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Periodicity and attractivity of a ratio-dependent Leslie system with impulses ☆

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ABSTRACT

A ratio-dependent Leslie system with impulses is studied. By using a comparison theorem, continuation theorem base on coincidence degree and constructing a suitable Lyapunov function, we establish sufficient and necessary conditions for the existence and global attractivity of periodic solution. Examples show that the obtained criteria are easily verifiable.

1. Introduction

Leslie [1] introduced the famous Leslie predator–prey system

$$\dot{x}(t) = x(t)\big[a - bx(t)\big] - p(x)y(t), \qquad \dot{y}(t) = y(t)\left[e - f\frac{y(t)}{x(t)}\right], \tag{1.1}$$

where $x(t)$, $y(t)$ stand for the population (the density) of the prey and the predator at time t, respectively, and $p(x)$ is the so-called predator functional response to prey. In biomathematics, we define $p(x)$: When $p(x) = cx$, the functional response $p(x)$ is called type 1; When $p(x) = \frac{cx}{d+x}$, the functional response $p(x)$ is called type 2; When $p(x) = \frac{cx^2}{d+x^2}$, the functional response $p(x)$ is called type 3.

In (1.1), it has been assumed that the prey grows logistically with growth rate a and carries capacity $\frac{a}{b}$ in the absence of predation. The predator consumes the prey according to the functional response $p(x)$ and grows logistically with growth rate e and carrying capacity $\frac{x}{f}$ proportional to the population size of the prey (or prey abundance). The parameter f is a measure of the food quality that the prey provides and converted to predator birth. Leslie introduced a predator–prey model where the carrying capacity of the predator's environment is proportional to the number of prey, and still stressed the fact that there are upper limits to the rates of increasing of both prey x and predator y, which are not recognized in

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the Lotka–Volterra model. These upper limits can be approached under favorable conditions: for the predators, when the number of prey per predator is large; for the prey, when the number of predators (and perhaps the number of prey also) is small [2].

However, recently more and more obvious evidences of biology and physiology show that in many conditions, especially when the predators have to search for food (consequently, have to share or compete for food), a more realistic and general predator–prey system should rely on the theory of ratio-dependence, this theory is confirmed by lots of experimental results [3]. A ratio-dependent Leslie system with the functional response of Holling–Tanner type is as follows:

$$\dot{x}(t) = x(t)\big[a - bx(t)\big] - p\left(\frac{x(t)}{y(t)}\right)y(t), \qquad \dot{y}(t) = y(t)\left[e - f\frac{y(t)}{x(t)}\right], \tag{1.2}$$

where $p(x)$ has the same means as before.

In recent research on species dynamics of the Leslie system has important significance, see [4–34] for details. Motivated by [13], we consider a ratio-dependent Leslie predator–prey model with impulses

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[b(t) - a(t)x_1(t) - \frac{c(t)x_1(t)x_2(t)}{h^2x_2^2(t) + x_1^2(t)}\right], \\ \dot{x}_2(t) &= x_2(t)\left[e(t) - f(t)\frac{x_2(t)}{x_1(t)}\right], \quad t \neq t_k, \\ x_i\big(t_k^+\big) &= \big(1 + h_k^i\big)x_i(t_k), \quad x_i(0) > 0,\ i = 1, 2, \end{aligned} \tag{1.3}$$

where $x_i(t)$, $i = 1, 2$, denote the density of prey and predator at time t, respectively. $b, a, c, d, e, f, p, \alpha_i, \beta_i, \gamma_i \in C(R, R+)$, $i = 1, 2$, are all ω-periodic functions of t; h^2 is a positive constant, denoting the constant of capturing half-saturation. Assume that h_k^i, $i = 1, 2$, $k \in Z_+ = \{1, 2, \ldots\}$ are constants and there is an integer $q > 0$ such that $h_{k+q}^i = h_k^i$, $t_{k+q} = t_k + \omega$. With model (1.3) we can take into account the possible exterior effects under which the population densities change very rapidly. For instance, impulsive reduction of the population density of a given species is possible after its partial destruction by catching, a natural constraint in this case is $1 + h_k^i > 0$ for all $k \in Z_+$. An impulsive increase of the density is possible by artificial breeding of the species or release some species ($h_k^i > 0$). Our main objective of this paper is to give sufficient and necessary conditions for the existence and global attractivity of periodic solution for the above model (1.3). More knowledge about impulsive differential equations see [35,36] for details.

For convenience, throughout this paper, we shall use the following notations:

R denotes the real numbers. Z_+ denotes the positive integers. For a continuous ω-function $g(t)$, $g^u = \max_{t\in[0,\omega]} g(t)$, $g^l = \min_{t\in[0,\omega]} g(t)$, $\overline{g} = \frac{1}{\omega}\int_0^\omega g(t)\,dt$.

Throughout this paper, we suppose that the following conditions are satisfied:

(H_1) $b, e \in C(R, R)$ are all ω-periodic functions with $\bar{b} > 0$, $\bar{e} > 0$.
(H_2) $a, c, f \in C(R, R)$ are all nonnegative ω-periodic functions of t.
(H_3) $\overline{b - \frac{c}{2|h|}} + \frac{1}{\omega}\sum_{k=1}^{q} \ln(1 + h_k^1) > 0$, $\bar{e} + \frac{1}{\omega}\sum_{k=1}^{q} \ln(1 + h_k^2) > 0$.

This paper is organized as follows: In Section 2, some basic results are obtained. In Section 3, we shall give sufficient and necessary conditions for the existence of ω-periodic solution of system (1.3). In Section 4, we shall give sufficient conditions for the attractivity of ω-periodic solution of system (1.3). In Section 5, we give an example to verify that our result is correct.

2. Preliminaries

First we shall make some preparations. Let $J \subset R$. Denote by $PC(J, R^N)$ the space of functions $\phi : J \to R^N$ which are continuous for $t \in J$, $t \neq t_k$, are continuous from the left for $t \in J$ and have discontinuities of the first kind at the points $t = t_k \in J$. Let

$$\begin{aligned} PC'\big(J, R^N\big) &= \left\{\phi(t) \;\middle|\; \phi : J \to R^N,\ \frac{d\phi}{dt} : J \to R^N\right\}, \\ PC_\omega &= \big\{\phi(t) \in PC\big([0, \omega], R^N\big) \;\big|\; \phi(t + \omega) = \phi(t)\big\}, \\ PC'_\omega &= \big\{\phi(t) \in PC'\big([0, \omega], R^N\big) \;\big|\; \phi(t + \omega) = \phi(t)\big\}, \\ \|\phi\| &= \max\big\{\|\phi\|_{PC_\omega}, \|\phi\|_{PC'_\omega}\big\}, \end{aligned}$$

where $\|\phi\|_{PC_\omega} = \max_{0\in[0,\omega]}\{|\phi|\}$, $\|\phi\|_{PC'_\omega} = \max_{0\in[0,\omega]}\{|\phi'|\}$.

Definition 2.1. The set $\mathscr{F}$ is said to be quasi-equicontinuous in $[0, \omega]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathscr{F}, k \in N, t_1, t_2 \in (t_{k-1}, t_k) \cap [0, \omega], |t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \varepsilon$.

Lemma 2.1. *(See [35].) The set $\mathscr{F} \subset PC_\omega$ is relatively compact if and only if*

(1) *$\mathscr{F}$ is bounded, that is, $\|\phi\|_{PC_\omega} = \sup\{|\phi(t)|:\ t \in J\} \leqslant M$ for each $x \in \mathscr{F}$ and some $M > 0$;*
(2) *$\mathscr{F}$ is quasi-equicontinuous in J.*

Lemma 2.2. *(See [37].) Suppose $\psi \in PC'_\omega$. Then*

$$\Big|\sup_{s\in[0,\omega]}\psi(s) - \inf_{s\in[0,\omega]}\psi(s)\Big| \leqslant \frac{1}{2}\left[\int_0^\omega \big|\dot{\psi}(s)\big|\,ds + \sum_{k=1}^{p}\big|\Delta\psi(t_k)\big|\right].$$

From Lemma 2.2, we have:

Lemma 2.3. *Suppose $\psi \in PC'_\omega$. Then*

$$\big|\psi(t) - \psi(s)\big| \leqslant \frac{1}{2}\left[\int_0^\omega \big|\dot{\psi}(s)\big|\,ds + \sum_{k=1}^{p}\big|\Delta\psi(t_k)\big|\right], \quad \textit{for all } t, s \in [0,\omega].$$

3. Existence of positive periodic solutions

For the readers convenience, we first summarize a few concepts from the book by Gaines and Mawhin [38]. Let X and Y be normed vector spaces. Let $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\overline{\Omega}$ if $(QN)(\overline{\Omega})$ is bounded and $KP(I - Q)N : \overline{\Omega} \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$. In the proof of our existence result, we need the following continuation theorem.

Lemma 3.1. *(See [38].) Let $L : \operatorname{Dom} L \subset X \to Y$ be the Fredholm injection with index of 0, $N : \overline{\Omega} \to Y$ on $\overline{\Omega}$ L-compact. Suppose*

(1) *$\lambda \in (0,1)$, $x \in \operatorname{Dom} L \cap \partial\Omega$ holds, then $Lx \neq \lambda Nx$;*
(2) *suppose $x \in \operatorname{Ker} L \cap \partial\Omega$, $QNx \neq 0$ holds, and*

$$\deg(JQN|_{\operatorname{Ker} L\cap\partial\Omega}) \neq 0,$$

where $\deg(JQN|_{\operatorname{Ker} L\cap\partial\Omega})$, represents the Brouwer degree. Then $Lx = Nx$ has at least one solution on $\operatorname{Dom} L \cap \overline{\Omega}$.

Now, we state our main theorem.

Theorem 3.1. *Under assumptions (H_1), (H_2), system (1.3) has at least one positive ω-periodic solution if and only if (H_3).*

Proof. Let $x_i(t) = \exp(y_i(t))$, $i = 1, 2$, then the system (1.3) becomes

$$\begin{aligned}
&\dot{y}_1(t) = b(t) - a(t)e^{y_1(t)} - \frac{c(t)e^{y_1(t)+y_2(t)}}{h^2e^{2y_1(t)+e^{2y_2(t)}}} \equiv f_1,\\
&\dot{y}_2(t) = e(t) - f(t)e^{y_2(t)-y_1(t)} \equiv f_2, \quad t \neq t_k;\\
&y_i\big(t_k^+\big) = \ln\big(1 + h_k^i\big) + y_i(t_k), \quad i = 1, 2,\ t = t_k,\ k \in Z_+.
\end{aligned} \tag{3.1}$$

Only if part. If $(y_1(t), y_2(t))^T$ is a positive ω-periodic solution of system (3.1), then for $t \neq t_k$ $(k = 1, 2, \ldots, q)$ integrating (3.1) over the interval $[0,\omega]$ and using $y_i(0) = y_i(\omega)$, we have

$$\int_0^\omega \left\{b(t) - a(t)e^{y_1(t)} - \frac{c(t)e^{y_1(t)+y_2(t)}}{h^2e^{2y_1(t)+e^{2y_2(t)}}}\right\}dt = -\sum_{k=1}^{q}\ln\big(1 + h_k^1\big),$$

$$\int_0^\omega \big\{e(t) - f(t)e^{y_2(t)-y_1(t)}\big\}\,dt = -\sum_{k=1}^{q}\ln\big(1 + h_k^2\big).$$

Thus

$$0<\overline{\frac{c}{2|h|}}\omega<\left\{a(t)e^{y_1(t)}+\frac{c(t)e^{y_1(t)+y_2(t)}}{h^2e^{2y_1(t)}+e^{2y_2(t)}}\right\}dt=\sum_{k=1}^{q}\ln\left(1+h_k^1\right)+\bar{b}\omega,$$

$$\bar{e}\omega+\sum_{k=1}^{q}\ln\left(1+h_k^2\right)=\int_0^{\omega}f(t)e^{y_2(t)-y_1(t)}\,dt>0,$$

which give (H_3).

If part. In order to use Lemma 3.1 to system (3.1), we set

$$X=\left\{x(t)=\left(x_1(t),x_2(t)\right)^T \;\middle|\; x_i(t)\in PC_\omega,\ i=1,2\right\},$$

with norm

$$\|x\|=\left\|\left(x_1(t),x_2(t)\right)^T\right\|=\sum_{i=1}^{2}\left\|x_i(t)\right\|=\sum_{i=1}^{2}\max_{0\leqslant t\leqslant\omega}\left|x_i(t)\right|.$$

Then $(X,\|\cdot\|)$ is a Banach space. Moreover, let

$$Y=\left\{\tilde{y}=[\dot{y},\xi_1,\xi_2,\ldots,\xi_q]\right\}=X\times R^{2q},$$

where $y(t)=(y_1(t),y_2(t))^T\in PC'_\omega$, $\xi_k=(m_k^1,m_k^2)^T=(\Delta y_1(t_k),\Delta y_2(t_k))^T$ are constant vectors, $k=1,2,\ldots,q$. Let $\|\tilde{y}\|=\|\dot{y}(t)\|+\sum_{i=1}^{q}\|\xi_i\|$. Then $(Y,\|\cdot\|)$ is a Banach space.

Set $L:\operatorname{Dom}L\subset X\to Y$, as

$$L\begin{pmatrix}y_1\\y_2\end{pmatrix}=\left(\begin{pmatrix}y_1'(t)\\y_2'(t)\end{pmatrix},\quad\begin{pmatrix}\Delta y_1(t_1)\\\Delta y_2(t_1)\end{pmatrix},\quad\ldots,\quad\begin{pmatrix}\Delta y_1(t_q)\\\Delta y_2(t_q)\end{pmatrix}\right)$$

where

$$\operatorname{Dom}L=\left\{y(t)=(y_1,y_2)^T\in X\;\middle|\;y'(t)\in PC_\omega\right\}=\left\{y(t)=(y_1,y_2)^T\in X\;\middle|\;y(t)\in PC'_\omega\right\}.$$

At the same time, we denote

$$N\left(y(t)\right)=\left(\begin{pmatrix}f_1\\f_2\end{pmatrix},\quad\begin{pmatrix}\ln(1+h_1^1)\\\ln(1+h_1^2)\end{pmatrix},\quad\ldots,\quad\begin{pmatrix}\ln(1+h_q^1)\\\ln(1+h_q^2)\end{pmatrix}\right)$$

and define two projectors P and Q as $P:X\to X$,

$$P\left(y(t)\right)=\frac{1}{\omega}\begin{pmatrix}\int_0^{\omega}y_1(t)\,dt\\\int_0^{\omega}y_2(t)\,dt\end{pmatrix};$$

$Q:Y\to Y$, as

$$Q\left(\begin{pmatrix}f_1(t)\\f_2(t)\end{pmatrix},\quad\begin{pmatrix}h_1^1\\h_1^2\end{pmatrix},\quad\ldots,\quad\begin{pmatrix}h_q^1\\h_q^2\end{pmatrix}\right)=\left(\frac{1}{\omega}\begin{pmatrix}\int_0^{\omega}f_1(t)\,dt+\sum_{k=1}^{q}h_k^1\\\int_0^{\omega}f_2(t)\,dt+\sum_{k=1}^{q}h_k^2\end{pmatrix},\quad\begin{pmatrix}0\\0\end{pmatrix},\quad\ldots,\quad\begin{pmatrix}0\\0\end{pmatrix}\right)$$

obviously

$$\operatorname{Im}L=\left\{z=\left(\begin{pmatrix}f_1(t)\\f_2(t)\end{pmatrix},\quad\begin{pmatrix}h_1^1\\h_1^2\end{pmatrix},\quad\ldots,\quad\begin{pmatrix}h_q^2\\h_q^2\end{pmatrix}\right)\in Y\;\middle|\;\int_0^{\omega}f_i(t)\,dt+\sum_{k=1}^{q}h_k^i=0,\ i=1,2\right\}$$

and

$$\operatorname{Ker}L=\left\{x\;\middle|\;x\in X,\ y=\begin{pmatrix}e_1\\e_2\end{pmatrix}\in R^2\right\}=\operatorname{Im}P,$$

$$\operatorname{Im}L=\left\{x\in Y\;\middle|\;\int_0^{\omega}f_i(t)\,dt+\sum_{k=1}^{q}h_k^i=0,\ i=1,2\right\}=\operatorname{Ker}Q$$

are closed sets in Y and $dim\operatorname{Ker}L=\operatorname{codim}\operatorname{Im}L=2$. Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse of L:

$$K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$$

has the form

$$K_P(z) = K_P\left(\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix}, \begin{pmatrix} h_1^1 \\ h_1^2 \end{pmatrix}, \ldots, \begin{pmatrix} h_q^1 \\ h_q^2 \end{pmatrix} \right)$$
$$= \begin{pmatrix} \int_0^t f_1(s)\,ds + \sum_{0<t_k<t} h_k^1 - \sum_{k=1}^q h_k^1 - \frac{1}{\omega}\int_0^\omega \int_0^t f_1(s)\,ds\,dt \\ \int_0^t f_2(s)\,ds + \sum_{0<t_k<t} p_k^2 - \sum_{k=1}^q h_k^2 - \frac{1}{\omega}\int_0^\omega \int_0^t f_2(s)\,ds\,dt \end{pmatrix}.$$

Thus

$$QN\big(y(t)\big) = \left(\frac{1}{\omega}\begin{pmatrix} \int_0^\omega f_1(t)\,dt + \sum_{k=1}^q \ln(1+h_k^1) \\ \int_0^\omega f_2(t)\,dt + \sum_{k=1}^q \ln(1+h_k^2) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

and

$$K_P(I-Q)N\big(y(t)\big) = \begin{pmatrix} \int_0^t f_1(t)\,dt + \sum_{0<t_k<t} \ln(1+h_k^1) \\ \int_0^t f_2(t)\,dt + \sum_{0<t_k<t} \ln(1+h_k^2) \end{pmatrix} + \left(\frac{1}{2} - \frac{t}{\omega}\right)\begin{pmatrix} \int_0^\omega f_1(t)\,dt + \sum_{k=1}^q \ln(1+h_k^1) \\ \int_0^\omega f_2(t)\,dt + \sum_{k=1}^q \ln(1+h_k^2) \end{pmatrix}$$
$$- \frac{1}{\omega}\begin{pmatrix} \int_0^\omega \int_0^t f_1(s)\,ds\,dt + \sum_{k=1}^q \ln(1+h_k^1) \\ \int_0^\omega \int_0^t f_2(s)\,ds\,dt + \sum_{k=1}^q \ln(1+h_k^2) \end{pmatrix}$$

by *Lebesque* convergence theorem, we know QN and $K_P(I-Q)N$ are continuous. Moreover, by Lemma 2.1, we get $QN(\overline{\Omega}), K_P(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L-compact on $\overline{\Omega}$, here Ω is any open bounded set in X.

Now we are in a position to search for an appropriate open bounded subset Ω for the application of Lemma 3.1, corresponding to equation $Ly = \lambda Ny$, $\lambda \in (0,1)$, we have

$$\begin{cases} \dot{y}_i(t) = \lambda f_i(t), & t \neq t_k, \\ y_i(t_k^+) = \lambda\big(\ln(1+h_k^i) + y_i(t_k)\big), & t = t_k,\ i = 1,2, \end{cases} \tag{3.2}$$

where $f_i(t)$ are defined as (3.1). Suppose that $y(t) = (y_1(t), y_2(t))^T \in X$ is a solution of system (3.2) for a certain $\lambda \in (0,1)$. By integrating system (3.2) over the interval $[0,\omega]$, we can obtain

$$\int_0^\omega \left\{ b(t) - a(t)e^{y_1(t)} - \frac{c(t)e^{y_1(t)} + e^{y_2(t)}}{h^2 e^{2y_1(t)+2y_2(t)}} \right\} dt = -\sum_{k=1}^q \ln\big(1+h_k^1\big). \tag{3.3}$$

Thus

$$\int_0^\omega \left\{ a(t)e^{y_1(t)} + \frac{c(t)e^{y_1(t)+y_2(t)}}{h^2 e^{2y_1(t)} + e^{2y_2(t)}} \right\} dt = \sum_{k=1}^q \ln\big(1+h_k^1\big) + \bar{b}\omega. \tag{3.4}$$

From (3.2), (3.4), we can obtain

$$\int_0^\omega \big|\dot{y}_1(t)\big|\,dt \leqslant \big(|\bar{b}| + \bar{b}\big)\omega + \sum_{k=1}^q \big|\ln\big(1+h_k^1\big)\big| + \sum_{k=1}^q \ln\big(1+p_k^1\big) \equiv A_1. \tag{3.5}$$

Similarly, we have

$$\int_0^\omega \big|\dot{y}_2(t)\big|\,dt \leqslant \big(|\bar{e}| + \bar{e}\big)\omega + \sum_{k=1}^q \big|\ln\big(1+h_k^2\big)\big| + \sum_{k=1}^q \ln\big(1+p_k^2\big) \equiv A_2. \tag{3.6}$$

Note that $y = (y_1(t), y_2(t))^T \in X$, then there exist $\xi_i, \eta_i \in [0,\omega]$, $i = 1,2$, such that

$$y_i(\xi_i) = \inf_{t\in[0,\omega]} y_i(t), \qquad y_i(\eta_i) = \sup_{t\in[0,\omega]} y_i(t). \tag{3.7}$$

It follows from (3.4), (3.7) that

$$\bar{a}\exp\bigl(y_1(\xi_1)\bigr)\omega \leqslant \int_0^{\omega} a(t)\exp\bigl(y_1(t)\bigr)\,dt \leqslant \sum_{k=1}^{q}\ln\bigl(1+h_k^1\bigr)+\bar{b}\omega,$$

which implies that

$$y_1(\xi_1) \leqslant \ln\Biggl(\frac{1}{\omega}\sum_{k=1}^{q}\ln\bigl(1+h_k^1\bigr)+\bar{b}\Biggr)-\ln\bar{a} \equiv B_1. \tag{3.8}$$

This combined with (3.5) and Lemma 2.2 gives

$$y_1(t) \leqslant y_1(\xi_1)+\frac{1}{2}\Biggl(\int_0^{\omega}\bigl|\dot{y}_1(t)\bigr|\,dt+\sum_{k=1}^{q}\bigl|\ln\bigl(1+h_k^1\bigr)\bigr|\Biggr) \leqslant B_1+\frac{A_1}{2}+\frac{1}{2}\sum_{k=1}^{q}\bigl|\ln\bigl(1+h_k^1\bigr)\bigr| \equiv C_1. \tag{3.9}$$

In particular, we have $y_1(\eta_1) \leqslant C_1$.

On the other hand, from (3.2), (3.4) and (3.9), we have

$$\sum_{k=1}^{q}\ln\bigl(1+h_k^1\bigr)+\bar{b}\omega \leqslant \bar{a}\omega\exp\bigl(y_1(\eta_1)\bigr)+\frac{\bar{c}\omega}{2|h|}. \tag{3.10}$$

Then we have

$$y_1(\eta_1) \geqslant \ln\frac{\frac{1}{\omega}\sum_{k=1}^{q}\ln(1+h_k^1)+\overline{b-\frac{c}{2|h|}}}{\bar{a}} = D_1. \tag{3.11}$$

Then we derive from Lemma 2.2, (3.5), and (3.11) that

$$y_1(t) \geqslant y_1(\eta_1)-\frac{1}{2}\Biggl(\int_0^{\omega}\bigl|\dot{y}_1(t)\bigr|\,dt+\sum_{k=1}^{q}\bigl|\ln\bigl(1+h_k^1\bigr)\bigr|\Biggr) \geqslant D_1-\frac{A_1}{2}-\frac{1}{2}\sum_{k=1}^{q}\bigl|\ln\bigl(1+h_k^1\bigr)\bigr| \equiv E_1. \tag{3.12}$$

From (3.9) and (3.12), it follows that

$$\max_{t\in[0,\omega]}\bigl|y_1(t)\bigr| \leqslant \max\bigl\{|C_1|,|E_1|\bigr\} = H_1.$$

Similarly, we have

$$\max_{t\in[0,\omega]}\bigl|y_2(t)\bigr| \leqslant \max\bigl\{|C_2|,|E_2|\bigr\} = H_2,$$

where

$$C_2 = H_1+\ln\bar{e}-\ln\bar{f}+\frac{1}{2}\Biggl(\overline{|e|}+\bar{e}+\frac{1}{\omega}\sum_{k=1}^{q}\ln\bigl(1+h_k^2\bigr)\Biggr)+\sum_{k=1}^{q}\bigl|\ln\bigl(1+h_k^2\bigr)\bigr|,$$

$$E_2 = \ln\bar{e}-\ln\bar{f}-H_1-\frac{1}{2}\Biggl(\overline{|e|}+\bar{e}+\frac{1}{\omega}\sum_{k=1}^{q}\ln\bigl(1+h_k^2\bigr)\Biggr)-\sum_{k=1}^{q}\bigl|\ln\bigl(1+h_k^2\bigr)\bigr|.$$

It is obvious that H_1, H_2 are independent of λ.

Let $H = \max\{H_1, H_2\}+c$, where c is sufficiently large such that the solution $(\ln u_1^*, \ln u_2^*)^T$ of

$$\bar{b}-\bar{a}e^{y_1}-\frac{\bar{c}e^{y_1+y_2}}{h^2e^{2y_1}+e^{2y_2}}+\frac{1}{\omega}\sum_{k=1}^{q}\ln\bigl(1+h_k^1\bigr)=0,$$

$$\bar{e}-\bar{f}e^{y_2-y_1}+\frac{1}{\omega}\sum_{k=1}^{q}\ln\bigl(1+h_k^2\bigr)=0 \tag{3.13}$$

satisfies $\max\{|\ln u_i^*|\} < c$, then $\|y\| < M$. Let $\Omega = \{y=(y_1,y_2)\in(PC'_\omega)^2\colon\ \|x\| \leqslant H\}$ that Ω verifies the requirement (1) in Lemma 3.1, when $x\in\partial\Omega\cap R^2$, x is a constant vector in R^2 with $\|x\| = H$, then

$$QNy = \left(\begin{pmatrix} \bar{b}-\bar{a}e^{y_1}-\frac{\bar{c}e^{y_1+y_2}}{h^2e^{2y_1}+e^{2y_2}}+\frac{1}{\omega}\sum_{k=1}^{q}\ln(1+h_k^1) \\ \bar{e}-\bar{f}e^{y_2-y_1}+\frac{1}{\omega}\sum_{k=1}^{q}\ln(1+h_k^2)\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)_{2\times 1}$$

and for $x \in \operatorname{Ker} L \cap \Omega$. We have

$$JQNy = \begin{pmatrix} \bar{b} - \bar{a}e^{y_1} - \frac{\bar{c}e^{y_1+y_2}}{h^2e^{2y_1}+e^{2y_2}} + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1+h_k^1) \\ \bar{e} - \bar{f}e^{y_2-y_1} + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1+h_k^2) \end{pmatrix}_{2\times 1},$$

a direct computation gives

$$\deg(JQN|_{\operatorname{Ker} L\cap\partial\Omega}) \neq 0.$$

Here J is taken as the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. So far we have proved that Ω satisfies all the assumptions in Lemma 3.1. Hence (3.1) has at least one ω-periodic solutions $\bar{y}$ with $\bar{x} \in \overline{\Omega} \cap \operatorname{Dom} L$. Let $\bar{x}_i(t) = \exp(\bar{y}_i)$, $i = 1, 2$. Then $(\bar{x}_1(t), \bar{x}_2(t))$ are positive ω-periodic solutions of (1.3). The proof is completed. □

4. Attractivity of positive periodic solutions

Lemma 4.1. *(See [39].) Let f be a nonnegative function defined on $[0, +\infty)$ such that f is integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$. Then $\lim_{t\to+\infty} f(t) = 0$.*

Theorem 4.1. *In addition to the conditions (H_1)–(H_3), if the following condition also holds:*

(H_4) *There exist positive constants s_i, ω_i, $i = 1, 2$, and ρ such that $\min_{t\in[0,\omega]}\{\psi_i(t), \zeta_i(t)\} > \rho$, $i = 1, 2$, with*

$$\psi_i(t) = \left(s_1a(t) - \frac{s_2f(t)x_2^*(t)}{x_{*1}^2(t)} + \frac{2s_1c(t)x_{*1}^2(t)x_{*2}(t)}{(h^2x_2^{*2}(t)+x_1^{*2}(t))^2} + \frac{s_1c(t)x_{*2}(t)}{h^2x_2^{*2}(t)+x_1^{*2}(t)}\right),$$
$$\zeta_i(t) = \left(\frac{s_2f(t)}{x_1^*(t)} - \frac{s_1c(t)x_1^*(t)}{h^2x_{*2}^2(t)+x_{*1}^2(t)} - \frac{2h^2s_1c(t)x_1^*(t)x_2^{*2}(t)}{(h^2x_{*2}^2(t)+x_{*1}^2(t))^2}\right), \tag{4.1}$$

where $x_{*i}(t)$, $x_i^*(t)$, $i = 1, 2$, are defined as Theorem 3.1. Then system (1.3) has a unique positive ω-periodic solution which is globally attractive.

Proof. We shall show that the periodic solution $(x_1(t), x_2(t))$ of system (1.3) is globally attractive. Let $(y_1(t), y_2(t))$ be any other solution of system (1.3). Consider the following Lyapunov function:

$$W(t) = \sum_{i=1}^{2} s_i\left|\ln x_i(t) - \ln y_i(t)\right|. \tag{4.2}$$

Calculating the upper right derivative $D^+W(t)$ of $W(t)$ along the solution of (4.2), by simplifying, for $t \neq t_k$, we have

$$\begin{aligned}
D^+W(t) \leqslant{}& -s_1c(t)\operatorname{sgn}\{x_1(t)-y_1(t)\}\left[\frac{x_1(t)x_2(t)-y_1(t)y_2(t)}{h^2x_2^2(t)+x_1^2(t)} + \frac{y_1(t)y_2(t)}{h^2x_2^2(t)+x_1^2(t)} - \frac{y_1(t)y_2(t)}{h^2y_2^2(t)+y_1^2(t)}\right] \\
& - s_1a(t)\left|x_1(t)-y_1(t)\right| - s_2f(t)\operatorname{sgn}\{x_2(t)-y_2(t)\}\left[\frac{x_2(t)}{x_1(t)} - \frac{y_2(t)}{y_1(t)} + \frac{y_2(t)}{x_1(t)} - \frac{y_2(t)}{y_1(t)}\right] \\
\leqslant{}& -s_1c(t)\operatorname{sgn}\{x_1(t)-y_1(t)\}\left[\frac{x_2(t)(x_1(t)-y_1(t))}{h^2x_2^2(t)+x_1^2(t)} + \frac{y_1(t)(x_2(t)-y_2(t))}{h^2x_2^2(t)+x_1^2(t)}\right. \\
& \left. + \frac{y_1(t)y_2(t)(h^2y_2^2(t)-h^2x_2^2(t))+y_1^2(t)-x_1^2(t)}{(h^2x_2^2(t)+x_1^2(t))(h^2y_2^2(t)+y_1^2(t))}\right] - \left(s_1a(t) - \frac{s_2f(t)y_2(t)}{x_1(t)y_1(t)}\right)\left|x_1(t)-y_1(t)\right| \\
& - \frac{s_2f(t)}{x_1(t)}\left|x_2(t)-y_2(t)\right| \\
\leqslant{}& -\left(s_1a(t) - \frac{s_2f(t)x_2^*(t)}{x_{*1}^2(t)} + \frac{2s_1c(t)x_{*1}^2(t)x_{*2}(t)}{(h^2x_2^{*2}(t)+x_1^{*2}(t))^2} + \frac{s_1c(t)x_{*2}(t)}{h^2x_2^{*2}(t)+x_1^{*2}(t)}\right)\left|x_1(t)-y_1(t)\right| \\
& - \left(\frac{s_2f(t)}{x_1^*(t)} - \frac{s_1c(t)x_1^*(t)}{(h^2x_{*2}^2(t)+x_{*1}^2)^2(t)} - \frac{2h^2s_1c(t)x_1^*(t)x_2^{*2}(t)}{(h^2x_{*2}^2(t)+x_{*1}^2(t))^2}\right)\left|x_2(t)-y_2(t)\right| \\
\leqslant{}& -\rho\sum_{i=1}^{2}\left|x_i(t)-y_i(t)\right|,
\end{aligned} \tag{4.3}$$

where ρ is defined as (4.1).

On the other hand, for $t = t_k$, we have

$$W(t_k^+) = \sum_{i=1}^{2} s_i \left| \ln x_i(t_k^+) - \ln y_i(t_k^+) \right| = W(t_k). \tag{4.4}$$

From (4.3) and (4.4), we have

$$D^+ W(t) \leqslant 0, \qquad \Delta W(t_k) \leqslant 0.$$

An integration of (4.3) over $[T_0, t]$, we obtain that

$$\rho \int_{T_0}^{t} \sum_{j=1}^{2} \left| x_i(s) - y_i(s) \right| ds \leqslant W(T_0) - W(t), \quad \text{for all } t \geqslant T_0.$$

Therefore, by Lemma 4.1, we have

$$\int_{T_0}^{t} \sum_{j=1}^{2} \left| x_i(s) - y_i(s) \right| ds \leqslant \frac{W(T_0)}{\rho} < +\infty. \tag{4.5}$$

Then we have

$$\lim_{t \to \infty} \left| x_i(t) - y_i(t) \right| = 0, \quad i = 1, 2,$$

which implies the global attractivity of system (1.3). This completes the proof of Theorem 4.1. □

5. An example

Consider the following system

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left(9 + \sin 2t - 4x_1(t) - \frac{\frac{1}{2}x_1(t)x_2(t)}{x_2^2(t) + x_1^2(t)}\right),\\
\dot{x}_2(t) &= x_2(t)\left(9 + \cos 2t - 4\frac{x_2(t)}{x_1(t)}\right), \quad t \neq t_k,\\
x_i(t_k^+) &= \frac{1}{2}x_i(t_k), \quad t_k = k\pi,\ t_{k+2} = t_k + \pi,\ i = 1, 2,\ k \in Z_+ = \{1, 2, \ldots, \}
\end{aligned} \tag{5.1}$$

then the conditions (H_1)–(H_4) of Theorem 4.1 are satisfied. Thus (5.1) has a positive periodic solution, which is globally attractive.

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