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Abstract

We prove that a locally cobipartite graph on n vertices contains a family of at most n cliques that cover its edges. This is related to Opsut's conjecture that states the competition number of a locally cobipartite graph is at most two. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A *food web* D is an digraph (V,A), where V is a collection of species and there is an arc from species x to species y if x preys on y. We shall follow common practice and assume that D is acyclic. The *competition graph of* D is an undirected graph $G=(V,E)$ defined on the vertices of D with an edge between species x and species y if x and y share a common prey. Cohen [1] introduced competition graphs during his study of the ecological phase spaces of food webs. Many researchers have since studied competition graphs (see, for example, Kim's thesis [2] or Lundgren's article [4]).

In this paper we consider a problem motivated by a conjecture of Opsut [5]. An undirected graph is a *competition graph* if it is the competition graph of some acyclic food web. Any graph G can be made into a competition graph by adding $|E(G)|$ isolated vertices because this expanded graph is the competition graph of the food web

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that contains a unique prey for each pair of adjacent vertices in the graph G. Roberts [7] defined the *competition number* of a graph G, denoted $k(G)$, as the least number of isolated vertices that must be added to G to produce a competition graph. Roberts observed that chordal graphs have competition number at most one, and Opsut proved that line graphs have competition number at most two. A graph is *locally cobipartite* if the vertices in the neighborhood of any vertex can be covered with at most two cliques. Opsut [5] conjectured that the competition number of a locally cobipartite graph is at most two. This conjecture remains open. Others have considered Opsut's conjecture, for example Kim and Roberts [3] and Wang [9]. Opsut proved that, for any graph $G=(V,E)$, the competition number satisfies $k(G) \geqslant \theta'(G) - |V(G)| + 2$, where $\theta'(G)$ is the *edge clique covering number of* G and is equal to the least number of cliques of G that cover E. Thus, Opsut's conjecture implies that a locally cobipartite graph on n vertices contains a family of at most n cliques that cover its edges. We prove this corollary of Opsut's conjecture in Section 2. The result is sharp, as seen by considering chordless cycles.

The edge clique covering number has been widely studied in the literature. It has applications to many important problems including representing intersections of sets using graphs and many assignment-type problems. In general it is hard to calculate because, given a graph G and an integer k, deciding whether $\theta'(G) \leqslant k$, is NP-complete (see [5]). For a view of the edge clique covering number from an extremal graph theory perspective see the survey by Pyber [6]. For applications of the edge clique covering number to assignment type problems such as the traffic phasing problem, the reader is referred to the survey by Roberts [8].

2. Edge clique covering

In this section we prove the main theorem. We first introduce some notation.

Let G be an undirected graph with vertex set V and edge set E. All of the graphs in this paper have neither multiple edges nor loops. For $X, Y \subseteq V$, set $E(X,Y) = \{xy \in E : x \in X, y \in Y\}$. We use $E(X)$ as an abbreviation for $E(X,X)$. The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V : uv \in E\}$. The *closed neighborhood* of v is the set $N[v] = \{v\} \cup N(v)$. For $S \subseteq V$, the graph *induced* by S is the graph with vertex set S and edge set $E(S)$. Let $G[v]$ be the graph induced by $N[v]$. The *distance* in G from the vertex u to the vertex v is the number of edges in the shortest path connecting u with v; it is denoted $d(u,v)$.

The *clique cover number* of G, denoted $\theta(G)$, is the minimum nonnegative integer k such that there is a family of k of G cliques covering the vertices of G. A graph G is *cobipartite* if $\theta(G)=2$. A graph G is *locally cobipartite* if $\theta(G[v]) \leqslant 2$, for all $v \in V$. The *edge clique cover number* of G, denoted $\theta'(G)$, is the least k such that there is a family of k cliques of G covering the edges of G. The complete bipartite graph $K_{1,3}$ is referred to as the *claw*. We shall use the observation that locally cobipartite graphs do not contain an induced claw; that is, they are claw-free.

The notation $G[X,Y]$ represents a cobipartite graph G with clique partition X and Y. A cobipartite graph $G[X,Y]$ with $X=\{x_1,\dots,x_n\}$ satisfying $Y=N(x_1)$ and

$$N(x_1)\supseteq\cdots\supseteq N(x_n)$$

is called a *nested graph*. The number of distinct nonempty sets in the chain above is the *depth* of the nested graph.

Lemma 1. *If $G[X,Y]$ is a nested graph with depth r, then*
(i) *there is a family of r cliques covering all edges in $E(X,Y)\cup E(Y)$, and*
(ii) *there is a set $\{y_1,\dots,y_r\}\subseteq Y$ such that $N(y_i)\neq N(y_j)$, for $1\leqslant i<j\leqslant r$.*

Proof. Define equivalence classes $[x_i]=\{x\in X: N(x)=N(x_i)\}$, for $1\leqslant i\leqslant n$. Observe $[x_i]\cup(N(x_i)\cap Y)$ induces a clique. Moreover, there are precisely r distinct cliques $Q_1,\dots,Q_r$ defined in this way because the depth of G is r. These r cliques $Q_1,\dots,Q_r$ cover the edges $E(X,Y)\cup E(Y)$. This proves (i). To prove (ii), consider the r distinct cliques $Q_1,\dots,Q_r$. These can be arranged so that $(Q_1\cap Y)\supset\cdots\supset(Q_r\cap Y)$. Now define $y_i\in(Q_i\cap Y)\setminus(Q_{i+1}\cap Y)$, for $1\leqslant i<r$, and $y_r\in(Q_r\cap Y)$. Clearly $y_1,\dots,y_r$ have distinct neighborhoods. □

For disjoint vertex subsets S and T, let $\theta'(S,T)$ represent the least k such that there are k cliques of G that cover $E(S,T)$. Observe that, if S and T induce cliques in G, then $\theta'(S,T)\leqslant\min\{|S|,|T|\}$.

Suppose v is a vertex in a locally cobipartite graph. Define

$$t(v)=\min\{\theta'(A,B): A \text{ and } B \text{ are cliques that partition } N(v)\}.$$

Set $t(G)=\min\{t(v)\colon v\in V\}$.

Finally, let A and B be two disjoint subsets of vertices inducing cliques. Define $s(A,B)$ to be the number of equivalence classes in the partition of A determined by the equivalence relation in which two vertices of A are equivalent if they have exactly the same neighbors in B. Observe carefully the asymmetry: $s(A,B)$ and $s(B,A)$ need not be equal. For example, consider a traingle with a pendent edge with vertices a_1,a_2, b_1,b_2. Suppose $\{a_1,a_2,b_1\}$ induces a triangle and b_2 is a pendent vertex adjacent to b_1. If $A=\{a_1,a_2\}$ and $B=\{b_1,b_2\}$, then $s(A,B)=1$ whereas $S(B,A)=2$.

Define

$$s(v)=\min\{s(A,B): A,B \text{ cliques partitioning } N(v) \text{ and } \theta'(A,B)=t(v)\}.$$

It is important to note that $s(v)\geqslant t(v)$.

We are now ready to prove the main result.

Theorem 2. *A locally cobipartite graph on n vertices contains a family of at most n cliques that cover its edges.*

Proof. It suffices to prove the theorem for connected graphs. Suppose $G=(V,E)$ is a connected locally cobipartite graph with $|V|=n$ vertices. We may assume $n\geqslant 3$. For convenience, set $t=t(G)$. Choose v_0 from $\{u\in V\colon t(u)=t\}$ so that $s(v_0)$ is minimum. Set $\ell=\max\{d(u,v_0)\colon u\in V\}$. Define the *level sets* $V_i=\{u\in V\colon d(u,v_0)=i\}$, for $0\leqslant i\leqslant\ell$. Because G is connected, $V=\bigcup_{i=0}^{\ell} V_i$. For $1\leqslant i\leqslant\ell$ and $u\in V_i$, define $N^-(u)=N(u)\cap V_{i-1}$. For $0\leqslant i<\ell$, define $N^+(u)=N(u)\cap V_{i+1}$, and for convenience, set $N^+(u)=\emptyset$, for $u\in V_\ell$. Extend this notation naturally to subsets $S\subset V$ by defining, $N(S)=\bigcup_{u\in S} N(u)$, $N^-(S)=\bigcup_{u\in S} N^-(u)$, and $N^+(S)=\bigcup_{u\in S} N^+(s)$.

An edge $xy\in E$ is *free* if $d(x,v_0)=d(y,v_0)$ and $N^-(x)\cap N^-(y)=\emptyset$. Note that if xy is a free edge, then x and y are in the same level set. No edges in $G[v_0]$ are free by definition. Define

$$C[x]=\{x\}\cup N^+(x)\cup\{u\in V\colon ux \text{ is a free edge}\}.$$

Two important facts to notice: if xy is a free edge, then $\{x,y\}\subseteq C[x]\cap C[y]$, and $C[u]=\{u\}\cup N^+(u)$, for $u\in V_1$.

We now present several useful consequences of the claw-freeness of G.

Claim 1. (a) *For all $x\neq v_0, N^+(x)$ is a clique.*
(b) *If xy is free, then $N^+(x)=N^+(y)$.*
(c) *If xy and yz are free, then $xz\in E(G)$.*
(d) *For all $x\neq v_0, C[x]$ is clique.*
(e) *The $n-1$ cliques $\{C[x]\}_{x\neq v_0}$ cover all edges of G not in $G[v_0]$.*
(f) *If $xy\in E(V_i)$, then $N^-(x)\not\subseteq N^-(y)$ implies $N^+(x)\subseteq N^+(y)$.*

Proof. To see that $N^+(x)$ is a clique, for all $x\neq v_0$, it is enough to observe that $N^-(x)\neq\emptyset$ and G is claw-free. This proves (a). Suppose that xy is free. To prove (b), it suffices to show that $N^+(x)\subseteq N^+(y)$. If there were some $z\in N^+(x)\setminus N^+(y)$, then for any $w\in N^-(x)$, the four vertices w,x,y,z would form a claw centered at x. This proves (b). Suppose that xy and yz are free, and let $w\in N^-(y)$. Because $w\notin N^-(x)\cup N^-(z)$, the edge xz must be present in G to avoid inducing a claw on w,x,y, and z. So (c) is true. Part (d) follows immediately from the definition of $C[x]$ and (a)–(c). To prove (e) observe that there are only two types of edges in G: edges between level sets, called vertical edges, and edges inside a level set, called horizontal edges. Clearly all vertical edges not in $G[v_0]$ are covered by $\{C[x]\}_{x\neq v_0}$. A horizontal free edge xy is covered twice, once by $C[x]$ and once by $C[y]$. A horizontal edge xy that is not free is covered by $C[z]$, where $z\in N^-(x)\cap N^-(y)$. So all edges of G not in $G[v_0]$ are covered by $\{C[x]\}_{x\neq v_0}$, and (e) is proven. Suppose $N^-(x)\not\subseteq N^-(y)$ for some $x,y\in V_i$ and some $1\leqslant i\leqslant\ell$. Reasoning as in part (b), if there were some $z\in N^+(x)\setminus N^+(y)$, then for any $w\in N^-(x)\setminus N^-(y)$, the four vertices w,x,y,z would form a claw centered at x. This establishes (f) and concludes the proof of the claim. □

Let $A(v_0)$ and $B(v_0)$ be two cliques that partition $V_1=N(v_0)$ such that $\theta'(A(v_0),B(v_0))=t$ and, subject to this constraint, $s(A(v_0),B(v_0))$ is minimum. Because $\min\{|A(v_0)|,|B(v_0)|\}\geqslant\theta'(A(v_0),B(v_0))=t$, the number of vertices satisfies $n\geqslant 2t+1$. If $\ell=1$, then there is a cover of $E(G)$ using $t+2$ cliques, namely $\{v_0\}\cup A(v_0)$, $\{v_0\}\cup B(v_0)$, and the t cliques of G that cover the edges $E(A(v_0),B(v_0))$. Because $t+2\leqslant\max\{3,2t+1\}\leqslant n$, we may assume that $\ell\geqslant 2$.

Claim 1 suggests a cover of E using $n+t+1$ cliques, namely the cover $\mathscr{F}$ consisting of the cliques $\{v_0\}\cup A(v_0)$, $\{v_0\}\cup B(v_0)$, $C[x]$ for all $x\neq v_0$, and the t cliques of G that cover the edges $E(A(v_0),B(v_0))$. Our goal is to modify $\mathscr{F}$ by removing some cliques and adding others so that eventually the new family is a cover using at most n cliques. The reader should keep in mind that $\mathscr{F}$ is the base cover from which we work. The remainder of the proof describes how to modify $\mathscr{F}$. As an example, the reader can verify that the graph $\overline{3K_2}$ is a cobipartite graph with six vertices and $\mathscr{F}$ contains eight cliques.

For any $u\in V_1$, let $A(u)$ and $B(u)$ be two cliques that partition $N(u)$ such that $\theta'(A(u),B(u))=t(u)$ and $v_0\in B(u)$. Also define $X(u)=A(u)\cap V_1$. A vertex $u\in V_1$ is *maximal* if there is no vertex $v\in V_1$ such that $N^+(u)\subsetneq N^+(v)$. Because $v_0\in B(u)$, it follows that $B(u)\subseteq\{v_0\}\cup V_1$ and $A(u)$ is the disjoint union of $X(u)$ and $N^+(u)$.

A set of vertices $W\subset V_k$ is *diverse* (from level $k\geqslant 2$) if $N^-(x)\neq N^-(y)$, for all $x,y\in W$. For $u\in V$, define $m(u)$ to be the maximum cardinality of a diverse subset of $N^+(u)$. Suppose $u\in V_1$ is a maximal vertex. Let $W(u)=\{w_1,\dots,w_{m(u)}\}$ denote a maximum diverse subset of $N^+(u)$. Part (a) of Claim 1 implies that $W(u)$ induces a clique. It then follows from part (f) of Claim 1 that $W(u)$ can be ordered so that $N^+(w_1)\supseteq\cdots\supseteq N^+(w_m)$. We shall use this property during the proof of Claim 3.

First we establish an important sequence of inequalities in the following claim.

Claim 2. *For any maximal* $u\in V_1$,

$$m(u)+|X(u)|\geqslant s(u)\geqslant t(u)\geqslant t.$$

Proof. To prove Claim 2, it suffices to prove $m(u)+|X(u)|\geqslant s(u)$ because the other two inequalities follow directly from the definitions of $s(u)$, $t(u)$, and t. Recall that $v_0\in B(u)$ and $A(u)$ is the disjoint union of $X(u)$ and $N^+(u)$. Partition $N^+(u)$ into $m=m(u)$ sets $H_i=\{v\in N^+(u):N^-(v)=N^-(w_i)\}$, for $1\leqslant i\leqslant m$. Let $X(u)=\{x_1,\dots,x_{|X(u)|}\}$, and define $H_{m+j}=\{x_j\}$, for $1\leqslant j\leqslant|X(u)|$. Clearly $H_1,\dots,H_m,H_{m+1},\dots,H_{m+|X(u)|}$ is a partition of $A(u)$ into $m(u)+|X(u)|$ cliques such that $x,y\in H_i$ implies $N(x)\cap B(u)=N(y)\cap B(u)$. Hence $m(u)+|X(u)|\geqslant s(u)$. This concludes the proof of the claim. □

The next claim presents a procedure, based on Lemma 1, to reduce the number of cliques in $\mathscr{F}$.

Claim 3. *For any maximal* $u\in V_1$, *there is a family*

$\mathscr{F}'(u)$ *of* $n+t+1-(m(u)+|X(u)|)$ *cliques covering* $E(G)$.

Proof. To prove Claim 3, we first describe a sequence of pairs $(\mathscr{F}_1,W_1),\dots,(\mathscr{F}_p,W_p)$, where $\mathscr{F}_i$ is a family of cliques of G covering $E(G)$, W_i is a diverse set from level

$i+1$ that induces a clique, $|\mathscr{F}_i| = n+t+1-(m(u)+|X(u)|)+|W_i|$, and W_p satisfies $N^+(W_p)=\emptyset$.

We define $\mathscr{F}_1$ by modifying $\mathscr{F}$. Recall that $\mathscr{F}$ contains, among others, the cliques $\{C[x]\}_{x\neq v_0}$. Remove from $\mathscr{F}$ the cliques $C[x]$, for all $x \in \{u\}\cup X(u)$. Now, to cover the edges of the cliques we have just removed, consider the set $Q=\{u\}\cup N^+(u)\cup X(u)$. If $x\in X(u)$, then $N^+(u)\subseteq N^+(x)$ which, by the maximality of u, implies that $N^+(u)=N^+(x)$. Therefore Q is a clique containing $N^+(x)$, for all $x\in\{u\}\cup X(u)$. Add Q to create the family $\mathscr{F}_1$. This family contains $n+t+1-|X(u)|$ cliques and still covers $E(G)$ because there are no free edges between vertices of V_1 so $C[x]\subseteq\{u\}\cup X(u)\cup N^+(u)$, for all $x\in\{u\}\cup X(u)$. Recall that $W(u)$ is a maximum diverse subset of $N^+(u)$ satisfying $|W(u)|=m(u)$. Set $W_1=W(u)$. If $N^+(W_1)=\emptyset$, then the sequence terminates with $(\mathscr{F}_1, W_1)$. Because $|W_1|=m(u)$, the family $\mathscr{F}_1$ has size $|\mathscr{F}_1|=n+t+1-|X(u)|=n+t+1-(m(u)+|X(u)|)+|W_1|$, as desired.

Assume that $(\mathscr{F}_i, W_i)$ is defined at some stage $i\geqslant 1$ and assume $N^+(W_i)\neq\emptyset$. We now explain how to define $(\mathscr{F}_{i+1}, W_{i+1})$. Let $Y_i=N^+(W_i)$. Because W_i is a diverse set, part (f) of Claim 1 implies that the graph $G[W_i, Y_i]$ induced by $W_i\cup Y_i$ is a nested graph with depth r, say. Apply Lemma 1 to $G[W_i,Y_i]$. Part (i) of the lemma guarantees r cliques that cover $E(W_i,Y_i)\cup E(Y_i)$. To produce the family $\mathscr{F}_{i+1}$ from $\mathscr{F}_i$, add these r cliques and remove the $|W_i|$ cliques $C[x]$ for all $x\in W_i$. Let W_{i+1} be the set of r vertices of Y_i guaranteed by part (ii) of the lemma. Because $|\mathscr{F}_i|=n+t+1-(m(u)+|X(u)|)+|W_i|$, it is clear that $|\mathscr{F}_{i+1}|=n+t+1-(m(u)+|X(u)|)+|W_{i+1}|$. It is also clear that W_{i+1} is a diverse set from level $i+2$. Part (a) of Claim 1 guarantees that W_{i+1} is a clique. We must verify that $\mathscr{F}_{i+1}$ is a cover of $E(G)$. The only edges covered by the cliques $C[x]$ $(x\in W_i)$ that do not appear in $E(W_i,Y_i)\cup E(Y_i)$ are free edges incident to precisely one vertex from W_i. Such a free edge $e=xw$, $x\in V_{i+1}\setminus W_i$, $w\in W_i$ is still covered by $\mathscr{F}_{i+1}$ because the clique $C[x]$ remains in $\mathscr{F}_{i+1}$ and covers e. Hence $\mathscr{F}_{i+1}$ is a cover of $E(G)$. This completes the proof of the existence of the sequence $(\mathscr{F}_1, W_1),\ldots,(\mathscr{F}_p, W_p)$ described above.

The family $\mathscr{F}'(u)$ is obtained from $\mathscr{F}_p$ by removing the cliques $C[x]$ for all $x\in W_p$. Thus, the number of cliques in $\mathscr{F}'(u)$ is $n+t+1-(m(u)+|X(u)|)$. Furthermore, $\mathscr{F}'(u)$ is a cover of $E(G)$ because the edges not in $E(W_p)$ that were covered by the cliques $C[x]$ $(x\in W_p)$ are free edges with exactly one endpoint in W_p. This concludes the proof of Claim 3. □

If $m(u)+|X(u)|>t$ for some maximal $u\in V_1$, then $|\mathscr{F}'(u)|\leqslant n$ by Claim 3, so in this case $\mathscr{F}'(u)$ is the desired cover. So we may assume that $m(u)+|X(u)|\leqslant t$, for all maximal $u\in V_1$. This, together with Claim 2, implies $m(u)+|X(u)|=s(u)=t(u)=t$, for all maximal $u\in V_1$. The last two equalities and the choice of v_0 imply $s(v_0)=t(v_0)=t$. Because $A(v_0)$ and $B(v_0)$ were chosen so that $\theta'(A(v_0),B(v_0))=t$ and, subject to this constraint, $s(A(v_0),B(v_0))$ is minimum, it follows that $A(v_0)$ has a partition $A_1,\ldots,A_t$ such that, for $1\leqslant i\leqslant t$ and all $x,y\in A_i$, the neighborhoods of x and y satisfy $N(x)\cap B(v_0)=N(y)\cap B(v_0)$. Observe that this means we may assume that the t cliques that cover the edges $E(A(v_0),B(v_0))$ have the form $A_i\cup(N(A_i)\cap B(v_0))$.

Since $m(u)+|X(u)|=t$ for any maximal $u \in V_1$, Claim 3 also implies $|\mathscr{F}'(u)|=n+1$. So to conclude the proof of the theorem, it suffices to reduce the number of cliques in some $\mathscr{F}'(u)$ by just one. The remainder of the proof is devoted to this. If $V_2=\emptyset$, then $V(G)=A(v_0)\cup B(v_0)\cup\{v_0\}$ (say $|A(v_0)|\leqslant|B(v_0)|$), so it follows that all of the edges of G can be covered with at most $t+2\leqslant|A(v_0)|+2\leqslant n$ cliques. Hence we may assume $V_2\neq\emptyset$. So there is a maximal $u\in V_1$ such that $m(u)\geqslant 1$. This implies that $t\geqslant 1$.

Choose a maximal $v_1\in V_1$ with the following properties:

(α) $|X(v_1)|$ is minimum, and

(β) subject to (α), $|N^+(v_1)\cap N^+(A(v_0))|$ is minimum.

For convenience, set $m=m(v_1)$, $X=X(v_1)$, and $\mathscr{F}'=\mathscr{F}'(v_1)$. Let $Q=\{v_1\}\cup N^+(v_1)\cup X(v_1)$ denote the clique we have added to $\mathscr{F}$ to produce $\mathscr{F}_1$ in Claim 3. In particular, Q is a clique in $\mathscr{F}'$. Let $W=W(v_1)=\{w_1,\dots,w_m\}$ be a maximum diverse subset of $N^+(v_1)$ and let

$$H_i=\{w\in N^+(v_1)\colon N^-(w)=N^-(w_i)\}\quad \text{for } 1\leqslant i\leqslant m$$

be the corresponding partition of $N^+(v_1)$.

If $V_1\subseteq N(A_i)$ for some i, then we can reduce the number of cliques in $\mathscr{F}'$ by replacing the two cliques $\{v_0\}\cup B(v_0)$ and $A_i\cup(N(A_i)\cap B(v_0))$ with their union. Therefore, we may assume that

$$\text{No vertex } u\in A(v_0) \text{ is adjacent to all vertices of } V_1. \tag{1}$$

Similarly if, for some i, all vertices in $N(A_i)\cap B(v_0)$ are adjacent to all vertices of V_1, then we can reduce the number of cliques in $\mathscr{F}'$ by replacing the two cliques $\{v_0\}\cup A(v_0)$ and $A_i\cup(N(A_i)\cap B(v_0))$ with their union. Therefore, we may assume that

$$\text{For each } A_i, \text{ there exists } b\in N(A_i)\cap B(v_0) \text{ such that } V_1\nsubseteq N(b). \tag{2}$$

If $t=1$ then $t=s(v_0)$ implies $N(x)\cap B(v_0)=N(y)\cap B(v_0)$ for all $x,y\in A(v_0)$. Furthermore, $N(A(v_0))\cap B(v_0)\neq\emptyset$ so $V_1\subset N(b)$ for some $b\in B(v_0)$, contradicting (2). Thus $t>1$. Now by definition of t, $N(A_i)\cap B(v_0)\neq\emptyset$, for all $1\leqslant i\leqslant t$. Our proof now splits into two cases.

Case 1: $N^+(v_1)\neq V_2$. In this case there must be a maximal vertex $u\in V_1\setminus\{v_1\}$ such that $N^+(u)\neq N^+(v_1)$. In particular, $X(u)\cap X=\emptyset$ which means that the cliques $C[z]$ ($z\in X(u)$) are in $\mathscr{F}'$. Recall that, because u is maximal, $N^+(u)=N^+(z)$, for all $z\in X(u)$. Hence, if there is at least one vertex $z\in X(u)$, then we can reduce the number of cliques in $\mathscr{F}'$ by replacing the two cliques $C[u]$ and $C[z]$ with their union. So we may assume that $X(u)=\emptyset$. This and property (α) in the choice of v_1 implies $|X|=0$. Consequently, for any maximal vertex $y\in V_1$, we have $X(y)=\emptyset$ and $A(y)=N^+(y)$. We conclude that, for any maximal $y\in V_1$, the set $N(y)\cap V_1$ induces a clique because it is a subset of $B(y)$.

Observe that $|X|=0$ means that the cliques $C[x]$ ($x\in V_1$) are all in $\mathscr{F}'$. So we may assume $N^+(x)\neq N^+(y)$, for all distinct x and y in V_1 such that $xy\in E$ because if $N^+(x)=N^+(y)$, then we can reduce the number of cliques in $\mathscr{F}'$ by replacing the two cliques $C[x]$ and $C[y]$ with their union.

Suppose there exists a maximal vertex $y \in A(v_0)$. Without loss of generality, $y \in A_1$. Because $N(y) \cap V_1$ is a clique, every vertex in $N(A_1) \cap B(v_0)$ is adjacent to every vertex in V_1, contradicting (2). Therefore, there are no maximal vertices in $A(v_0)$. Suppose there exists a maximal vertex $y \in N(A(v_0)) \cap B(v_0)$. Because $N(y) \cap V_1$ is a clique, vertices in $N(y) \cap A(v_0)$ are adjacent to all vertices in V_1, contradicting (1). So we may assume that $B(v_0) \setminus N(A(v_0))$ contains all maximal vertices. In particular, $v_1 \in B(v_0) \setminus N(A(v_0))$.

We now claim that $N^+(x) \subseteq N^+(v_1)$, for all $x \in A(v_0)$. It suffices to prove this for $x \in A(v_0)$ with the property that there is no $z \in A(v_0)$ such that $N^+(x) \subset N^+(z)$. Consider such an x. Now (2) guarantees that there is a vertex $y \in N(x) \cap B(v_0)$ and a vertex $w \in A(v_0) \setminus N(y)$. If $N^+(x) \not\subseteq N^+(y)$, then $y \notin A(x)$ so $w \in A(x)$. This implies $N^+(w) = N^+(x)$ by the choice of x. Therefore we may assume that $N^+(x) \subset N^+(y)$. Now x and v_1 are both neighbors of y. However $N^+(x) \subset N^+(y)$ implies $x \in B(y)$. Because $xv_1 \notin E$, we conclude that $v_1 \in A(y)$ and $N^+(y) \subseteq N^+(v_1)$. Hence $N^+(x) \subset N^+(y) \subseteq N^+(v_1)$, as desired.

The previous paragraph implies $N^+(A(v_0)) \subseteq N^+(v_1)$. A maximal vertex $u \neq v_1$ satisfies $|X(u)| = 0$, as observed in the first paragraph of this case. If $N^+(A(v_0)) = N^+(v_1)$, then because $N^+(v_1) \neq V_2$, maximality implies $|N^+(u) \cap N^+(A(v_0))| < |N^+(v_1) \cap N^+(A(v_0))|$, contradicting ($\beta$) in the choice of v_1. Therefore, $N^+(A(v_0)) \subset N^+(v_1)$. Recall that $W = \{w_1, \dots, w_m\}$ is a maximum diverse subset of $N^+(v_1)$ and H_i for $1 \leqslant i \leqslant m$, is the corresponding partition of $N^+(v_1)$. Without loss of generality, $w_m \notin N^+(A(v_0))$. Note that, since $w_m \notin N^+(A(v_0))$, we have $N^-(H_m) \cap A(v_0) = \emptyset$. Replace the $|A(v_0)|$ cliques $C[x]$ ($x \in A(v_0)$) with the $m-1$ cliques $H_i \cup (N^-(H_i) \cap A(v_0))$, for $1 \leqslant i \leqslant m-1$. Because $|X| = 0$ and $m + |X| = t$, it follows that $m - 1 = t - 1 < |A(v_0)|$. In particular, this replacement reduces the number of cliques in $\mathscr{F}'$. The added cliques cover all of the edges in $E(A(v_0), N^+(A(v_0)))$. The edges between all of the vertices of $N^+(A(v_0))$ are still covered by $C[v_1]$, whereas the edges in $E(A(v_0), A(v_0))$ are still covered by $\{v_0\} \cup A(v_0)$, so the resulting family of cliques is a cover of $E(G)$ and contains at most n cliques.

Case 2: $N^+(v_1) = V_2$. Recall that $\mathscr{F}'$ contains, among others, the clique $Q = \{v_1\} \cup N^+(v_1) \cup X(v_1)$ and all cliques $C[x]$ such that $x \in V_1 \setminus (\{v_1\} \cup X)$. Also recall that $W = \{w_1, \dots, w_m\}$ is a maximum diverse subset of $N^+(v_1)$ and H_i for $1 \leqslant i \leqslant m$, is the corresponding partition of $N^+(v_1)$.

Suppose there is a set $U \subseteq V_1 \setminus (\{v_1\} \cup X)$ such that $|U| > m$ and either $U \subseteq A(v_0)$ or $U \subseteq B(v_0)$. Without loss of generality $U \subseteq A(v_0)$. We can replace the $|U|$ cliques $C[x]$ ($x \in U$) with the m cliques $H_i \cup (N^-(H_i) \cap A(v_0))$, for $1 \leqslant i \leqslant m$. Because $|U| > m$, this replacement reduces the number of cliques in $\mathscr{F}'$. Moreover, $N^+(v_1) = V_2$ implies that the added cliques cover all of the edges in $E(U, N^+(U))$. The edges between vertices of $N^+(A(v_0))$ are still covered by Q whereas the edges in $E(U, A(v_0))$ are still covered by $\{v_0\} \cup A(v_0)$. Thus the resulting family of cliques is a cover of $E(G)$ and contains at most n cliques. So we may assume that there is no such set U.

If $|V_1| - |X| - 1 > 2m$, then there is a set $U \subseteq V_1 \setminus (\{v_1\} \cup X)$ such that $|U| > m$ and either $U \subseteq A(v_0)$ or $U \subseteq B(v_0)$. Therefore we may assume that $|V_1| - |X| - 1 \leqslant 2m$.

Because $2t \leqslant |V_1|$ and $m+|X|=t$, we find that $2t \leqslant |V_1| \leqslant 2m+|X|+1 \leqslant 2t-|X|+1$. In particular, $|X| \leqslant 1$.

Suppose that $\emptyset \neq X=\{x\}$. From the previous paragraph we find that $|X|=1$ implies $m=t-1$ and $|A(v_0)|=|B(v_0)|=t$. If $\{x,v_1\} \subset A(v_0)$, then $U=B(v_0)$ is a subset of $V_1 \setminus (\{v_1\} \cup X)$ such that $|U|>m$. A similar argument applies if $\{x,v_1\} \subset B(v_0)$. So we may assume that one of these two vertices is an element of $A(v_0)$ and the other is an element of $B(v_0)$. Now the vertices x and v_1 are symmetric in the sense that when we chose v_1 we could have chosen x, since x and v_1 are both maximal, $|X(x)|=|X|=1$, and $N^+(v_1)=N^+(x)=V_2$. Hence we may assume $x \in A(v_0)$ and $v_1 \in B(v_0)$. If there is a vertex $w \in N(v_1) \cap A(v_0)$ such that $w \neq x$, then $w \notin X$ so $w \notin A(v_1)=X \cup N^+(v_1)$. This implies that $w \in B(v_1)$. Since $wv_1 \in E$ and $B(v_0) \setminus \{v_1\} \subset B(v_1)$, it follows that $B(v_0) \subset N(w)$, contradicting (1). Hence $N(v_1) \cap A(v_0)=\{x\}$. In this case, the clique Q already covers the one edge of $E(A(v_0),B(v_0))$ with endpoint v_1. Therefore, we can reduce the number of cliques in $\mathscr{F}'$ by removing the t cliques $A_i \cup (N(A_i) \cap B(v_0))$ and replacing them with the $t-1$ cliques $\{b\} \cup (N(b) \cap A(v_0))$ such that $b \in B(v_0) \setminus \{v_1\}$.

So we may assume $|X|=0$. An argument similar to the one given at the beginning of case 1 shows that $v_1 \in B(v_0) \setminus N(A(v_0))$. This means $|V_1|=2t+1$, $|A(v_0)|=t$, and $|B(v_0)|=t+1$. Let $B=B(v_0) \setminus \{v_1\}=\{b_1,b_2,\ldots,b_t\}$, ordered so that $|N(b_i) \cap A(v_0)| \geqslant |N(b_j) \cap A(v_0)|$, for $1 \leqslant i<j \leqslant t$. Recall that, by definition, $v_0 \in B(b_i)$, for all i. If $v_1 \in B(b_i)$ for some i, then $(N(b_i) \cap A(v_0)) \cup N^+(b_i)$ is a clique (because it is a subset of $A(b_i)$) so, in this case we can reduce the number of cliques in $\mathscr{F}'$ by replacing $C[b_i]$ and the t cliques covering $E(A(v_0),B(v_0))$ with $\{b_i\} \cup (N(b_i) \cap A(v_0)) \cup N^+(b_i)$ and the $t-1$ cliques $\{b_j\} \cup (N(b_j) \cap A(v_0))$, for $j \neq i$. Therefore we may assume $v_1 \in A(b_i)$, for all $1 \leqslant i \leqslant t$. Since $v_1 \in B(v_0) \setminus N(A(v_0))$, it follows that $N(b_i) \cap A(v_0) \subset B(b_i)$, for all i. If $N(b_i) \cap A(v_0) \subseteq N(b_j) \cap A(v_0)$ for $i<j$, then $|N(b_i) \cap A(v_0)| \geqslant |N(b_j) \cap A(v_0)|$ implies $N(b_i) \cap A(v_0)=N(b_j) \cap A(v_0)$, contradicting $\theta'(A(v_0),B(v_0))=t$. Therefore, $N(b_i) \cap A(v_0) \nsubseteq N(b_j) \cap A(v_0)$, for all $i<j$. From this we may deduce $b_j \notin B(b_i)$. Thus, $b_j \in A(b_i)$ and consequently, $N^+(b_i) \subseteq N^+(b_j)$, for $i<j$. In particular, $N^+(B) \subseteq N^+(b_t)$. Because $|X|=0$, we may assume that $N^+(b_t) \neq N^+(v_1)$ since otherwise $N^+(b_t)=N^+(v_1)$ and the number of cliques in $\mathscr{F}'$ can be reduce by replacing $C[b_t]$ and Q with their union. Hence $N^+(B) \subset N^+(v_1)$. We can now replace the t cliques $C[b_i]$ $(1 \leqslant i \leqslant t)$ in $\mathscr{F}'$ with the at most $m-1$ cliques $H_i \cup (N^-(H_i) \cap B(v_0))$, for values of i satisfying $N^-(H_i) \cap B \neq \emptyset$. $\square$

3. Conclusion

Opsut's conjecture seems to result from an attempt to extend the family of line graphs while maintaining competition number at most two. A natural family to consider in this case is the family of claw-free graphs. However, Opsut proved that for any graph $G=(V,E)$, the competition number satisfies $k(G) \geqslant \min_{v \in V(G)} \theta(N(v))$. This implies that the icosahedron — a claw-free graph — has competition number at least three. Therefore, claw-free graphs do not necessarily have competition number at most two.

The locally cobipartite graphs then seem to be the next natural family to consider, which may explain the motivation for Opsut's conjecture.

Although the family of claw-free graphs does not seem to be significant in the study of the competition number, it may still be important for the edge clique covering number. We do not know an example of a claw-free graph whose edge clique covering number exceeds the number of vertices. Perhaps then Theorem 2 can be extended to claw-free graphs.

It may also be possible to extend Theorem 2 in another direction. A graph is *locally co-k-partite* if the neighborhood of every vertex can be partitioned into at most k cliques. We conjecture that a locally co-k-partite graph on n vertices as edge clique cover number at most $\lceil kn/2 \rceil$.

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