-1}$.

By Lemma 5.5 we have that $(\theta_{i-1})_{L_i} \in \mathrm{Irr}(L_i)$. Since L_i/N_i is an elementary abelian p_i-group for some prime p_i, we have

$$\theta_{i-1}(1) = \theta_i(1) p_i^{m_i}$$

for some integer $m_i \geqslant 1$. Here $m_i = 1$ in the case that G is supersolvable. By Lemma 5.5(b), we have that $\theta_k(1) = 1$. By Lemma 5.5(c), $k \leqslant n$. We conclude that $\chi(1)$ has at most $k \leqslant n$ distinct prime divisors.

If G is supersolvable, by Lemma 5.5(d) we have $k \leqslant n - 1$. Thus $\chi(1)$ has at most $n - 1$ prime divisors. $\square$

6. Proof of Theorem A

Hypotheses 6.1. *Assume Hypotheses* 5.3. *For each i, let L_i/N_i be a chief factor of G where $L_i \leqslant N_{i-1}$.*

Lemma 6.2. *Assume Hypotheses* 6.1. *There exists a subgroup U of L_i and a character $\phi \in \mathrm{Irr}(U)$, such that*

$$(N_i, \theta_i) \leqslant (U, \phi) < (L_i, \psi). \tag{6.3}$$

Proof. Suppose that the lemma is false. Then for any U and $\phi \in \mathrm{Irr}(U)$ such that (6.3) holds, we have that $(L_i)_\phi = L_i$. Choose a chain

$$(N_i, \theta_i) = (U_s, \phi_s) < \cdots < (U_1, \phi_1) < (U_0, \phi_0) = (L_i, \psi)$$

such that $|U_{j-1} : U_j|$ is a prime number for all $j = 1, 2, \dots, s$. We can do that since L_i/N_i is an elementary abelian group. Since $(L_i)_{\phi_j} = L_i$ for all $j = 1, 2, \dots, s$, we have $(U_{j-1})_{\phi_j} = U_{j-1}$. Since $|U_{j-1} : U_j|$ is a prime number, it follows that $(\phi_{j-1})_{U_j} = \phi_j$ for $j = 1, \dots, s$. But then $(\theta_{i-1})_{N_i} \in \mathrm{Irr}(N_i)$, a contradiction with Definition 5.2(i). Therefore there exist $U < L_i$ and a character $\phi \in \mathrm{Irr}(U)$ such that (6.3) holds and $(L_i)_\phi \neq L_i$.

Since $N_i \leqslant (L_i)_\phi < L_i$, and L_i/N_i is an elementary abelian subgroup, the subgroup $(L_i)_\phi$ is normal in L_i. By Clifford Theory ψ is induced from some character $\psi_\phi \in \mathrm{Irr}((L_i)_\phi)$. Since $(L_i)_\phi$ is normal in L_i, and $(\psi_\phi)^{L_i} = \psi$, we have $\psi(g) = 0$ for any $g \in L_i \setminus (L_i)_\phi$. □

Lemma 6.4. *Assume Hypotheses* 6.1. *Let* $r_i = |\{\alpha_j \mid N_i \leqslant \mathrm{Ker}(\alpha_j)$ *and* $N_{i-1} \nleqslant \mathrm{Ker}(\alpha_j)\}|$. *Then we have*

$$\mathrm{dl}\big(N_{i-1}/\mathbf{C}_{N_{i-1}}(L_i/N_i)\big) \leqslant h(r_i),$$

where h *is as in Definition* 2.3.

Proof. By Lemma 5.5(a), we have that

$$\psi = (\theta_{i-1})_{L_i} \in \mathrm{Irr}(L_i). \tag{6.5}$$

Let $V(\psi)$ be the "vanishing-off subgroup of ψ" (see p. 200 of [1]), the smallest subgroup $V(\psi)$ of L_i such that ψ vanishes on $L_i \setminus V(\psi)$. Since $\psi = (\theta_{i-1})_{L_i}$ and $\theta_{i-1} \in \mathrm{Irr}(N_{i-1})$, the subgroup $V(\psi)$ is N_{i-1}-invariant. Therefore $N_i V(\psi)$ is a normal subgroup of N_{i-1}. Let U and $\phi \in \mathrm{Irr}(U)$ be as in Lemma 6.2. Observe that $V(\psi) \leqslant (L_i)_\phi$ since for all $g \in L_i \setminus (L_i)_\phi$ we have that $\psi(g) = 0$. Also observe that $N_i \leqslant (L_i)_\phi$. Thus $N_i V(\psi) \leqslant (L_i)_\phi$. Therefore $N_i V(\psi)$ is a proper subgroup of L_i.

Let M be a subgroup such that $N_i V(\psi) \leqslant M < L_i$ and L_i/M is a chief factor of N_{i-1}. So we have the following relations:

$$N_i \leqslant N_i V(\psi) \leqslant M < L_i \leqslant N_{i-1}.$$

Since L_i is a normal subgroup of G, the quotient L_i/M^g is also a chief factor of N_{i-1}, for any $g \in G$. Hence for any $g \in G$

$$MM^g = M \quad \text{or} \quad MM^g = L_i. \tag{6.6}$$

Lemma 4.1 gives us that

$$\psi\overline{\psi} = 1_M^{L_i} + \Phi,$$

where Φ is either 0 or a character of L_i. Since $[(\chi)_{L_i}, \psi] \neq 0$, this implies that

$$\big(\chi\overline{\chi}\big)_{L_i} = 1_M^{L_i} + \Theta,$$

where Θ is either 0 or a character of L_i. This and (1.1) imply that

$$1_{L_i} + \sum_{j=1}^{n} a_j (\alpha_j)_{L_i} = 1_M^{L_i} + \Theta.$$

Thus

$$\mathrm{Irr}(L_i/M)^{\#} = \bigcup_{j=1}^{n} \bigl\{\gamma \in \mathrm{Irr}(L_i/M)^{\#} \mid \bigl[(\alpha_j)_{L_i}, \gamma\bigr] \neq 0\bigr\}.$$

Let $X = \{\alpha_j | [(\alpha_j)_{L_i}, \gamma] \neq 0$ for some $\gamma \in \mathrm{Irr}(L_i/M)^{\#}\}$. Observe that X is a subset of the set

$$\bigl\{\alpha_j \mid N_i \leqslant \mathrm{Ker}(\alpha_j) \text{ and } N_{i-1} \nleqslant \mathrm{Ker}(\alpha_j)\bigr\}.$$

Thus

$$|X| \leqslant r_i. \tag{6.7}$$

Let $\gamma, \delta \in \mathrm{Irr}(L_i/M)^{\#}$. Suppose that γ and δ lie below the same $\alpha_j \in X$, i.e., $[(\alpha_j)_{L_i}, \gamma] \neq 0$ and $[(\alpha_j)_{L_i}, \delta] \neq 0$ for some $j = 1, \ldots, n$. Since L_i is a normal subgroup of G and $\alpha_j \in \mathrm{Irr}(G)$, by Clifford Theory there exists $g \in G$ such that $\gamma^g = \delta$. By definition we have that $M \leqslant \mathrm{Ker}(\delta)$. Observe that

$$M^g \leqslant \bigl(\mathrm{Ker}(\gamma)\bigr)^g = \mathrm{Ker}\bigl(\gamma^g\bigr).$$

Since $\gamma^g = \delta$, we have $MM^g \leqslant \mathrm{Ker}(\delta)$. By (6.6) we have that $M^g = M$, i.e., $g \in N_G(M)$. We conclude that γ and δ lie below the same α_j if and only if $\gamma^g = \delta$ for some $g \in N_G(M)$, i.e., the set $\{\gamma \in \mathrm{Irr}(L_i/M)^{\#} \mid [(\alpha_j)_L, \gamma] \neq 0\}$ is an $N_G(M)$-orbit in $\mathrm{Irr}(L_i/M)^{\#}$. Set $H = N_G(M)$. Each H-orbit in $\mathrm{Irr}(L_i/M)^{\#}$ lies under at least one character α_j in X, and any each α_j lies over a single H-orbit $\mathrm{Irr}(L_i/M)$. Hence H acts on $\mathrm{Irr}(L_i/M)^{\#}$ with at most $|X|$ orbits. By (6.7) we conclude that H acts on $\mathrm{Irr}(L_i/M)^{\#}$ with at most r_i orbits. By Lemma 2.2 we have that

$$\mathrm{dl}\bigl(H/\mathbf{C}_H(L_i/M)\bigr) \leqslant h(r_i).$$

Since $N_{i-1} \leqslant H = N_G(M)$ and $\mathbf{C}_H(L_i/M) \cap N_{i-1} = \mathbf{C}_{N_{i-1}}(L_i/M)$, we have

$$\mathrm{dl}\bigl(N_{i-1}/\mathbf{C}_{N_{i-1}}(L_i/M)\bigr) \leqslant h(r_i).$$

For any $g \in G$, we can check that

$$(G, \chi) = \bigl(N_0, (\theta_0)^g\bigr) > \bigl(N_1, (\theta_1)^g\bigr) > \cdots > \bigl(N_k, (\theta_k)^g\bigr)$$

is a maximal (G,Ω)-reducing chain. Thus, as before we can conclude that

$$\mathrm{dl}\big(N_{i-1}/\mathbf{C}_{N_{i-1}}\big(L_i/M^g\big)\big) \leqslant h(r_i). \tag{6.8}$$

Since L_i/N_i is a chief factor of G and $N_i \leqslant M < L_i$, we have that

$$\mathrm{core}_G(M) = \bigcap_{g\in G} M^g = N_i.$$

Therefore

$$\bigcap_{g\in G} \mathbf{C}_{N_{i-1}}\big(L_i/M^g\big) = \mathbf{C}_{N_{i-1}}(L_i/N_i). \tag{6.9}$$

Observe that the lemma follows from (6.8) and (6.9). □

Lemma 6.10. *Assume Hypotheses* 5.3.

$$\mathrm{dl}(N_{i-1}/N_i) \leqslant \mathrm{dl}\big(N_{i-1}/\mathbf{C}_{N_{i-1}}(L_i/N_i)\big) + 1.$$

Proof. Set $C = \mathbf{C}_{N_{i-1}}(L_i/N_i)$. Observe that $L_i \leqslant C$ and that C is a normal subgroup of G. We want to prove that C/N_i is abelian. We may assume that $C > L_i$. Observe that if U is a group and $N_i \leqslant U \leqslant L_i$, then U is normal in C. By Lemma 6.2, there exist U and $\phi \in \mathrm{Irr}(U)$, where

$$(N_i,\theta_i) \leqslant (U,\phi) < (L_i,\psi) \tag{6.11}$$

and $(L_i)_\phi < L_i$. In particular we have that $C_\phi \neq C$. Since $(\theta_{i-1})_{L_i} = \psi \in \mathrm{Irr}(L_i)$ and $U < L_i \leqslant C \leqslant N_{i-1}$, we have that $(\theta_{i-1})_C \in \mathrm{Irr}(C)$ and $(\theta_{i-1})_C$ lies above ϕ. By Clifford Theory, there exists $\zeta \in \mathrm{Irr}(C_\phi)$ such that $\zeta^C = (\theta_{i-1})_C$. Since $(\zeta^C)_{L_i} \in \mathrm{Irr}(L_i)$, we have that $C = C_\phi L_i$ (see Exercise 5.7 of [1]). Observe that C_ϕ is normal in C since L_i/N_i is central in $C = \mathbf{C}_{N_{i-1}}(L_i/N_i)$. Since L_i/N_i is abelian, so is C/C_ϕ. Since C is normal in G, for any $g \in G$ we have that C/C_ϕ^g is abelian.

Since $(\theta_{i-1})_C \in \mathrm{Irr}(C)$, while $[(\theta_{i-1})_U,\phi] \neq 0$ and $(L_i)_\phi < L_i$, we have that $(\theta_{i-1})_{C_\phi}$ is a reducible character. Set $P = \bigcap_{g\in G} C_\phi^g$. Observe that P is a normal subgroup of G with $N_i \leqslant P < N_{i-1}$. Observe also that $(\theta_{i-1})_P$ is reducible since $P \leqslant C_\phi$. By Lemma 5.5(a), we have that $P = N_i$. Therefore C/N_i is abelian and the lemma follows. □

Lemma 6.12. *Assume Hypotheses* 6.1. *Then*

$$\mathrm{dl}(N_{i-1}/N_i) \leqslant h(r_i) + 1.$$

Proof. It follows from Lemmas 6.4 and 6.10 □

Lemma 6.13. *Let $n > 1$ be an integer. Set* $\mathbf{N} = \{1, 2, \ldots\}$. *Define*

$$p(n) = \max\{n_1 \cdot n_2 \cdot \cdots \cdot n_s \mid n_1, n_2, \ldots, n_s \in \mathbf{N} \text{ and } n_1 + n_2 + \cdots + n_s = n\}. \tag{6.14}$$

Then

$$p(n+1) \leqslant 2p(n).$$

Therefore

$$p(n) \leqslant 2^{n-1}. \tag{6.15}$$

Proof. Observe that $n \leqslant p(n)$ since we can take $s = 1$ and $n_1 = n$ in (6.14). Thus if $p(n+1) = m_1 \cdot m_2 \cdot \cdots \cdot m_t$, where $m_1, m_2, \ldots, m_t$ are non-zero positive integers and $m_1 + m_2 + \cdots + m_t = n + 1$, then $m_i > 1$ for some $i \in \{1, \ldots, t\}$. Assume that $m_1 \geqslant 2$. Then $m_1 - 1 \geqslant 1$, $(m_1 - 1) + m_2 + \cdots + m_t = n$. By definition we have that $(m_1 - 1) \cdot m_2 \cdot \cdots \cdot m_t \leqslant p(n)$. Thus

$$\begin{aligned} p(n+1) &= m_1 \cdot m_2 \cdot \cdots \cdot m_t \\ &= (m_1 - 1) \cdot m_2 \cdot \cdots \cdot m_t + 1 \cdot m_2 \cdot \cdots \cdot m_t \\ &\leqslant p(n) + 1 \cdot m_2 \cdot \cdots \cdot m_t \\ &\leqslant p(n) + (m_1 - 1) \cdot m_2 \cdot \cdots \cdot m_t \\ &\leqslant p(n) + p(n) = 2p(n). \end{aligned}$$

Since $p(2) = 2$, inequality (6.15) follows. □

Proof of Theorem A. Working with the group $G/\mathrm{Ker}(\chi)$, by induction on the order of G we can assume that $\mathrm{Ker}(\chi) = 1$. So we may assume Hypotheses 5.3. Let

$$(G, \chi) = (N_0, \theta_0) > (N_1, \theta_1) > \cdots > (N_k, \theta_k)$$

be a maximal (Ω, χ)-reducing chain. Set $n = \eta(\chi)$. By Lemma 5.5(b) and (c), we have that N_k is abelian and $k \leqslant n$. By Lemma 6.12, we have that, for $i = 1, \ldots, k$,

$$\mathrm{dl}(N_{i-1}/N_i) \leqslant h(r_i) + 1,$$

where $r_i = |\{\alpha_j \mid N_i \leqslant \mathrm{Ker}(\alpha_j) \text{ and } N_{i-1} \nleqslant \mathrm{Ker}(\alpha_j)\}|$. The definition of a maximal reducing chain and the definition of r_i implies that

$$r_1 + r_2 + \cdots + r_k \leqslant n. \tag{6.16}$$

By Lemma 6.13 we have that

$$\prod_{i=1}^{k} r_i \leqslant 2^{n-1}.$$

Thus

$$\mathrm{dl}(G) \leqslant \sum_{i=1}^{k} \mathrm{dl}(N_{i-1}/N_i) + \mathrm{dl}(N_k) \leqslant \sum_{i=1}^{k} \bigl(h(r_i)+1\bigr) + 1.$$

Since $h(r_i) = C_1 \log(r_i) + C_2$ by Definition 2.3, we have that

$$\begin{aligned} \mathrm{dl}(G) &\leqslant \sum_{i=1}^{k} \bigl(C_1 \log(r_i) + C_2 + 1\bigr) + 1 = C_1 \left[\sum_{i=1}^{k} \log(r_i)\right] + (C_2+1)k + 1 \\ &\leqslant C_1 \log\left(\prod_{i=1}^{k} r_i\right) + (C_2+1)k + 1. \end{aligned}$$

Let $s = \sum_{i=1}^{k} r_i$. By (6.16) we have $s \leqslant n$. By Lemma 6.15 we have that

$$\prod_{i=1}^{k} r_i \leqslant 2^{s-1} \leqslant 2^{n-1}.$$

Thus

$$\mathrm{dl}(G) \leqslant C_1 \log\bigl(2^{n-1}\bigr) + (C_2+1)k + 1 \leqslant (n-1)C_1 \log(2) + (C_2+1)n + 1,$$

where the last inequality follows from $k \leqslant n$ (see Lemma 5.5(c)). Set $C = C_1 \log(2) + C_2 + 1$ and $D = 1 + C_1 \log(2)$. Then

$$\mathrm{dl}(G) \leqslant Cn + D. \qquad \square$$

Theorem 6.17. *Let G be a supersolvable group. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) > 1$. Then*

$$\mathrm{dl}\bigl(G/\mathrm{Ker}(\chi)\bigr) \leqslant 2\eta(\chi) - 1.$$

Proof. Working with the group $G/\mathrm{Ker}(\chi)$, by induction on the order of G we can assume that $\mathrm{Ker}(\chi) = 1$.

Let

$$(G, \chi) = (N_0, \theta_0) > (N_1, \theta_1) > \cdots > (N_k, \theta_k)$$

be a maximal (Ω, χ)-reducing chain. Let L_i/N_i be a chief factor of G, where $L_i \leqslant N_{i-1}$. Since G is a supersolvable group, L_i/N_i is a cyclic group of prime order. Set $H = N_{i-1}/\mathbf{C}_{N_{i-1}}(L_i/N_i)$. Observe that H acts faithfully on L_i/N_i as automorphisms. Since L_i/N_i is cyclic, H is abelian, i.e.,

$$\mathrm{dl}\big(N_{i-1}/\mathbf{C}_{N_{i-1}}(L_i/N_i)\big) \leqslant 1.$$

By Lemma 6.10 we conclude that

$$\mathrm{dl}(N_{i-1}/N_i) \leqslant 2.$$

By Lemma 5.5(d) we have that $k \leqslant \eta(\chi) - 1$. Also N_k is abelian by Lemma 5.5(b). Thus

$$\mathrm{dl}(G) \leqslant 2\big(\eta(\chi) - 1\big) + 1. \qquad \square$$

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