\textbf{$\gamma$-Bounded representations of amenable groups}

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\begin{abstract}
Let $G$ be an amenable group, let $X$ be a Banach space and let $\pi : G \to B(X)$ be a bounded representation. We show that if the set $\{\pi(t) : t \in G\}$ is $\gamma$-bounded then $\pi$ extends to a bounded homomorphism $w : C^*(G) \to B(X)$ on the group $C^*$-algebra of $G$. Moreover $w$ is necessarily $\gamma$-bounded. This extends to the Banach space setting a theorem of Day and Dixmier saying that any bounded representation of an amenable group on Hilbert space is unitarizable. We obtain additional results and complements when $G = \mathbb{Z}, \mathbb{R}$ or $\mathbb{T}$, and/or when $X$ has property ($\alpha$).
\end{abstract}

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\section{1. Introduction}

The notions of $R$-boundedness and $\gamma$-boundedness play a prominent role in various recent developments of operator valued harmonic analysis and multiplier theory, see for example [42,39,1,4,21–23,26]. These notions are also now central in the closely related fields of functional calculi (see [27,13,30]), abstract control theory in Banach spaces [20,19], or vector valued stochastic integration, see [41] and the references therein. This paper is devoted to another aspect of harmonic analysis, namely Banach space valued group representations. Our results will show that $\gamma$-boundedness is the key concept to understand certain behaviors of such representations.

Throughout we let $G$ be a locally compact group, we let $X$ be a complex Banach space and we let $B(X)$ denote the Banach algebra of all bounded operators on $X$. By a representation of $G$

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on $X$, we mean a strongly continuous mapping $\pi : G \to B(X)$ such that $\pi(tt') = \pi(t)\pi(t')$ for any $t, t'$ in $G$, and $\pi(e) = I_X$. Here $e$ and $I_X$ denote the unit of $G$ and the identity operator on $X$, respectively. We say that $\pi$ is bounded if moreover $\sup_{t \in G} \|\pi(t)\| < \infty$. Assume that $G$ is amenable and that $X = H$ is a Hilbert space. Then it follows from the Day–Dixmier unitarization Theorem (see e.g. [38, Chap. 0]) that any bounded representation of $G$ on $H$ extends to a bounded homomorphism $C^*(G) \to B(H)$ from the group $C^*$-algebra $C^*(G)$ into $B(H)$. In general this extension property is no longer possible when $H$ is replaced by an arbitrary Banach space. To see a simple example, let $G$ be an infinite abelian group, let $1 \leq p < \infty$ and let $\lambda_p : G \to B(L^p(G))$ be the regular representation defined by letting $[\lambda_p(t)f](s) = f(s-t)$ for any $f \in L^p(G)$. Recall that $C^*(G) = C_0(\hat{G})$, where $\hat{G}$ denotes the dual group of $G$. Hence if $\lambda_p$ extends to a bounded homomorphism $C^*(G) \to B(L^p(G))$, then any function in $C_0(\hat{G})$ is a bounded multiplier on $L^p(G)$. As is well known, this implies that $p = 2$, see e.g. [32, Thm. 4.5.2]. (See also Corollary 6.2 for more on this.) This leads to the problem of finding conditions on a Banach space representation $\pi : G \to B(X)$ ensuring that its extension to a bounded homomorphism $C^*(G) \to B(X)$ is indeed possible.

We recall the definitions of $\gamma$-boundedness and $R$-boundedness. The latter is more classical (see [7]), but the two notions are completely similar. Let $(g_k)_{k \geq 1}$ be a sequence of complex valued, independent standard Gaussian variables on some probability space $\Sigma$. For any $x_1, \ldots, x_n$ in $X$, we let

$$\left\| \sum_k g_k \otimes x_k \right\|_{G(X)} = \left( \int_\Sigma \left\| \sum_k g_k(\lambda)x_k \right\|_X^2 d\lambda \right)^{\frac{1}{2}}.$$ 

Next we say that a set $F \subset B(X)$ is $\gamma$-bounded if there is a constant $C \geq 0$ such that for any finite families $T_1, \ldots, T_n$ in $F$, and $x_1, \ldots, x_n$ in $X$, we have

$$\left\| \sum_k g_k \otimes T_k x_k \right\|_{G(X)} \leq C \left\| \sum_k g_k \otimes x_k \right\|_{G(X)}.$$ 

In this case, we let $\gamma(F)$ denote the smallest possible $C$. This constant is called the $\gamma$-bound of $F$. Now let $(\epsilon_k)_{k \geq 1}$ be a sequence of independent Rademacher variables on some probability space. Then replacing the sequence $(g_k)_{k \geq 1}$ by the sequence $(\epsilon_k)_{k \geq 1}$ in the above definitions, we obtain the notion of $R$-boundedness. The corresponding $R$-bound constant of $F$ is denoted by $R(F)$. Using the symmetry of Gaussian variables, it is easy to see (and well known) that any $R$-bounded set $F \subset B(X)$ is automatically $\gamma$-bounded, with $\gamma(F) \leq R(F)$. If further $X$ has a finite cotype, then Rademacher averages and Gaussian averages are equivalent (see e.g. [37, Chap. 3]), hence the notions of $R$-boundedness and $\gamma$-boundedness are equivalent. Clearly any $\gamma$-bounded set is bounded and if $X$ is isomorphic to a Hilbert space, then any bounded set is $\gamma$-bounded. We recall that conversely if $X$ is not isomorphic to a Hilbert space, then there exist bounded sets $F \subset B(X)$ which are not $\gamma$-bounded (see [1, Prop. 1.13]).

Our main result asserts that if $G$ is amenable and if $\pi : X \to B(X)$ is a representation such that $[\pi(t) : t \in G]$ is $\gamma$-bounded, then there exists a (necessarily unique) bounded homomorphism $w : C^*(G) \to B(X)$ extending $\pi$ (see Definition 2.4 for the precise meaning). Moreover $w$ is $\gamma$-bounded, i.e. it maps the unit ball of $C^*(G)$ into a $\gamma$-bounded set of $B(X)$.

If $X$ has property $(\alpha)$, we obtain the following analog of the Day–Dixmier unitarization Theorem: a representation $\pi : G \to B(X)$ extends to a bounded homomorphism $C^*(G) \to B(X)$ if and only if $[\pi(t) : t \in G]$ is $\gamma$-bounded. As an illustration, consider the case $G = \mathbb{Z}$ and recall that
Let $T : X \to X$ be an invertible operator on a Banach space with property $(\alpha)$. We obtain that there exists a constant $C \geq 1$ such that

$$\left\| \sum_k c_k T^k \right\| \leq C \sup \left\{ \left\| \sum_k c_k z^k \right\| : z \in \mathbb{C}, \ |z| = 1 \right\}$$

for any finite sequence $(c_k)_{k \in \mathbb{Z}}$ of complex numbers, if and only if the set

$$\{ T^k : k \in \mathbb{Z} \}$$

is $\gamma$-bounded.

The main result presented above is established in Section 4. Its proof makes crucial use of the transference methods available on amenable groups (see [8]) and of the Kalton–Weis $\ell$-spaces introduced in the unpublished paper [28]. Sections 2 and 3 are devoted to preliminary results and background on these spaces and on group representations. In Section 5 we give a proof of the following result: if a Banach space $X$ has property $(\alpha)$, then any bounded homomorphism $w : A \to B(X)$ defined on a nuclear $C^*$-algebra $A$ is automatically $R$-bounded (and even matricially $R$-bounded). This result is due to Éric Ricard (unpublished). In the case when $A$ is abelian, it goes back to De Pagter and Ricker [10] (see also [29]). Section 6 contains examples and illustrations, some of them using the above theorem. We pay a special attention to the $\gamma$-bounded representations of the classical abelian groups $\mathbb{Z}, \mathbb{R}, \mathbb{T}$.

We end this introduction with some notation and general references. First, we will use vector valued integration and Bochner $L^p$-spaces for which we refer to [15]. We let $G(X) \subset L^2(\Sigma; X)$ be the closed subspace spanned by the finite sums $\sum_k g_k \otimes x_k$, with $x_k \in X$. Next the space $\text{Rad}(X)$ is defined similarly, using the Rademacher sequence $(\varepsilon_k)_{k \geq 1}$. For any $n \geq 1$, we let $\text{Rad}_n(X) \subset \text{Rad}(X)$ be the subspace of all sums $\sum_{k=1}^n \varepsilon_k \otimes x_k$. It follows from classical duality on Bochner spaces that we have a natural isometric isomorphism

$$\text{Rad}_n(X)^{**} = \text{Rad}_n(X^{**}). \quad (1.1)$$

Second, we refer to [17] for general background on classical harmonic analysis. Given a locally compact group $G$, we let $dt$ denote a fixed left Haar measure on $G$. For any $p \geq 1$, we let $L^p(G) = L^p(G, dt)$ denote the corresponding $L^p$-space. We recall that the convolution on $G$ makes $L^1(G)$ a Banach algebra. Finally we will use basic facts on $C^*$-algebras and Hilbert space representations, for which [38] and [35] are relevant references.

For any Banach spaces $X, Y$, we let $B(Y, X)$ denote the space of all bounded operators from $Y$ into $X$, equipped with the operator norm, and we set $B(X) = B(X, X)$. Given any set $V$, we let $\chi_V$ denote the indicator function of $V$.

2. Preliminaries on $\gamma$-bounded representations

We let $M_{n,m}$ denote the space of $n \times m$ scalar matrices equipped with its usual operator norm. We start with the following well-known tensor extension property, for which we refer e.g. to [14, Cor. 12.17].

**Lemma 2.1.** Let $a = [a_{ij}] \in M_{n,m}$ and let $x_1, \ldots, x_m \in X$. Then

$$\left\| \sum_{i,j} a_{ij} g_i \otimes x_j \right\|_{G(X)} \leq \|a\|_{M_{n,m}} \left\| \sum_j g_j \otimes x_j \right\|_{G(X)}.$$
This result does not remain true if we replace Gaussian variables by Rademacher variables and this defect is the main reason why it is sometimes easier to deal with $\gamma$-boundedness than with $R$-boundedness.

An extremely useful property proved in [7, Lem. 3.2] is that if $F \subset B(X)$ is any $R$-bounded set, then its strongly closed absolute convex hull $\text{aco}(F)$ is $R$-bounded as well, with an estimate $R(\text{aco}(F)) \leq 2R(F)$. It turns out that a similar property holds for $\gamma$-bounded sets without the extra factor 2.

**Lemma 2.2.** Let $F \subset B(X)$ be any $\gamma$-bounded set. Then its closed absolute convex hull $\text{aco}(F)$ with respect to the strong operator topology is $\gamma$-bounded as well, and

$$\gamma(\text{aco}(F)) = \gamma(F).$$

**Proof.** Consider the set

$$\tilde{F} = \{zT: T \in F, z \in \mathbb{C}, |z| \leq 1\}.$$ 

Applying Lemma 2.1 to diagonal matrices, we see that $\tilde{F}$ is $\gamma$-bounded and that $\gamma(\tilde{F}) = \gamma(F)$. Moreover $\text{aco}(F)$ is equal to $\text{co}(\tilde{F})$, the convex hull of $\tilde{F}$. Hence the argument in [7, Lem. 3.2] shows that $\text{aco}(F)$ is $\gamma$-bounded and that $\gamma(\text{aco}(F)) = \gamma(\tilde{F})$. The result follows at once. \qed

Let $Z$ be an arbitrary Banach space. Following [29], we say that a bounded linear map $v: Z \to B(X)$ is $\gamma$-bounded (resp. $R$-bounded) if the set

$$\{v(z): z \in Z, \|z\| \leq 1\}$$

is $\gamma$-bounded (resp. $R$-bounded). In this case, we let $\gamma(v)$ (resp. $R(v)$) denote the $\gamma$-bound (resp. the $R$-bound) of the latter set.

Next we say that a representation $\pi: G \to B(X)$ is $\gamma$-bounded (resp. $R$-bounded) if the set

$$\{\pi(t): t \in G\}$$

is $\gamma$-bounded (resp. $R$-bounded). In this case, we let $\gamma(\pi)$ (resp. $R(\pi)$) denote the $\gamma$-bound (resp. the $R$-bound) of the latter set.

For any bounded representation $\pi: G \to B(X)$, we let $\sigma_\pi : L^1(G) \to B(X)$ denote the associated bounded homomorphism defined by

$$\sigma_\pi(k) = \int_G k(t)\pi(t)dt, \ k \in L^1(G),$$

where the latter integral is defined in the strong sense. It turns out that $\sigma_\pi$ is nondegenerate, that is,

$$\text{Span}\{\sigma_\pi(k)x: k \in L^1(G), x \in X\}$$

(2.1)

is dense in $X$. Moreover, for every nondegenerate bounded homomorphism $\sigma : L^1(G) \to B(X)$, there exists a unique representation $\pi: G \to B(X)$ such that $\sigma = \sigma_\pi$, see [11, Lem. 2.4 and Rem. 2.5].
Lemma 2.3. Let $\pi : G \to B(X)$ be a bounded representation. Then $\pi$ is $\gamma$-bounded if and only if $\sigma_\pi$ is $\gamma$-bounded. Moreover $\gamma(\pi) = \gamma(\sigma_\pi)$ in this case.

Proof. For any $k \in L^1(G)$ such that $\|k\|_1 \leq 1$, the operator $\sigma_\pi(k)$ belongs to the strongly closed absolute convex hull of $\{\pi(t) : t \in G\}$. Hence the ‘only if’ part follows from Lemma 2.2, and we have $\gamma(\sigma_\pi) \leq \gamma(\pi)$.

For the converse implication, we let $(h_\iota)$ be a contractive approximate identity of $L^1(G)$. For any $t \in G$, let $\delta_t$ denote the point mass at $t$. Then for any $k \in L^1(G)$ and any $x \in X$, we have

\[
\pi(t)\sigma_\pi(k)x = \sigma_\pi(\delta_t \ast k)x
= \lim_{\iota} \sigma_\pi(h_\iota \ast \delta_t \ast k)x
= \lim_{\iota} \sigma_\pi(h_\iota \ast \delta_t)\sigma_\pi(k)x.
\]

Hence if we let $Y \subset X$ be the dense subspace defined by (2.1), we have that

\[
\lim_{\iota} \sigma_\pi(h_\iota \ast \delta_t)y = \pi(t)y, \quad y \in Y, \quad t \in G.
\]

Now assume that $\sigma_\pi$ is $\gamma$-bounded and let $y_1, \ldots, y_n \in Y$ and $t_1, \ldots, t_n \in G$. For any $\iota$ and any $k = 1, \ldots, n$, we have $\|h_\iota \ast \delta_{t_\iota}\|_1 \leq 1$. Hence

\[
\left\| \sum_k g_k \otimes \sigma_\pi(h_\iota \ast \delta_{t_\iota})y_k \right\|_{G(X)} \leq \gamma(\sigma_\pi) \left\| \sum_k g_k \otimes y_k \right\|_{G(X)}.
\]

Passing to the limit when $\iota \to \infty$, this yields

\[
\left\| \sum_k g_k \otimes \pi(t_\iota)y_k \right\|_{G(X)} \leq \gamma(\sigma_\pi) \left\| \sum_k g_k \otimes y_k \right\|_{G(X)}.
\]

Since $Y$ is dense in $X$, this implies that $\pi$ is $\gamma$-bounded, with $\gamma(\pi) \leq \gamma(\sigma_\pi)$. \hfill $\Box$

Let $\lambda : G \to B(L^2(G))$ denote the left regular representation. We recall that for any $f \in L^2(G)$,

\[
\lambda(t)f = \delta_t \ast f \quad \text{and} \quad \sigma_\lambda(k) = k \ast f
\]

for any $t \in G$ and any $k \in L^1(G)$. The reduced $C^*$-algebra of $G$ is defined as

\[
C^*_\lambda(G) = \sigma_\lambda(L^1(G)) \subset B(L^2(G)).
\]

We recall that $C^*_\lambda(G)$ is equal to the group $C^*$-algebra $C^*(G)$ if and only if $G$ is amenable, see e.g. [34, (4.21)]. The notion on which we will focus on in Section 4 and beyond is the following.

Definition 2.4. We say that a bounded representation $\pi : G \to B(X)$ extends to a bounded homomorphism $w : C^*_\lambda(G) \to B(X)$ if $w \circ \sigma_\lambda = \sigma_\pi$. 
Note that there exists a bounded operator $w : C^\ast_\lambda(G) \to B(X)$ such that $w \circ \sigma_\lambda = \sigma_\pi$ if and only if there is a constant $C \geqslant 0$ such that

$$
\| \sigma_\pi(f) \| \leqslant C \| \sigma_\lambda(f) \|, \quad f \in L^1(G),
$$

that this extension is unique and is necessarily a homomorphism.

We refer the reader to [11] for some results concerning representations $\pi : G \to B(X)$ extending to an $R$-bounded homomorphism $w : C^\ast_\lambda(G) \to B(X)$ in the case when $G$ is abelian, and their relationships with $R$-bounded spectral measures (see also Remark 4.5).

3. Multipliers on the Kalton–Weis $\ell$-spaces

We will need abstract Hilbert space valued Banach spaces, usually called $\ell$-spaces, which were introduced by Kalton and Weis in the unpublished paper [28]. These $\ell$-spaces allow to define abstract square functions and were used in [28] to deal with relationships between $H^\infty$ calculus and square function estimates. Similar spaces are constructed in [24] for the same purpose, in the setting of noncommutative $L^p$-spaces. Recently, $\ell$-spaces played an important role in the development of vector valued stochastic integration (see in particular [40,41]) and for control theory in a Banach space setting [19]. In this section, we first recall some definitions and basics of $\ell$-spaces, and then we develop specific properties which will be useful in the next section.

Let $X$ be a Banach space and let $H$ be a Hilbert space. We let $\overline{H}$ denote the conjugate space of $H$. We will identify the algebraic tensor product $\overline{H} \otimes X$ with the subspace of $B(H, X)$ of all bounded finite rank operators in the usual way. Namely for any finite families $(\xi_k)_k$ in $H$ and $(x_k)_k$ in $X$, we identify the element $\sum_k \xi_k \otimes x_k$ with the operator $u : H \to X$ defined by letting $u(\eta) = \sum_k \langle \eta, \xi_k \rangle x_k$ for any $\eta \in H$.

For any $u \in \overline{H} \otimes X$, there exists a finite orthonormal family $(e_k)_k$ of $H$ and a finite family $(x_k)_k$ of $X$ such that $u = \sum_k e_k \otimes x_k$. Then we set

$$
\| u \|_G = \left\| \sum_k g_k \otimes x_k \right\|_{G(X)}.
$$

Using Lemma 2.1, it is easy to check that this definition does not depend on the $e_k$’s and $x_k$’s representing $u$. Next for any $u \in B(H, X)$, we set

$$
\| u \|_\ell = \sup\{ \| uP \|_G \mid P : H \to H \text{ finite rank orthogonal projection}\}.
$$

Note that the above quantity may be infinite. Then we denote by $\ell_+(H, X)$ the space of all bounded operators $u : H \to X$ such that $\| u \|_\ell < \infty$. This is a Banach space for the norm $\| \cdot \|_\ell$. We let $\ell(H, X)$ denote the closure of $\overline{H} \otimes X$ in $\ell_+(H, X)$. It is observed in [28] that $\ell(H, X) = \ell_+(H, X)$ provided that $X$ does not contain $c_0$ (we will not use this fact in this paper).

**Proposition 3.1.** Let $S \in B(H)$.

1. For any finite rank operator $u : H \to X$, we have $\| u \circ S \|_G \leqslant \| u \|_G \| S \|$.
2. For any $u \in \ell_+(H, X)$, the operator $u \circ S$ belongs to $\ell_+(H, X)$ and $\| uS \|_\ell \leqslant \| u \|_\ell \| S \|$. 
Proof. Part (1) is a straightforward consequence of Lemma 2.1. Indeed suppose that \( u = \sum_i \overline{e_i} \otimes x_i \) for some finite orthonormal family \((e_i)_i\) of \(H\) and some \(x_i \in X\). Then if \((e'_j)_j\) is an orthonormal basis of \( \text{Span}\{S^*(e_i): i = 1, \ldots, n\} \), we have

\[
u \circ S = \sum_{i,j} \langle e_i, S(e'_j) \rangle \overline{e'_j} \otimes x_i.
\]

Hence

\[
\|u \circ S\|_G = \left\| \sum_{i,j} \langle e_i, S(e'_j) \rangle g_j \otimes x_i \right\|_{G(X)} \leq \left\| \sum_{i,j} \langle e_i, S(e'_j) \rangle \right\|_{\ell^2 \to \ell^2} \left\| \sum_{i} g_i \otimes x_i \right\|_{G(X)} \leq \|S\| \|u\|_G.
\]

To prove (2), consider an arbitrary \( u : H \to X \) and let \( P : H \to H \) be a finite rank orthogonal projection. Then \( SP \) is finite rank hence there exists a finite rank orthogonal projection \( Q : H \to H \) such that \( SP = QSP \). Applying the first part of this proof to \( uQ \), we infer that

\[
\|uSP\|_G = \|uQSP\|_G \leq \|uQ\|_G \|QSP\| \leq \|u\|_\ell \|S\|.
\]

The result follows by passing to the supremum over \( P \). \( \Box \)

Remark 3.2. (1) It is clear from above that for any finite rank \( u : H \to X \), we have \( \|u\|_G = \|u\|_\ell \). More generally for any \( u : H \to X \), we have \( \|u\|_\ell = \sup\{\|uw\|_G\} \), where the supremum runs over all finite rank operators \( w : H \to H \) with \( \|w\| \leq 1 \).

(2) Let \( S \in B(H) \) and let \( \varphi_S : B(H, X) \to B(H, X) \) be defined by \( \varphi_S(u) = u \circ S \). It is easy to check (left to the reader) that the restriction of \( \varphi_S \) to \( \overline{H} \otimes X \) coincides with \( S^* \otimes I_X \).

We will now focus on the case when \( H = L^2(\Omega, \mu) \), for some arbitrary measure space \((\Omega, \mu)\). We will identify \( H \) and \( \overline{H} \) in the usual way. We let \( L^2(\Omega; X) \) be the associated Bochner space and we recall that \( L^2(\Omega) \otimes X \) is dense in \( L^2(\Omega; X) \). There is a natural embedding of \( L^2(\Omega; X) \) into \( B(L^2(\Omega), X) \) obtained by identifying any \( F \in L^2(\Omega; X) \) with the operator

\[
u_F : f \longmapsto \int_\Omega F(t)f(t) \, d\mu(t), \quad f \in L^2(\Omega).
\]

Thus we have the following diagram of embeddings, that we will use without any further reference. For example, it will make sense through these identifications to compute \( \|F\|_\ell \) for any \( F \in L^2(\Omega; X) \).
By a subpartition of \( \Omega \), we mean a finite set \( \theta = \{ I_1, \ldots, I_m \} \) of pairwise disjoint measurable subsets of \( \Omega \) such that \( 0 < \mu(I_i) < \infty \) for any \( i = 1, \ldots, m \). We will use the natural partial order on subpartitions, obtained by saying that \( \theta \leq \theta' \) if and only if each set in \( \theta \) is a union of some sets in \( \theta' \). For any subpartition \( \theta = \{ I_1, \ldots, I_m \} \), we let \( E_\theta : L^2(\Omega) \to L^2(\Omega) \) be the orthogonal projection defined by

\[
E_\theta(f) = \sum_{i=1}^{m} \frac{1}{\mu(I_i)} \left( \int_{I_i} f(t) \, d\mu(t) \right) \chi_{I_i}, \quad f \in L^2(\Omega).
\]

It is plain that \( \lim_{\theta \to \infty} \| E_\theta(f) - f \|_2 = 0 \) for any \( f \in L^2(\Omega) \). Now let \( E^X_\theta : B(L^2(\Omega), X) \to L^2(\Omega) \otimes X \) be defined by \( E^X_\theta(u) = u E_\theta \). Then the above approximation property extends as follows.

**Lemma 3.3.**

(1) For any \( u \in \ell(L^2(\Omega), X) \), \( \lim_{\theta \to \infty} \| E^X_\theta(u) - u \|_\ell = 0 \).

(2) For any \( u \in L^2(\Omega; X) \), \( \lim_{\theta \to \infty} \| E^X_\theta(u) - u \|_{L^2(\Omega; X)} = 0 \).

**Proof.** By Remark 3.2, (2), the restriction of \( E^X_\theta \) to \( L^2(\Omega) \otimes X \) coincides with \( E_\theta \otimes I_X \), hence (1) holds true if \( u \in L^2(\Omega) \otimes X \). According to Proposition 3.1, we have

\[
\| E^X_\theta : \ell(L^2(\Omega), X) \to \ell(L^2(\Omega), X) \| \leq 1.
\]

Since \( L^2(\Omega) \otimes X \) is dense in \( \ell(L^2(\Omega), X) \), part (1) follows by equicontinuity. The proof of (2) is identical. \( \Box \)

**Lemma 3.4.** For any \( u \in B(L^2(\Omega), X) \) and any subpartition \( \theta_0 \) of \( \Omega \),

\[
\| u \|_\ell = \sup \left\{ \| u E_\theta \|_G : \theta \text{ subpartition of } \Omega, \theta \geq \theta_0 \right\}.
\]

**Proof.** Let \( P : L^2(\Omega) \to L^2(\Omega) \) be a finite rank orthogonal projection, and let \( (h_1, \ldots, h_n) \) be an orthonormal basis of its range. Then

\[
u P = \sum_k h_k \otimes u(h_k) \quad \text{and} \quad u E_\theta P = \sum_k h_k \otimes u E_\theta(h_k)
\]

for any subpartition \( \theta \). Since \( E_\theta(h_k) \to h_k \) for any \( k = 1, \ldots, n \), we deduce that

\[
\| u P \|_G = \lim_{\theta \to \infty} \| u E_\theta P \|_G.
\]

By Proposition 3.1, this implies that \( \| u P \|_G \leq \sup_{\theta \geq \theta_0} \| u E_\theta \|_G \) and the result follows at once. \( \Box \)
Let \( \phi : \Omega \to B(X) \) be a bounded strongly measurable function. We may define a multiplication operator \( T_\phi : L^2(\Omega ; X) \to L^2(\Omega ; X) \) by letting
\[
[T_\phi(F)](t) = \phi(t)F(t), \quad F \in L^2(\Omega ; X).
\]
Consider the associated bounded set
\[
F_\phi = \left\{ \frac{1}{\mu(I)} \int_I \phi(t) \, d\mu(t) : I \subset \Omega, \ 0 < \mu(I) < \infty \right\}.
\]
(3.1)
The following is an analog of [24, Prop. 4.4] and extends [28, Prop. 4.11].

**Proposition 3.5.** If the set \( F_\phi \) is \( \gamma \)-bounded, there exists a (necessarily unique) bounded operator
\[
M_\phi : \ell(L^2(\Omega), X) \to \ell_+(L^2(\Omega), X),
\]
such that \( M_\phi \) and \( T_\phi \) coincide on the intersection \( \ell(L^2(\Omega), X) \cap L^2(\Omega ; X) \). Moreover we have
\[
\|M_\phi\| \leq \gamma(F_\phi).
\]

**Proof.** Let \( \mathcal{E} \subset L^2(\Omega) \) be the dense subspace of all simple functions and let \( u \in \mathcal{E} \otimes X \). There exists a subpartition \( \theta_0 = (A_1, \ldots, A_N) \) and some \( x_1, \ldots, x_N \) in \( X \) such that
\[
u = \sum_j \chi_{A_j} \otimes x_j.
\]
Let \( \theta = (I_1, \ldots, I_m) \) be another subpartition and assume that \( \theta_0 \preceq \theta \). Thus there exist \( \alpha_{ij} \in \{0, 1\} \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, N \) such that \( \chi_{A_j} = \sum_i \alpha_{ij} \chi_{I_i} \) for any \( j \). Consequently, we have
\[
u = \sum_{i,j} \alpha_{ij} \chi_{I_i} \otimes x_j \quad \text{and} \quad [T_\phi(u)](t) = \sum_{i,j} \alpha_{ij} \chi_{I_i}(t) \phi(t) x_j.
\]
For any \( i = 1, \ldots, m \), let
\[
T_i = \frac{1}{\mu(I_i)} \int_{I_i} \phi(t) \, d\mu(t).
\]
Then a thorough look at the definition of \( E_\theta^X \) shows that
\[
E_\theta^X(T_\phi(u)) = \sum_{i,j} \alpha_{ij} \chi_{I_i} \otimes T_i(x_j).
\]
Since \( (\mu(I_i)^{-\frac{1}{2}} \chi_{I_i})_i \) is an orthonormal family of \( L^2(\Omega) \), this implies that
\[
\|E_\theta^X(T_\phi(u))\|_G = \left\| \sum_{i,j} \alpha_{ij} \mu(I_i)^{\frac{1}{2}} g_i \otimes T_i(x_j) \right\|_{G(X)}.
\]
Likewise,
\[ \|u\|_G = \left\| \sum_{i,j} \alpha_{ij} \mu(I_i)^{1/2} g_i \otimes x_j \right\|_{G(X)}. \]

Since each \( T_i \) belongs to the set \( F_\phi \), this implies that \( \|E_\phi^X(T_i(u))\|_G \leq \gamma(F_\phi)\|u\|_G \). Taking the supremum over \( \theta \) and applying Lemma 3.4, we obtain that \( T_\phi(u) \in \ell_+(L^2(\Omega), X) \), with
\[ \|T_\phi(u)\|_\ell \leq \gamma(F_\phi)\|u\|_G. \]

This induces a bounded operator \( M_\phi : \ell(L^2(\Omega), X) \to \ell_+(L^2(\Omega), X) \) coinciding with \( T_\phi \) on \( E \otimes X \) and verifying \( \|M_\phi\| \leq \gamma(F_\phi) \).

To show that \( M_\phi \) and \( T_\phi \) coincide on \( \ell(L^2(\Omega), X) \cap L^2(\Omega; X) \), let \( u \) belong to this intersection and note that by construction, \( M_\phi(E_\phi^X(u)) = T_\phi(E_\phi^X(u)) \) for any subpartition \( \theta \). Then the equality \( M_\phi(u) = T_\phi(u) \) follows from Lemma 3.3.

In the rest of this section, we consider natural tensor extensions of the spaces and multipliers considered so far. Let \( N \geq 1 \) be a fixed integer and let \( (e_1, \ldots, e_N) \) denote the canonical basis of \( \ell^2_N \). We let \( \ell^2_N \otimes L^2(\Omega) \) be the Hilbert space tensor product of \( \ell^2_N \) and \( L^2(\Omega) \). For any bounded operator \( u : \ell^2_N \otimes L^2(\Omega) \to X \) and any \( k = 1, \ldots, N \), let \( u_k : L^2(\Omega) \to X \) be defined by \( u_k(f) = u(e_k \otimes f) \). Then the mapping \( u \mapsto \sum_k e_k \otimes u_k \) induces an algebraic isomorphism
\[ B(\ell^2_N \otimes L^2(\Omega), X) \simeq \ell^2_N \otimes B(L^2(\Omega), X). \] (3.2)

Let us now see the effects of this isomorphism on the special spaces considered so far. Let \( \Omega_N = \Omega \times \{1, \ldots, N\} \), so that we have a natural isometric isomorphism
\[ \ell^2_N \otimes L^2(\Omega) = L^2(\Omega_N). \]

Then it is clear that under the identification (3.2), an operator \( u : L^2(\Omega_N) \to X \) belongs to \( L^2(\Omega_N; X) \) if and only if \( u_k \) belongs to \( L^2(\Omega; X) \) for any \( k = 1, \ldots, N \). Moreover this induces an isometric isomorphism identification
\[ L^2(\Omega_N; X) = \ell^2_N(L^2(\Omega; X)). \]

Likewise it is easy to check (left to the reader) that \( u : L^2(\Omega_N) \to X \) belongs to \( \ell_+(L^2(\Omega_N), X) \) (resp. \( \ell(L^2(\Omega_N), X) \)) if and only if \( u_k \) belongs to \( \ell_+(L^2(\Omega), X) \) (resp. \( \ell(L^2(\Omega), X) \)) for any \( k = 1, \ldots, N \), which leads to algebraic isomorphisms
\[ \ell_+(L^2(\Omega_N), X) \simeq \ell^2_N \otimes \ell_+(L^2(\Omega), X) \quad \text{and} \quad \ell(L^2(\Omega_N), X) \simeq \ell^2_N \otimes \ell(L^2(\Omega), X). \] (3.3)

Now let \( \phi : \Omega \to B(X) \) be a bounded strongly measurable function as before and let \( \phi_N : \Omega_N \to B(X) \) be defined by
\[ \phi_N(t, k) = \phi(t), \quad t \in \Omega, \ k = 1, \ldots, N. \]
As in (3.1), we may associate a set \( F_{\phi N} \subset B(X) \) to \( \phi N \). A moment’s thought shows that \( F_{\phi} \subset F_{\phi N} \subset \text{co}(F_{\phi}) \). Hence \( F_{\phi N} \) is \( \gamma \)-bounded if and only if \( F_{\phi} \) is \( \gamma \)-bounded and we have

\[
\gamma(F_{\phi N}) = \gamma(F_{\phi})
\]

in this case. It is clear that under the identifications (3.3), the associated multiplier operator \( M_{\phi N} : \ell(L^2(\Omega_N), X) \to \ell^+(L^2(\Omega_N), X) \) satisfies

\[
M_{\phi N} = I_{\ell^*_N} \otimes M_{\phi}.
\]

(3.4)

4. Characterization of \( \gamma \)-bounded representations of amenable groups

Throughout we let \( G \) be a locally compact group equipped with a left Haar measure and for any measurable \( I \subset G \), we simply let \( |I| \) denote the measure of \( I \). If \( \pi : G \to B(X) \) is any bounded representation and \( \|\pi\| = \sup_t \|\pi(t)\| \), it is plain that for any \( I \subset G \) and any \( z \in X \), we have

\[
\|\pi\|^{-1}|I|^\frac{1}{2}\|z\| \leq \left( \int_I \|\pi(t)z\|^2 dt \right)^{\frac{1}{2}} \leq \|\pi\||I|^\frac{1}{2}\|z\|.
\]

The first part of the following lemma is an analog of this double estimate when the space \( L^2(G; X) \) is replaced by \( \ell^+(L^2(G), X) \). In the second part, we apply the principles explained at the end of the previous section.

Lemma 4.1. Let \( \pi : G \to B(X) \) be a \( \gamma \)-bounded representation and let \( I \subset G \) be any measurable subset of \( G \) with finite measure.

(1) For any \( z \in X \), the function \( t \mapsto \chi_I(t)\pi(t)z \) belongs to \( \ell^+(L^2(G), X) \) and we have

\[
\gamma(\pi)^{-1}|I|^\frac{1}{2}\|z\| \leq \left\| t \mapsto \chi_I(t)\pi(t)z \right\|_\ell \leq \gamma(\pi)|I|^\frac{1}{2}\|z\|.
\]

(2) Let \( N \geq 1 \) be an integer. Let \( z_1, \ldots, z_N \in X \) and let \( F_k(t) = \chi_I(t)\pi(t)z_k \) for any \( k = 1, \ldots, N \). Then

\[
\gamma(\pi)^{-1}|I|^\frac{1}{2}\left\| \sum_k e_k \otimes z_k \right\|_G \leq \left\| \sum_k e_k \otimes F_k \right\|_\ell \leq \gamma(\pi)|I|^\frac{1}{2}\left\| \sum_k e_k \otimes z_k \right\|_G.
\]

Proof. Part (1) is a special case of part (2) so we only need to prove the second statement. The upper estimate is a simple consequence of Proposition 3.5 applied with \( \pi = \phi \), and the discussion at the end of Section 3. Indeed, let \( F_\pi \) be the set associated with \( \pi : G \to B(X) \) as in (3.1). For any \( I \subset G \) with \( 0 < |I| < \infty \), the operator \( |I|^{-1} \int_I \pi(t) dt \) belongs to the strong closure of the absolute convex hull of \( \{\pi(t) : t \in G\} \). Hence \( \gamma(F_\pi) \leq \gamma(\pi) \) by Lemma 2.2. Let

\[
M_\pi : \ell(L^2(G), X) \to \ell^+(L^2(G), X)
\]
be the multiplier operator associated with $\pi$. Then for any $I \subset G$ and any $z \in X$, the function $t \mapsto \chi_I(t)\pi(t)z$ is equal to $M_\pi(\chi_I \otimes z)$. Thus according to (3.4), we have

$$\sum_k e_k \otimes F_k = M_\pi \left( \sum_k e_k \otimes \chi_I \otimes z_k \right).$$

Moreover $(|I|^{-\frac{1}{2}} e_k \otimes \chi_I)_k$ is an orthonormal family of $\ell^2_n \otimes L^2(G)$, hence

$$\left\| \sum_k e_k \otimes \chi_I \otimes z_k \right\|_\ell = |I|^{\frac{1}{2}} \left\| \sum_k e_k \otimes z_k \right\|_G.$$

Consequently we have

$$\left\| \sum_k e_k \otimes F_k \right\|_\ell \leq \gamma(F_{\pi N}) \left\| \sum_k e_k \otimes \chi_I \otimes z_k \right\|_\ell \leq \gamma(\pi) |I|^{\frac{1}{2}} \left\| \sum_k e_k \otimes z_k \right\|_G.$$

We now turn to the lower estimate, for which we will use duality. For any $\varphi_1, \ldots, \varphi_N$ in $X^*$, we set

$$\left\| \varphi_1, \ldots, \varphi_N \right\|_{\ell^*} = \sup \left\{ \sum_{k=1}^N \langle \varphi_k, x_k \rangle : x_1, \ldots, x_N \in X, \left\| \sum_{k=1}^N g_k \otimes x_k \right\|_{G(X)} \leq 1 \right\}.$$ 

We fix some $I \subset G$ with $0 < |I| < \infty$. Then we consider $z_1, \ldots, z_N$ in $X$ and the functions $F_1, \ldots, F_N$ in $L^2(G; X)$ given by $F_k(t) = \chi_I(t)\pi(t)z_k$. By Hahn–Banach there exist $\varphi_1, \ldots, \varphi_N$ in $X^*$ such that

$$\left\| \varphi_1, \ldots, \varphi_N \right\|_{\ell^*} = 1 \quad \text{and} \quad \left\| \sum_k e_k \otimes z_k \right\|_G = \sum_k \langle \varphi_k, z_k \rangle.$$ 

Using the latter equality and Lemma 3.3, (2), we thus have

$$\left|I\right| \left\| \sum_k e_k \otimes z_k \right\|_G = \sum_k \int_I \langle \varphi_k, z_k \rangle \, dt$$

$$= \sum_k \int_I \langle \pi(t^{-1})^* \varphi_k, \pi(t)z_k \rangle \, dt$$

$$= \sum_k \int_G \langle \chi_I(t)\pi(t^{-1})^* \varphi_k, F_k(t) \rangle \, dt$$

$$= \lim_{\theta \to \infty} \sum_k \int_G \langle \chi_I(t)\pi(t^{-1})^* \varphi_k, \left[ E_\theta^X (F_k) \right](t) \rangle \, dt.$$
Let \( \theta = (I_1, \ldots, I_m) \) be a subpartition of \( G \) such that \( I = I_1 \cup \cdots \cup I_n \) for some \( n \leq m \) and let

\[
J_\theta = \sum_k \int_G \langle \chi_I(t)\pi(t^{-1})^* \phi_k, [E^X_\theta(F_k)](t) \rangle \, dt
\]

be the above sum of integrals. For any \( i = 1, \ldots, n \), let

\[
T_i = \frac{1}{|I_i|} \int_{I_i} \pi(t) \, dt \quad \text{and} \quad S_i = \frac{1}{|I_i|} \int_{I_i} \pi(t^{-1}) \, dt.
\]

For any \( k \) we have

\[
E^X_\theta(F_k) = \sum_{i=1}^n \chi_{I_i} \otimes T_i(z_k).
\] (4.1)

We deduce that

\[
J_\theta = \sum_{k=1}^N \sum_{i=1}^n \int_{I_i} \langle \pi(t^{-1})^* \phi_k, T_i(z_k) \rangle \, dt
\]

\[
= \sum_{k=1}^N \sum_{i=1}^n \int_{I_i} \langle \phi_k, \pi(t^{-1})T_i(z_k) \rangle \, dt
\]

\[
= \sum_{k=1}^N \sum_{i=1}^n |I_i| \langle \phi_k, S_i T_i(z_k) \rangle.
\]

According to the definition of the \( \ell^* \)-norm, this identity implies that

\[
|J_\theta| \leq \left\| \sum_k g_k \otimes \left( \sum_i |I_i| S_i T_i(z_k) \right) \right\|_{G(X)}.
\]

Let \( a : \ell^2_{nN} \rightarrow \ell^2_N \) be defined by

\[
a((c_{ik})_{1 \leq i \leq n}) = \left( \sum_i c_{ik} |I_i|^{1/2} \right)_k, \quad c_{ik} \in \mathbb{C}.
\]

Let \( c = (c_{ik}) \) in \( \ell^2_{nN} \). Using Cauchy–Schwarz and the fact that \( |I| = \sum_i |I_i| \), we have

\[
\|a(c)\|^2 = \sum_k \left| \sum_i c_{ik} |I_i|^{1/2} \right|^2
\]

\[
\leq \sum_k \left( \sum_i |c_{ik}|^2 \right) \left( \sum_i |I_i| \right) = |I| \|c\|_2.
\]
Hence \( \|a\| \leq |I|^{\frac{1}{2}} \). Let \((g_{ik})_{i,k \geq 1}\) be a doubly indexed family of independent standard Gaussian variables. According to Lemma 2.1, the latter estimate implies that

\[
\left\| \sum_{i,k} g_k \otimes |I_i|^{\frac{1}{2}} y_{ik} \right\|_{G(X)} \leq \|I\|^{\frac{1}{2}} \sum_{i,k} g_{ik} \otimes y_{ik} \right\|_{G(X)}
\]

for any \( y_{ik} \) in \( X \). We deduce that

\[
|J_\theta| \leq \|I\|^{\frac{1}{2}} \left\| \sum_{i,k} g_{ik} \otimes |I_i|^{\frac{1}{2}} S_i T_i(z_k) \right\|_{G(X)}.
\]

Next observe that by convexity again, we have \( \gamma(\{S_1, \ldots, S_n\}) \leq \gamma(\pi) \). The latter estimate therefore implies that

\[
|J_\theta| \leq \gamma(\pi) |I|^{\frac{1}{2}} \left\| \sum_{i,k} g_{ik} \otimes |I_i|^{\frac{1}{2}} T_i(z_k) \right\|_{G(X)}.
\]

Since \( \left\{ |I_i|^{-\frac{1}{2}} \chi_{I_i} \right\}_i \) is an orthonormal family of \( L^2(G) \), we have, using (4.1),

\[
\left\| \sum_{i,k} g_{ik} \otimes |I_i|^{\frac{1}{2}} T_i(z_k) \right\|_{G(X)} = \left\| \sum_{i,k} e_k \otimes \chi_{I_i} \otimes T_i(z_k) \right\|_G
\]

\[
= \left\| (I_{\ell_\infty^N} \otimes E_{\theta}^X) \left( \sum_{k} e_k \otimes F_k \right) \right\|_G
\]

\[
\leq \left\| \sum_{k} e_k \otimes F_k \right\|_{\ell_\infty}.
\]

Hence

\[
|J_\theta| \leq \gamma(\pi) |I|^{\frac{1}{2}} \left\| \sum_{k} e_k \otimes F_k \right\|_{\ell_\infty},
\]

and passing to the limit when \( \theta \to \infty \), this yields the lower estimate. \( \square \)

**Remark 4.2.** The above lemma remains true if \( \pi(t) \) is replaced by \( \pi(t^{-1}) \). This follows either from the proof itself, or by considering the representation \( \pi^{\text{op}} : G^{\text{op}} \to B(X) \) defined by \( \pi^{\text{op}}(t) = \pi(t^{-1}) \). Here \( G^{\text{op}} \) denotes the opposite group of \( G \), i.e. \( G \) equipped with the reverse product.

The following notion was introduced in [29]. For any \( C^* \)-algebra \( A \), the space \( M_N(A) \) of \( N \times N \) matrices with entries in \( A \) is equipped with its unique \( C^* \)-norm.

**Definition 4.3.** Let \( A \) be a \( C^* \)-algebra and let \( w : A \to B(X) \) be a bounded linear map.
(1) We say that \( w \) is matricially \( \gamma \)-bounded if there is a constant \( C \geq 0 \) such that
\[
\left\| \sum_{i,j=1}^{N} g_i \otimes w(a_{ij}) x_j \right\|_{G(X)} \leq C \left\| [a_{ij}] \right\|_{M_N(A)} \sum_{j=1}^{N} g_j \otimes x_j \right\|_{G(X)}
\]
for any \( N \geq 1 \), for any \( [a_{ij}] \in M_N(A) \) and for any \( x_1, \ldots, x_N \in X \). In this case we let \( \| w \|_{\text{Mat-}\gamma} \) denote the smallest possible \( C \).

(2) We say that \( w \) is matricially \( R \)-bounded if (4.2) holds when the Gaussian sequence \((g_k)_{k}^{\infty}\) is replaced by a Rademacher sequence \((\varepsilon_k)_{k}^{\infty}\), and we let \( \| w \|_{\text{Mat-R}} \) denote the smallest possible constant in this case.

Two simple comments are in order (see [29, Remark 4.2] for details). First, restricting (4.2) to the case when \([a_{ij}]\) is a diagonal matrix, we obtain that any matricially \( \gamma \)-bounded map \( w : \mathcal{A} \to B(X) \) is \( \gamma \)-bounded, with
\[
\gamma(w) \leq \| w \|_{\text{Mat-}\gamma}.
\]
Second, if \( X = H \) is a Hilbert space, then \( \gamma \)-matricial boundedness coincides with complete boundedness and we have \( \| w \|_{\text{Mat-}\gamma} = \| w \|_{\text{cb}} \) (the completely bounded norm of \( w \)). Similar comments apply to \( R \)-boundedness.

The proof of our main result below uses transference techniques from [8] in the framework of \( \ell \)-spaces.

**Theorem 4.4.** Let \( G \) be an amenable locally compact group and let \( \pi : G \to B(X) \) be a bounded representation. The following assertions are equivalent.

(i) \( \pi \) is \( \gamma \)-bounded.

(ii) \( \pi \) extends to a bounded homomorphism \( w : C^\ast_{\lambda}(G) \to B(X) \) (in the sense of Definition 2.4) and \( w \) is \( \gamma \)-bounded.

In this case, \( w \) is matricially \( \gamma \)-bounded and
\[
\gamma(\pi) \leq \gamma(w) \leq \| w \|_{\text{Mat-}\gamma} \leq \gamma(\pi)^2.
\]

**Proof.** Assume (ii) and let \( \sigma_\pi : L^1(G) \to B(X) \) be induced by \( \pi \). Then \( \sigma_\pi = w \circ \sigma_\lambda \) and \( \sigma_\lambda \) is a contraction. Hence \( \sigma_\pi \) is \( \gamma \)-bounded, with \( \gamma(\sigma_\pi) \leq \gamma(\sigma_\lambda) \). Then (i) follows from Lemma 2.3 and we have \( \gamma(\pi) \leq \gamma(w) \).

Assume (i). Our proof of (ii) will be divided into two parts. We first show that for any \( k \in L^1(G) \), we have
\[
\left\| \sigma_\pi(k) \right\| \leq \gamma(\pi)^2 \left\| \sigma_\lambda(k) \right\|.
\]
This implies the existence of \( w : C^\ast_{\lambda}(G) \to B(X) \) extending \( \pi \). Then we will show (4.6), which implies that \( w \) is actually \( \gamma \)-bounded. Although (4.3) is a special case of (4.6), establishing that estimate first makes the proof easier to read.

Let \( k \in L^1(G) \) and assume that \( k \) has a compact support \( \Gamma \subset G \). Let \( V \subset G \) be an arbitrary open neighborhood of the unit \( e \), with \( 0 < |V| < \infty \). We let \( T : L^2(G) \to L^2(G) \) be
the multiplication operator defined by letting \( T(f) = \chi_V f \) for any \( f \in L^2(G) \). Then we let \( S : L^2(G) \to L^2(G) \) be defined by

\[
(S g)(s) = \int_G k(t) g(ts) \, dt, \quad g \in L^2(G), \ s \in G.
\]

Under the natural duality between \( L^2(G) \) and itself, \( S \) is the transposed map of \( \sigma_\lambda(k) \), hence

\[
\| S \| = \| \sigma_\lambda(k) \|. \tag{4.4}
\]

Let \( x \in X \). The set \( \Gamma^{-1} V \subset G \) has a positive and finite measure, hence applying Lemma 4.1 (and Remark 4.2), we see that the function

\[
F : s \mapsto \chi_{\Gamma^{-1} V} (s) \pi (s^{-1}) x
\]

belongs to \( L^2(G; X) \cap \ell_+(L^2(G), X) \). Let \( u : L^2(G) \to X \) be the bounded operator associated to \( F \) and let \( \tilde{u} = u \circ S \circ T \in B(L^2(G), X) \). Consider an arbitrary \( f \in L^2(G) \), for any \( h \in L^2(G) \),

\[
u(h) = \int_G h(s) \chi_{\Gamma^{-1} V} (s) \pi (s^{-1}) x \, ds,
\]

hence according to the definitions of \( T \) and \( S \), we have

\[
\tilde{u}(f) = \int_G \left( \int_G k(t) \chi_V (ts) f(ts) \, dt \right) \chi_{\Gamma^{-1} V} (s) \pi (s^{-1}) x \, ds.
\]

Using Fubini (which is applicable because \( \chi_V f \) is integrable) and the left invariance of \( ds \), this implies

\[
\tilde{u}(f) = \int_G k(t) \left( \int_G \chi_V (ts) f(ts) \chi_{\Gamma^{-1} V} (s) \pi (s^{-1}) x \, ds \right) \, dt
\]

\[
= \int_G k(t) \left( \int_G \chi_V (s) f(s) \chi_{\Gamma^{-1} V} (t^{-1}s) \pi (s^{-1}) x \, ds \right) \, dt
\]

\[
= \int_G \chi_V (s) f(s) \left( \int_G k(t) \chi_{\Gamma^{-1} V} (t^{-1}s) \pi (s^{-1}) x \, dt \right) \, ds.
\]

Since \( k \) is supported in \( \Gamma \) we deduce that

\[
\tilde{u}(f) = \int_G \chi_V (s) f(s) \left( \int_G k(t) \pi (s^{-1}t) x \, dt \right) \, ds. \tag{4.5}
\]
Let $y = \sigma_\pi(k)x$. For any $s \in G$, we have

$$\int_G k(t)\pi(s^{-1}t)x\ dt = \int_G k(t)\pi(s^{-1})\pi(t)x\ dt = \pi(s^{-1})y.$$ 

Thus (4.5) shows that $\tilde{u}$ is the bounded operator associated to the function $\tilde{F}: s \mapsto \chi_V(s)\pi(s^{-1})y$.

By Proposition 3.1, we have $\|\tilde{F}\|_{\ell^\infty} \leq \|S T\|\|F\|_{\ell^1}$. Applying (4.4) and the fact that $T$ is a contraction, we therefore obtain that

$$\|s \mapsto \chi_V(s)\pi(s^{-1})y\|_{\ell^\infty} \leq \|\sigma_\lambda(k)\|\|s \mapsto \chi_{\Gamma^{-1}V}(s)\pi(s^{-1})x\|_{\ell^1}.$$ 

Applying Lemma 4.1 (and Remark 4.2) twice we deduce that

$$|V|^{\frac{1}{2}}\|y\| \leq \gamma(\pi)^2\|\sigma_\lambda(k)\|\|\Gamma^{-1}V\|^{\frac{1}{2}}\|x\|,$$

and hence

$$\|\sigma_\pi(k)x\| \leq \gamma(\pi)^2 \left(\frac{|\Gamma^{-1}V|}{|V|}\right)^{\frac{1}{2}}\|\sigma_\lambda(k)\|\|x\|.$$ 

We now apply the assumption that $G$ is amenable. According to Folner’s condition (see e.g. [8, Chap. 2]), we can choose $V$ such that $|\Gamma^{-1}V|/|V|$ is arbitrarily close to 1. This yields (4.3) when $k$ is compactly supported. Since $\sigma_\lambda$ and $\sigma_\pi$ are continuous, this actually implies (4.3) for any $k \in L^1(G)$.

We now aim at showing that $w:C^*_\gamma(G) \to B(X)$ is matricially $\gamma$-bounded and that $\|w\|_{\text{Mat-}\gamma} \leq \gamma(\pi)^2$. In fact the argument is essentially a repetition of the above one, modulo standard matrix manipulations. We fix some integer $N \geq 1$ and consider $x_1, \ldots, x_N$ in $X$. According to Definition 4.3, it suffices to show that for any $[k_{ij}] \in M_N \otimes L^1(G)$, we have

$$\left\| \sum_{i,j} g_{ij} \otimes \sigma_\pi(k_{ij})x_j \right\|_{G(X)} \leq \gamma(\pi)^2 \left\| [\sigma_\lambda(k_{ij})] \right\|_{M_N(C^*_\gamma(G))} \left\| \sum_j g_j \otimes x_j \right\|_{G(X)}. \quad (4.6)$$

In the sequel we let

$$x = \sum_{j=1}^N e_j \otimes x_j \in \ell^2_N \otimes X.$$ 

Let us identify $M_N \otimes L^1(G)$ with $L^1(G; M_N)$ in the natural way and let $k \in L^1(G; M_N)$ be the $M_N$-valued function corresponding to $[k_{ij}]$. Then

$$(I_{M_N} \otimes \sigma_\pi)([k_{ij}]) = \int_G (k(t) \otimes \pi(t))\ dt \quad \text{in } M_N \otimes B(X).$$
Using the isometric identification
\[ \ell^2_N \otimes L^2(G) = L^2(G; \ell^2_N), \] (4.7)
we can regard \( M_N(C^*_\lambda(G)) \) as a \( C^* \)-subalgebra of \( B(L^2(G; \ell^2_N)) \). In this situation, it is easy to check that the matrix \([\sigma_{\lambda}(k_{ij})]\) corresponds to the operator valued convolution \( g \mapsto k * g \) defined by
\[ (k * g)(s) = \int_G k(t)g(t^{-1}s)dt, \quad g \in L^2(G; \ell^2_N), \ s \in G. \]

Thus showing (4.6) amounts to show that
\[ \left\| \sum_{i=1}^N e_i \otimes \tilde{F}_i \right\|_\ell \leq \gamma(\pi) \left\| \sum_{i=1}^N e_i \otimes F_i \right\|_\ell. \] (4.8)

As in the first part of the proof, we may and do assume that \( k \) has a compact support, which we denote by \( \Gamma \), and we fix an arbitrary open neighborhood \( V \subset G \) of \( e \), with \( 0 < |V| < \infty \). We let \( \hat{T} = I_{\ell^2_N} \otimes T : L^2(G; \ell^2_N) \rightarrow L^2(G; \ell^2_N) \) be the multiplication operator by \( \chi_V \) and we let \( \hat{S} : L^2(G; \ell^2_N) \rightarrow L^2(G; \ell^2_N) \) be the transposed map of \( g \mapsto k * g \). Let \( y_1, \ldots, y_N \) in \( X \) such that
\[ \int_G (k(t) \otimes \pi(t))xdt = \sum_{k=1}^N e_k \otimes y_k. \]

Next for any \( k = 1, \ldots, N \), let
\[ F_k(s) = \chi_{\Gamma^{-1}V}(s)\pi(s^{-1})x_k \quad \text{and} \quad \tilde{F}_k(s) = \chi_V(s)\pi(s^{-1})y_k. \]

Then the argument in the first part of this proof and the identification (4.7) show that
\[ \left\| \sum_k e_k \otimes \tilde{F}_k \right\|_\ell \leq \| \hat{S} \hat{T} \| \left\| \sum_k e_k \otimes F_k \right\|_\ell, \]
and hence
\[ \left\| \sum_k e_k \otimes \tilde{F}_k \right\|_\ell \leq \left\| k * : L^2(G; \ell^2_N) \rightarrow L^2(G; \ell^2_N) \right\| \left\| \sum_k e_k \otimes F_k \right\|_\ell. \]

Now using Lemma 4.1, (2) and arguing as in the first part of the proof, we deduce (4.8). \( \square \)

**Remark 4.5.** If \( G \) is an abelian group and \( \hat{G} \) denotes its dual group, then the Fourier transform yields a natural identification \( C^*_\lambda(G) = C_0(\hat{G}) \). Since abelian groups are amenable, Theorem 4.4 provides a 1–1 correspondence between \( \gamma \)-bounded representations \( G \rightarrow B(X) \) and \( \gamma \)-bounded nondegenerate homomorphisms \( C_0(\hat{G}) \rightarrow B(X) \).
It is shown in [11, Prop. 2.2] (see also [10]) that any $\gamma$-bounded nondegenerate homomorphism $w: C_0(\hat{G}) \to B(X)$ is of the form

$$w(h) = \int_{\hat{G}} h \, dP, \quad h \in C_0(\hat{G}), \quad (4.9)$$

where $P$ is a regular strong operator $\sigma$-additive spectral measure from the $\sigma$-algebra $B(\hat{G})$ of Borel subsets of $\hat{G}$ into $B(X)$. Moreover the range of this spectral measure is $\gamma$-bounded. Conversely, for any such spectral measure, (4.9) defines a $\gamma$-bounded nondegenerate homomorphism $w: C_0(\hat{G}) \to B(X)$. (In [10,11], the authors consider $R$-boundedness only but their results hold as well for $\gamma$-boundedness.)

Hence we obtain a 1–1 correspondence between $\gamma$-bounded representations $G \to B(X)$ and regular, $\gamma$-bounded, strong operator $\sigma$-additive spectral measures $B(\hat{G}) \to B(X)$.

**Remark 4.6.** (1) The above theorem should be regarded as a Banach space version of the Day–Dixmier unitarization Theorem which asserts that any bounded representation of an amenable group $G$ on some Hilbert space $H$ is unitarizable (see [38, Chap. 0]). Indeed when $X = H$, the main implication ‘(i) $\Rightarrow$ (ii)’ of Theorem 4.4 says that any bounded representation $\pi: G \to B(H)$ extends to a completely bounded homomorphism $w: C_\ast\lambda(G) \to B(H)$, with $\|w\|_{cb} \leq \|\pi\|^2$. According to Haagerup’s similarity Theorem [18], this implies the existence of an isomorphism $S: H \to H$ such that $\|S^{-1}\|\|S\| \leq \|\pi\|^2$ and $S^{-1}w(\cdot)S: C_\ast\lambda(G) \to B(H)$ is a $*$-representation. Equivalently, $S^{-1}\pi(\cdot)S$ is a unitary representation.

(2) We cannot expect an extension of Theorem 4.4 for general (= nonamenable) groups. See [38, Chap. 2] for an account on nonunitarizable representations of groups on Hilbert space, and relevant open problems.

5. Representations of nuclear $C^\ast$-algebras on spaces with property $(\alpha)$

We say that a Banach space $X$ has property $(\alpha)$ if there is a constant $\alpha \geq 1$ such that

$$\left\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes t_{ij} x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \leq \alpha \sup_{i,j} |t_{ij}| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \quad (5.1)$$

for any finite families $(x_{ij})_{i,j}$ in $X$ and $(t_{ij})_{i,j}$ in $\mathbb{C}$. This class was introduced in [36] and has played an important role in several recent issues concerning functional calculi and unconditionality (see [7,10,12,27,29]). We note that Banach spaces with property $(\alpha)$ have a finite cotype (because they cannot contain the $\ell_\infty^n$’s uniformly). Thus Rademacher averages and Gaussian averages are equivalent on them. Hence $R$-boundedness and $\gamma$-boundedness (as well as matricial $R$-boundedness and matricial $\gamma$-boundedness) are equivalent notions on these spaces. The class of spaces with property $(\alpha)$ is stable under taking subspaces and comprises Banach lattices with a finite cotype. On the opposite, nontrivial noncommutative $L^p$-spaces do not belong to this class. For a space $X$ with property $(\alpha)$ we let $\alpha(X)$ denote the smallest constant $\alpha$ satisfying (5.1).

Let $A$ be a $C^\ast$-algebra and let $w: A \to B(X)$ be a bounded homomorphism. Assume that $X$ has property $(\alpha)$. It was shown in [11, Cor. 2.19] that if $A$ is abelian, then $w$ is automatically $R$-bounded. By [29], $w$ is actually matricially $R$-bounded. When $G$ is an amenable group, the $C^\ast$-algebra $C_\ast\lambda(G)$ is nuclear (see e.g. [34, (1.31)]). Thus in view of Theorem 4.4, the question
whether any bounded homomorphism \( w : A \to B(X) \) is automatically \( R \)-bounded (or matricially \( R \)-bounded) when \( A \) is nuclear became quite relevant. A positive answer to this question was shown to me by Éric Ricard. I thank him for letting me include this result in the present paper.

**Theorem 5.1.** Let \( X \) be a Banach space with property \( (\alpha) \) and let \( A \) be a nuclear \( C^* \)-algebra. Any bounded homomorphism \( w : A \to B(X) \) is matricially \( R \)-bounded. If further \( w \) is nondegenerate, then

\[
\| w \|_{\text{Mat-}R} \leq K_X \| w \|^2,
\]

where \( K_X \geq 1 \) is a constant only depending on \( \alpha(X) \).

We need two lemmas. In the sequel we let \((\epsilon_j)_{j \geq 1}, (\theta_i)_{i \geq 1}\) and \((\eta_k)_{k \geq 1}\) denote Rademacher sequences. For simplicity we will often use the same notations \( \epsilon_j, \theta_i, \eta_k \) to denote values of these variables. We start with a double estimate which will lead to the result stated in Theorem 5.1 in the case when \( A \) is finite-dimensional. When

\[
A = \bigoplus_{k=1}^N M_{n_k},
\]

we let \((E_{ij}^k)_{1 \leq i,j \leq n_k}\) denote the canonical basis of \( M_{n_k}\), for any \( k = 1, \ldots, N \).

**Lemma 5.2.** Let \( X \) be a Banach space with property \( (\alpha) \), let \( n_1, \ldots, n_N \) be positive integers, and let

\[
w : \bigoplus_{k=1}^N M_{n_k} \to B(X)
\]

be any unital homomorphism. Then for any \( x \in X \), we have

\[
C_X^{-1} \| w \|^{-2} \| x \| \leq \left\| \sum_{k=1}^N \sum_{j=1}^{n_k} \epsilon_j \otimes \eta_k \otimes w(E_{1j}^k) x \right\|_{\text{Rad}(\text{Rad}(X))} \leq C_X \| w \|^2 \| x \|,
\]

where \( C_X \geq 1 \) is a constant only depending on \( \alpha(X) \).

**Proof.** Let \( \epsilon_j = \pm 1, \theta_i = \pm 1 \) and \( \eta_k = \pm 1 \) for \( j, i, k \geq 1 \). We let

\[
\Delta_r = \sum_{k=1}^N \sum_{j=1}^{n_k} n_k^{-\frac{1}{2}} \theta_j E_{1j}^k \quad \text{and} \quad \Delta_c = \sum_{k=1}^N \sum_{i=1}^{n_k} n_k^{-\frac{1}{2}} \theta_i E_{i1}^k.
\]

It is plain that

\[
\| \Delta_r \| = \| \Delta_c \| = 1 \quad \text{and} \quad \Delta_r \Delta_c = \sum_{k=1}^N E_{11}^k.
\]
Since $w$ is a homomorphism, we have

$$w(\Delta_c) \left( \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x \right) = \left( \sum_{k,i} n_k^{-\frac{1}{2}} \theta_i w(E_{ii}^k) \right) \left( \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x \right)$$

$$= \sum_{k,j,i} n_k^{-\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x.$$

We deduce that

$$\left\| \sum_{k,j,i} n_k^{-\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x \right\| \leq \|w\| \left\| \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x \right\| \tag{5.3}$$

Continuing the above calculation, we obtain further that

$$w(\Delta_r) \left( \sum_{k,j,i} n_k^{-\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x \right) = w(\Delta_c) \left( \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x \right)$$

$$= \left( \sum_{k} w(E_{ii}^k) \right) \left( \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x \right)$$

$$= \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x.$$

Consequently,

$$\left\| \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x \right\| \leq \|w\| \left\| \sum_{k,j,i} n_k^{-\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x \right\| \tag{5.4}$$

Now let $U = [u_{ij}^1] \oplus \cdots \oplus [u_{ij}^N]$ be a fixed unitary of $\bigoplus_{k=1}^N M_{n_k}$. Then consider the diagonal (unitary) elements

$$V = \sum_{k=1}^N \sum_{i=1}^{n_k} \eta_k \theta_i E_{ii}^k \quad \text{and} \quad W = \sum_{k=1}^N \sum_{j=1}^{n_k} \varepsilon_j E_{jj}^k.$$

Then $VUW$ is a unitary and

$$w(VUW)x = \sum_{k,j,i} \varepsilon_j \eta_k \theta_i u_{ij}^k w(E_{ij}^k)x.$$  

Since $w$ is unital, we deduce that

$$\|w\|^{-1} \|x\| \leq \left\| \sum_{k,j,i} \varepsilon_j \eta_k \theta_i u_{ij}^k w(E_{ij}^k)x \right\| \leq \|w\| \|x\|. \tag{5.5}$$
Let us apply the above with the special unitary $U$ defined by

$$u^k_{ij} = n_k^{-\frac{1}{2}} \exp\left\{ \frac{2\pi \sqrt{-1}}{n_k} (ij) \right\}, \quad k = 1, \ldots, N, \ i, j = 1, \ldots, n_k.$$  

Its main feature is that $|u^k_{ij}| = n_k^{-\frac{1}{2}}$ for any $i, j, k$. Since $X$ has property $(\alpha)$, this implies that for some constant $C_X \geq 1$ only depending on $\alpha(X)$, we have

$$\left\| \sum_{k,j,i} n_k^{-\frac{1}{2}} \epsilon_j \otimes \eta_k \otimes \theta_i \otimes u^k_{ij} w(E^k_{ij}) x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))} \leq C_X \left\| \sum_{k,j,i} \epsilon_j \otimes \eta_k \otimes \theta_i \otimes u^k_{ij} w(E^k_{ij}) x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))}$$

and

$$\left\| \sum_{k,j,i} \epsilon_j \otimes \eta_k \otimes \theta_i \otimes u^k_{ij} w(E^k_{ij}) x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))} \leq C_X \left\| \sum_{k,j,i} n_k^{-\frac{1}{2}} \epsilon_j \otimes \eta_k \otimes \theta_i \otimes w(E^k_{ij}) x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))}.$$  

Combining with (5.3), (5.4) and (5.5), we get the result. \qed

For any integer $m \geq 1$, we let

$$\sigma_{m,X} : M_m \rightarrow B(\text{Rad}_m(X))$$

be the canonical homomorphism defined by letting $\sigma_{m,X}(a) = a \otimes I_X$ for any $a \in M_m$. According to [29, Lem. 4.3], the mappings $\sigma_{m,X}$ are uniformly $R$-bounded. The same proof shows they are actually uniformly matricially $R$-bounded. We record this fact for further use.

**Lemma 5.3.** Let $X$ be a Banach space with property $(\alpha)$. Then

$$D_X := \sup_{m \geq 1} \|\sigma_{m,X}\|_{\text{Mat}-R} < \infty.$$  

**Proof of Theorem 5.1.** Throughout we let $w : A \rightarrow B(X)$ be a bounded homomorphism. By standard arguments, it will suffice to consider the case when $w$ is nondegenerate. The proof will be divided into three steps.

**First step:** we assume that $A$ is finite-dimensional, $w$ is unital and $\|w\| = 1$. Thus (5.2) holds for some positive integers $n_1, \ldots, n_N$. Let $m = n_1 + \cdots + n_N$, so that $A \subset M_m$ in a canonical way. Let $(\epsilon_{jk})_{j,k \geq 1}$ be a doubly indexed family of independent Rademacher variables, and let

$$S : X \rightarrow \text{Rad}_m(X)$$
be defined by
\[ S(x) = \sum_{k=1}^{N} \sum_{j=1}^{n_k} \varepsilon_{jk} \otimes w(E_{1j}^k)x, \quad x \in X. \]

Let \( Y \subset \text{Rad}_m(X) \) be the range of \( S \). According to Lemma 5.2 and the assumption that \( X \) has property \((\alpha)\), \( S \) is an isomorphism onto \( Y \) and there exist a constant \( B_X \geq 1 \) only depending on \( \alpha(X) \) such that
\[ \|S\| \leq B_X \quad \text{and} \quad \|S^{-1}: Y \to X\| \leq B_X. \quad (5.6) \]

Let \( a = [a_{ij}^1] \oplus \cdots \oplus [a_{ij}^N] \in A \). For any \( x \in X \), we have
\[ \left[ \sigma_{m,X}(a) \right](S(x)) = \sum_{k,j,i} \varepsilon_{ik} \otimes a_{ij}^k w(E_{1j}^k)x. \]

On the other hand we have for any \( k, i \) that \( E_{1i}^k a = \sum_j a_{ij}^k E_{1j}^k \). Hence
\[ w(E_{1i}^k) w(a)x = \sum_j a_{ij}^k w(E_{1j}^k)x, \]
and then
\[ \sum_{k,j,i} \varepsilon_{ik} \otimes a_{ij}^k w(E_{1j}^k)x = \sum_{k,i} \varepsilon_{ik} \otimes w(E_{1i}^k) w(a)x = S(w(a)x). \]

This shows that \( \sigma_{m,X}(a)S = Sw(a) \). Thus \( Y \) is invariant under the action of \( \sigma_{m,X} | A \) and if we let \( \sigma : A \to B(Y) \) be the homomorphism induced by \( \sigma_{m,X} \), we have shown that
\[ w(a) = S^{-1}\sigma(a)S, \quad a \in A. \]

Appealing to (5.6), this implies that
\[ \|w\|_{\text{Mat-}R} \leq \|S^{-1}\| \|S\| \|\sigma\|_{\text{Mat-}R} \leq \|S^{-1}\| \|S\| \|\sigma_{m,X}\|_{\text{Mat-}R} \leq B_X^2 DX. \]

**Second step:** we merely assume that \( A \) is finite-dimensional and \( w \) is unital. Let \( U \) be the unitary group of \( A \) and let \( d\tau \) denote the Haar measure on \( U \). We define a new norm on \( X \) by letting
\[ \|x\| = \left( \int_U \|w(U)x\|^2 \, d\tau(U) \right)^{\frac{1}{2}}, \quad x \in X. \]

Since \( w \) is unital, this is an equivalent norm on \( X \) and
\[ \|w\|^{-1} \|x\| \leq \|x\| \leq \|w\| \|x\|, \quad x \in X. \quad (5.7) \]
Let $\tilde{X}$ be the Banach space $(X, \| \cdot \|)$ and let $\tilde{w} : A \to B(\tilde{X})$ be induced by $w$. It readily follows from (5.7) that

$$\|w\|_{\text{Mat-R}} \leq \|w\|^2 \|\tilde{w}\|_{\text{Mat-R}}.$$  

Using Fubini’s Theorem it is easy to see that we further have

$$\alpha(\tilde{X}) \leq \alpha(X).$$

The first step shows that we have $\|\tilde{w}\|_{\text{Mat-R}} \leq K$ for some constant $K$ only depending on $\alpha(\tilde{X})$. The above observation shows that $K$ does actually depend only on $\alpha(X)$, and we therefore obtain an estimate $\|w\|_{\text{Mat-R}} \leq K_X \|w\|^2$.  

**Third step:** $A$ is infinite-dimensional and $w$ is nondegenerate. We will use second duals in a rather standard way. However the fact that $X$ may not be reflexive leads to some technicalities.

Observe that using Connes’s Theorem [9] and arguing e.g. as in [38, p. 135] (see also [33]), we may assume that there exists a directed net $(A_\lambda)_\lambda$ of finite-dimensional von Neumann subalgebras of $A^{**}$ such that

$$A^{**} = \bigcup \lambda A_\lambda^{w^*}.$$  

Let $u : A \to B(X^{**})$ be the homomorphism defined by letting $u(a) = w(a)^{**}$ for any $a \in A$. According to [29, Lem. 2.3], there exists a (necessarily unique) $w^*$-continuous homomorphism $\hat{u} : A^{**} \to B(X^{**})$ extending $u$. We claim that $\hat{u}(1)x = x$, $x \in X$.

Indeed let $(a_i)_i$ be a contractive approximate identity of $A$ and note that since $w$ is nondegenerate, $w(a_i)$ converges strongly to $I_X$. This implies that $u(a_i)x = w(a_i)x \to x$. Since $a_i \to 1$ in the $w^*$-topology of $A^{**}$, we also have that $u(a_i)x \to \hat{u}(1)x$ weakly, which yields the above equality.

Let $Z \subset X^{**}$ be the range of the projection $\hat{u}(1) : X^{**} \to X^{**}$. The above property means that $X \subset Z$. For any $\lambda$, we let $\hat{u}_\lambda : A_\lambda \to B(Z)$ denote the unital homomorphism induced by the restriction of $\hat{u}$ to $A_\lambda$. Since $X$ has property $(\alpha)$, its second dual $X^{**}$ has property $(\alpha)$ as well and $\alpha(X^{**}) = \alpha(X)$, by (1.1). Moreover $\|\hat{u}_\lambda\| \leq \|\hat{u}\| = \|u\| = \|w\|$. Hence by the second step of this proof, we have a uniform estimate

$$\|\hat{u}_\lambda\|_{\text{Mat-R}} \leq K_X \|w\|^2. \quad (5.8)$$

Consider $[a_{ij}] \in M_n(A)$ and assume that $\|[a_{ij}]\| \leq 1$. Let us regard $[a_{ij}]$ as an element of $M_n(A^{**})$. Then by Kaplansky’s density Theorem (see e.g. [25, Thm. 5.3.5]), there exist a net $(\lambda_s)_s$, and, for any $s$, a matrix $[a_{ij}^s]$ belonging to the unit ball of $M_n(A_{\lambda_s})$, such that for any $i, j = 1, \ldots, n, a_{ij}^s \to a_{ij}$ in the $w^*$-topology of $A^{**}$. Then for any $x_1, \ldots, x_n$ in $X$ and $\varphi_1, \ldots, \varphi_n$ in $X^*$, we have

$$\lim_{s} \sum_{i,j} \langle \varphi_i, \hat{u}_{\lambda_s}(a_{ij}^s)x_j \rangle = \sum_{i,j} \langle \varphi_i, w(a_{ij})x_j \rangle.$$
Applying (5.8) we deduce that
\[
\left\| \sum_{i,j} \varepsilon_i \otimes w(a_{ij}) x_j \right\|_{\text{Rad}(X)} \leq K_X \| w \|^2 \left\| \sum_{j} \varepsilon_j \otimes x_j \right\|_{\text{Rad}(X)}.
\]

Remark 5.4. When \( X = H \) is a Hilbert space, the above proof yields \( K_H = 1 \), and we recover the classical result that any bounded homomorphism \( u : A \to B(H) \) on a nuclear \( C^* \)-algebra is completely bounded, with \( \| u \|_{cb} \leq \| u \|^2 \) (see [5,6,38]).

Remark 5.5. Let \( ||| \gamma \) be a cross-norm on \( \ell^2 \otimes \ell^2 \) (in the sense that \( ||| z_1 \otimes z_2 |||_\gamma = \| z_1 \| \| z_2 \| \) for all \( z_1, z_2 \) in \( \ell^2 \)) and let \( \ell^2 \otimes_\gamma \ell^2 \) denote the completion of the normed space \( (\ell^2 \otimes \ell^2, ||| \gamma ) \). Assume moreover that any bounded operator \( a : \ell^2 \to \ell^2 \) has a bounded tensor extension \( a \otimes I_{\ell^2} : \ell^2 \otimes_\gamma \ell^2 \to \ell^2 \otimes_\gamma \ell^2 \). It follows from the above results that if the Banach space \( \ell^2 \otimes_\gamma \ell^2 \) has property (\( \alpha \)), then \( ||| \gamma \) is equivalent to the Hilbert tensor norm \( |||_2 \), and hence
\[
\ell^2 \otimes_\gamma \ell^2 \approx S^2,
\]
the space of Hilbert–Schmidt operators on \( \ell^2 \). Indeed by the closed graph theorem, there is a constant \( K \geq 1 \) such that \( \| a \otimes I_{\ell^2} \| \leq K \| a \| \) for any \( a \in B(\ell^2) \). Let \( w : B(\ell^2) \to B(\ell^2 \otimes_\gamma \ell^2) \) be the bounded homomorphism defined by \( w(a) = a \otimes I_{\ell^2} \). According to Lemma 5.2, there is a constant \( C \geq 1 \) such that for any \( n \geq 1 \),
\[
C^{-1} \| x \| \leq \left\| \sum_{k=1}^n \varepsilon_k \otimes w(E_{1k}) x \right\|_{\text{Rad}(\ell^2 \otimes_\gamma \ell^2)} \leq C \| x \|
\]
whenever \( x \) is a linear combination of the \( e_i \otimes e_j \), with \( 1 \leq i, j \leq n \). For any scalars \( (s_{ij})_{1 \leq i, j \leq n} \) and any \( \varepsilon_k = \pm 1 \), we have
\[
\sum_{k=1}^n \varepsilon_k w(E_{1k}) \left( \sum_{i,j=1}^n s_{ij} e_i \otimes e_j \right) = e_1 \otimes \left( \sum_{i,j=1}^n \varepsilon_i s_{ij} e_j \right).
\]
Hence for \( x = \sum_{i,j=1}^n s_{ij} e_i \otimes e_j \), we have
\[
\left\| \sum_{k=1}^n \varepsilon_k \otimes w(E_{1k}) x \right\|_{\text{Rad}(\ell^2 \otimes_\gamma \ell^2)} = \left\| \sum_{i=1}^n \varepsilon_i \otimes \left( \sum_{j=1}^n s_{ij} e_j \right) \right\|_{\text{Rad}(\ell^2)} = \left( \sum_{i=1}^n \left\| \sum_{j=1}^n s_{ij} e_j \right\|^2 \right)^{\frac{1}{2}} = \left( \sum_{i,j=1}^n |s_{ij}|^2 \right)^{\frac{1}{2}} = \| x \|_2.
\]
This shows that \( \| x \| \approx \| x \|_2 \) and the result follows by density.
That result is a variant of [31, Thm 2.2], a classical unconditional characterization of \( S^2 \).
6. Examples and applications

In the case when \( X \) has property \((\alpha)\), Theorem 5.1 leads to a simplified version of Theorem 4.4, as follows.

**Corollary 6.1.** Let \( G \) be an amenable group and assume that \( X \) has property \((\alpha)\). Let \( \pi: G \to B(X) \) be a bounded representation. Then \( \pi \) is \( R \)-bounded if and only if it extends to a bounded homomorphism \( w: C_\lambda^*(G) \to B(X) \).

**Proof.** Since \( G \) is amenable, the \( C^* \)-algebra \( C_\lambda^*(G) \) is nuclear. Hence any bounded homomorphism \( w: C_\lambda^*(G) \to B(X) \) is \( R \)-bounded, by Theorem 5.1. The equivalence therefore follows from Theorem 4.4. \( \square \)

The following is a noncommutative generalization of the fact that if \( G \) is an infinite abelian group \( G \) and \( p \neq 2 \), there exist bounded functions \( \hat{G} \to \mathbb{C} \) which are not bounded Fourier multipliers on \( L^p(G) \).

**Corollary 6.2.** Let \( G \) be an infinite amenable group and let \( 1 \leq p < \infty \). Let \( \lambda_p: G \to B(L^p(G)) \) be the ‘left regular representation’ defined by letting \( [\lambda_p(t)f](s) = f(t^{-1}s) \) for any \( f \in L^p(G) \). Then \( \lambda_p \) extends to a bounded homomorphism \( C_\lambda^*(G) \to B(L^p(G)) \) (if and) only if \( p = 2 \).

**Proof.** Assume that \( \lambda_p \) has an extension to \( C_\lambda^*(G) \). Since \( L^p(G) \) has property \((\alpha)\), Corollary 6.1 ensures that \( \{\lambda_p(t): t \in G\} \) is \( R \)-bounded. According to [11, Prop. 2.11], this implies that \( p = 2 \). (The latter paper considers abelian groups only but the proof works as well in the nonabelian case.) \( \square \)

We will now focus on the three classical groups \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{T} \). We wish to mention the remarkable work of Berkson, Gillespie and Muhly [2,3] on bounded representations of these groups on UMD Banach spaces. Roughly speaking, their results say that when \( G = \mathbb{Z} \), \( \mathbb{R} \) or \( \mathbb{T} \), and \( X \) is UMD, any bounded representation \( \pi: G \to B(X) \) gives rise to a spectral family \( E_\pi \) of projections allowing a natural spectral decomposition of \( \pi \) (see [2,3] for a precise statement). According to Remark 4.5, our results imply that if \( \pi: G \to B(X) \) is actually \( \gamma \)-bounded, then \( E_\pi \) is induced by a spectral measure.

Representations \( \pi: \mathbb{Z} \to B(X) \) are of the form \( \pi(k) = T^k \), where \( T: X \to X \) is a bounded invertible operator. Furthermore \( C_\lambda^*(\mathbb{Z}) \) coincides with \( C(\mathbb{T}) \). In the next statement, we let \( \kappa \in C(\mathbb{T}) \) be the function defined by \( \kappa(z) = z \), and we let \( \sigma(T) \) denote the spectrum of \( T \). We refer to [16] for some background on spectral decompositions and scalar type operators.

**Proposition 6.3.** Let \( T: X \to X \) be a bounded invertible operator.

1. The set \( \{T^k: k \in \mathbb{Z}\} \) is \( \gamma \)-bounded if and only if there exists a \( \gamma \)-bounded unital homomorphism \( w: C(\mathbb{T}) \to B(X) \) such that \( w(\kappa) = T \).
2. Assume that \( X \) has property \((\alpha)\). Then the following are equivalent.
   i. The set \( \{T^k: k \in \mathbb{Z}\} \) is \( R \)-bounded.
   ii. There is a bounded unital homomorphism \( w: C(\mathbb{T}) \to B(X) \) such that \( w(\kappa) = T \).
   iii. \( T \) is a scalar type spectral operator and \( \sigma(T) \subset \mathbb{T} \).
Proof. Part (1) corresponds to Theorem 4.4 when \( G = \mathbb{Z} \) and in part (2), the equivalence between (i) and (ii) is given by Corollary 6.1. The implication ‘(iii) \( \Rightarrow \) (ii)’ follows from [16, Thm. 6.24]. Conversely, assume (ii). Then by [29, Lem. 3.8], \( \sigma(T) \subset \mathbb{T} \) and there is a bounded unital homomorphism \( v : C(\sigma(T)) \to B(X) \) (obtained by factorizing \( w \) through its kernel) such that \( v(\kappa) = T, \sigma(v(f)) = f(\sigma(T)) \) for any \( f \in C(\sigma(T)) \), and \( v \) is an isomorphism onto its range. Since \( X \) has property \( (\alpha) \), it cannot contain \( c_0 \). Hence by [15, VI, Thm. 15], any bounded map \( C(\sigma(T)) \to X \) is weakly compact. Applying [16, Thm. 6.24], we deduce the assertion (iii).

Turning to representations of the real line, let \((T_t)_{t \in \mathbb{R}}\) be a bounded \( c_0 \)-group on \( X \), and let \( A \) denote its infinitesimal generator. Its spectrum \( \sigma(A) \) is included in the imaginary axis \( i\mathbb{R} \). Let \( \text{Rat} \subset C_0(\mathbb{R}) \) denote the subalgebra of all rational functions \( g \) with poles lying outside the real line and such that \( \deg(g) \leq -1 \). Rational functional calculus yields a natural definition of \( g(iA) \) for any such \( g \). The following is the analog of Proposition 6.3 for the real line and has an identical proof. Note that a special case of that result is announced in [43, Cor. 7.6], as a consequence of some unpublished work of Kalton and Weis.

**Proposition 6.4.** Let \((T_t)_{t \in \mathbb{R}}\) be a bounded \( c_0 \)-group with generator \( A \).

1. The set \( \{T_t: t \in \mathbb{R}\} \) is \( \gamma \)-bounded if and only if there exists a \( \gamma \)-bounded nondegenerate homomorphism \( w : C_0(\mathbb{R}) \to B(X) \) such that \( w(g) = g(iA) \) for any \( g \in \text{Rat} \).

2. Assume that \( X \) has property \( (\alpha) \). Then the following are equivalent.
   (i) The set \( \{T_t: t \in \mathbb{R}\} \) is \( R \)-bounded.
   (ii) There is a bounded nondegenerate homomorphism \( w : C_0(\mathbb{R}) \to B(X) \) such that \( w(g) = g(iA) \) for any \( g \in \text{Rat} \).
   (iii) \( A \) is a scalar type spectral operator.

Let \((X_n)_{n \in \mathbb{Z}}\) be an unconditional decomposition of a Banach space \( X \). For any bounded sequence \( \theta = (\theta_n)_{n \in \mathbb{Z}} \) of complex numbers, let \( T_\theta : X \to X \) be the associated multiplier operator defined by

\[
T_\theta \left( \sum_n x_n \right) = \sum_n \theta_n x_n, \quad x_n \in X_n.
\]

We say that the decomposition \((X_n)_{n \in \mathbb{Z}}\) is \( \gamma \)-unconditional (resp. \( R \)-unconditional) if the set

\[
\{ T_\theta: \theta \in \ell^\infty_\mathbb{Z}, \|\theta\|_\infty \leq 1 \} \subset B(X)
\]

is \( \gamma \)-bounded (resp. \( R \)-bounded).

For any bounded representation \( \pi : \mathbb{T} \to B(X) \), and any \( n \in \mathbb{Z} \), we let \( \hat{\pi}(n) \) denote the \( n \)th Fourier coefficient of \( \pi \), defined by

\[
\hat{\pi}(n) = \frac{1}{2\pi} \int_0^{2\pi} \pi(t)e^{-int} \, dt.
\]

Equivalently, \( \hat{\pi}(n) = \sigma_\pi(t \mapsto e^{-int}) \). Each \( \hat{\pi}(n) : X \to X \) is a bounded projection, the ranges \( \hat{\pi}(n)X \) form a direct sum and \( \bigoplus_n \hat{\pi}(n)X \) is dense in \( X \). However \((\hat{\pi}(n)X)_{n \in \mathbb{Z}}\) is not a Schauder
decomposition in general. (Indeed, take $X = L^1(\mathbb{T})$ and let $\pi$ be the regular representation of $\mathbb{T}$ on $L^1(\mathbb{T})$. Then $\hat{\pi}(n)f = \hat{f}(-n)e^{-in\pi}$ for any $f$, and the Fourier decomposition on $L^1(\mathbb{T})$ is not a Schauder decomposition.)

**Proposition 6.5.** Let $\pi : \mathbb{T} \rightarrow B(X)$ be a bounded representation.

1. $\pi$ is $\gamma$-bounded if and only if $(\hat{\pi}(n)X)_{n \in \mathbb{Z}}$ is a $\gamma$-unconditional decomposition of $X$.
2. Assume that $X$ has property $(\alpha)$. Then $\pi$ is $R$-bounded if and only if $(\hat{\pi}(n)X)_{n \in \mathbb{Z}}$ is an unconditional decomposition of $X$.

**Proof.** Assume that $\pi$ extends to a bounded homomorphism $w : c_{0,\mathbb{Z}} \rightarrow B(X)$. Then for any finitely supported scalar sequence $(\theta_n)_{n \in \mathbb{Z}}$, we have

$$w((\theta_n)_n) = \sum_n \theta_{-n}\hat{\pi}(n).$$

Since $w$ is nondegenerate and bounded, this implies that $(\hat{\pi}(n)X)_{n \in \mathbb{Z}}$ is an unconditional decomposition of $X$. It is clear that $(\hat{\pi}(n)X)_{n \in \mathbb{Z}}$ is actually $\gamma$-unconditional if and only if $w$ is $\gamma$-bounded. The result therefore follows from Theorem 4.4 and Corollary 6.1.

In the last part of this section, we are going to discuss the failure of the equivalence $(i) \Leftrightarrow (ii)$ in Proposition 6.3, (2), when $X$ is not supposed to have property $(\alpha)$. We use ideas from [12] and [29]. Let $(P_n)_{n \geq 1}$ be a sequence of bounded projections on some Banach space $X$. We say that this sequence is unconditional if $(P_nX)_{n \geq 1}$ is an unconditional decomposition of $X$, and we say that $(P_n)_{n \geq 1}$ has property $(\alpha)$ if further there is a constant $\alpha \geq 1$ such that

$$\left\lVert \sum_{i,j} \varepsilon_i \otimes t_{ij} P_j(x_i) \right\rVert_{\text{Rad}(X)} \leq \alpha \sup_{i,j} |t_{ij}| \left\lVert \sum_{i,j} \varepsilon_i \otimes P_j(x_i) \right\rVert_{\text{Rad}(X)} \quad (6.1)$$

for any finite families $(x_j)_j$ in $X$ and $(t_{ij})_{i,j}$ in $\mathbb{C}$. If $(P_n)_{n \geq 1}$ is unconditional, then we have a uniform equivalence

$$\left\lVert \sum_{i,j} \varepsilon_i \otimes P_j(x_i) \right\rVert_{\text{Rad}(X)} \approx \left\lVert \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes P_j(x_i) \right\rVert_{\text{Rad}(\text{Rad}(X))}.$$

Hence if $X$ has property $(\alpha)$, any unconditional sequence $(P_n)_{n \geq 1}$ on $X$ has property $(\alpha)$. Conversely, let $P_n : \text{Rad}(X) \rightarrow \text{Rad}(X)$ be the canonical projection defined by letting

$$P_n \left( \sum_{j \geq 1} \varepsilon_j \otimes x_j \right) = \varepsilon_n \otimes x_n.$$

Then $(P_n)_{n \geq 1}$ is unconditional on $\text{Rad}(X)$ for any $X$, and this sequence has property $(\alpha)$ on $\text{Rad}(X)$ if and only if $X$ has property $(\alpha)$.

Here is another typical example. For any $1 \leq p < \infty$, let $S^p$ denote the Schatten $p$-class on $\ell^2$ and regard any element of $S^p$ as a bi-infinite matrix $a = [a_{ij}]_{i,j \geq 1}$ in the usual way. We let $E_{ij}$
denote the matrix units of $B(\ell^2)$ and write $a = \sum_{i,j} a_{ij}E_{ij}$ for simplicity. For any $n \geq 1$, let $P_n : S^p \to S^p$ be the ‘$n$th column projection’ defined by
\[
P_n \left( \sum_{i,j} a_{ij}E_{ij} \right) = \sum_i a_{in}E_{in}.
\]
It is clear that the sequence $(P_n)_{n \geq 1}$ is unconditional on $S^p$. However if $p \neq 2$, $(P_n)_{n \geq 1}$ does not have property $(\alpha)$. This follows from the lack of unconditionality of the matrix decomposition on $S^p$. Indeed, let $a = \sum_{i,j} a_{ij}E_{ij}$, let $(t_{ij})_{i,j}$ be a finite family of complex numbers and set $x_i = \sum_j a_{ij}E_{ij}$ for any $i \geq 1$. Then
\[
\left\| \sum_{i,j} \varepsilon_i \otimes P_j(x_i) \right\|_{\text{Rad}(X)} = \|a_{ij}\|_{S^p} \quad \text{and} \quad \left\| \sum_{i,j} \varepsilon_i \otimes t_{ij}P_j(x_i) \right\|_{\text{Rad}(X)} = \|t_{ij}a_{ij}\|_{S^p}.
\]
Hence (6.1) cannot hold true.

**Proposition 6.6.** Assume that $X$ has a finite cotype and admits a sequence $(P_n)_{n \geq 1}$ of projections which is unconditional but does not have property $(\alpha)$. Then there exists an invertible operator $T : X \to X$ such that the set $\{T^k : k \in \mathbb{Z}\}$ is not $R$-bounded, but there exists a bounded unital homomorphism $w : C(\mathbb{T}) \to B(X)$ such that $w(\kappa) = T$.

**Proof.** Let $(\zeta_j)_{j \geq 1}$ be a sequence of distinct points of $\mathbb{T}$. Since $(P_n)_{n \geq 1}$ is unconditional, one defines a bounded unital homomorphism $w : C(\mathbb{T}) \to B(X)$ by letting
\[
w(f) = \sum_{j=1}^{\infty} f(\zeta_j)P_j, \quad f \in C(\mathbb{T}).
\]
Arguing as in [29, Remark 4.6], we obtain that $w$ is not $R$-bounded.

Let $T = w(\kappa)$, this is an invertible operator. If $\{T^k : k \in \mathbb{Z}\}$ were $R$-bounded, then $w$ would be $R$-bounded as well, by Theorem 4.4 and the cotype assumption. \qed

According to the above discussion, Proposition 6.6 applies on $S^p$ for any $1 \leq p \neq 2 < \infty$, as well as on any space of the form $\text{Rad}(X)$ when $X$ does not have property $(\alpha)$ but has a finite cotype. This leads to the following general question:

When $X$ does not have property $(\alpha)$, find a characterization of bounded invertible operators $T : X \to X$ such that $\pi : k \in \mathbb{Z} \mapsto T^k$ extends to a bounded homomorphism $C(\mathbb{T}) \to B(X)$.

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**References**