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Positive Solutions for Nonlinear m-Point Boundary Value Problems of Dirichlet Type via Fixed-Point Index Theory

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Abstract—Let $a \in C[0,1]$, $b \in C([0,1],(-\infty,0))$. Let $\phi_1(t)$ be the unique solution of the linear boundary value problem

$$u''(t)+a(t)u'(t)+b(t)u(t)=0, \qquad t\in(0,1),$$
$$u(0)=0, \quad u(1)=1.$$

We study the multiplicity of positive solutions for the m-point boundary value problems of Dirichlet type

$$u''+a(t)u'+b(t)u+g(t)f(u)=0,$$
$$u(0)=0, \quad u(1)-\sum_{i=1}^{m-2}\alpha_i u(\xi_i)=0,$$

where $\xi_i\in(0,1)$ and $\alpha_i\in(0,\infty)$, $i\in\{1,\dots,m-2\}$, are given constants satisfying $\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)<1$. The methods employed are fixed-point index theory. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Multipoint boundary value problems, Existence, Positive solutions, Fixed-point index.

1. INTRODUCTION

Linear multipoint boundary value problems were studied by Il'in and Moiseev [1]. Since then, the existence of solutions to nonlinear multipoint boundary value problems has been extensively studied by several authors; see [2–4] and the references therein. Ma [5] applied Guo-Krasnosel'skii's

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fixed-point theorem to study the existence of positive solutions of the three-point boundary value problems

$$\begin{gathered} u'' + \bar{g}(t)\bar{f}(u) = 0, \\ u(0) = 0, \quad u(1) - \alpha u(\xi) = 0, \end{gathered} \tag{1.1}$$

where

- $\bar{f} : [0, \infty) \to [0, \infty)$ is continuous, and
- $\bar{g} : [0, 1] \to [0, \infty)$ is continuous and there exists $t_0 \in [\eta, 1]$ such that $x(t_0) > 0$.

In [5], (1.1) was written to the following integral equation:

$$\begin{aligned} u(t) = &- \int_0^t (t - s)\bar{g}(s)\bar{f}(u(s))\, ds - \frac{\alpha t}{1 - \alpha\eta} \int_0^\eta (\eta - s)\bar{g}(s)\bar{f}(u(s))\, ds \\ &+ \frac{t}{1 - \alpha\eta} \int_0^1 (1 - s)\bar{g}(s)\bar{f}(u(s))\, ds. \end{aligned} \tag{1.2}$$

We note that (1.2) is not *good* for studying *the existence of positive solutions* since it contains two negative terms and one nonnegative term.

In this paper, we study the existence and multiplicity of positive solutions for more general m-point boundary value problem of Dirichlet type

$$\begin{gathered} u'' + a(t)u' + b(t)u + h(t)f(u) = 0, \\ u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} \alpha_i u(\xi_i) = 0. \end{gathered} \tag{1.3}$$

We make the following assumptions:

(C1) For $i \in \{1, \dots, m-2\}$, $\xi_i \in (0, 1)$, and $\alpha_i \in (0, \infty)$ are given constants;
(C2) $a \in C[0, 1]$, $b \in C([0, 1], (-\infty, 0))$.

We remark that positive solutions to (1.1) are concave. But positive solutions of (1.3) are not concave in general. *The loss of concavity* brings about a difficulty in defining an appropriate cone on which the fixed-point theorems in cones was applied.

The main tools of this paper are some existence and multiplicity results established by Lan [6] and Lan and Webb [7] for Hammerstein integral equations. To use those tools and to overcome the above difficulty of *the loss of concavity*, we give a new equivalent integral equation to (1.3), which contains only two nonnegative terms! See Lemma 3.3 below.

2. MULTIPLICITY RESULTS FOR HAMMERSTEIN INTEGRAL EQUATIONS

In this section, we state some existence and multiplicity results of positive solutions for the Hammerstein integral equations, which were established by Lan and Webb [7] and Lan [6].

Consider

$$u(t) = \int_0^1 k(t, s)g(s)f(u(s))\, ds := Tu(t), \tag{2.1}$$

where k, f, and g satisfy the following assumptions:

(A1) $f : [0, \infty) \to [0, \infty)$ is continuous;
(A2) $g \in C[0, 1]$ and $g(s) \ge 0$ for $t \in [0, 1]$, and there exist $a_0, b_0 \in [0, 1]$ such that $\int_{a_0}^{b_0} g(s)\, ds > 0$;
(A3) $k : [0, 1] \times [0, 1] \to [0, \infty)$ is continuous, and there exist a continuous function $\Phi : [0, 1] \to R^+$ and a number $c \in (0, 1]$ such that

$$k(t, s) \le \Phi(s), \qquad \text{for } t, s \in [0, 1],$$

and

$$c\Phi(s) \le k(t, s), \qquad \text{for } t \in [a_0, b_0] \text{ and } s \in [0, 1].$$

Let

$$f_{cr}^r = \min\left\{\frac{f(u)}{r} : u \in [r, cr]\right\}, \qquad f_0^r = \min\left\{\frac{f(u)}{r} : u \in [0, r]\right\},$$

$$m = \left(\max_{0\le t\le 1}\int_0^1 k(t,s)g(s)\,ds\right)^{-1}, \qquad M = \left(\min_{a_0\le t\le b_0}\int_{a_0}^{b_0} k(t,s)g(s)\,ds\right)^{-1},$$

$$f^\delta = \limsup_{x\to\delta}\frac{f(x)}{x}, \qquad f_\delta = \liminf_{x\to\delta}\frac{f(x)}{x},$$

where δ denotes either 0 or $+\infty$. Set

$$K = \{u \in C[0,1] : u \ge 0 \text{ on } [0,1],\ \min\{u(t) : a_0 \le t \le b_0\} \ge c\|u\|\}$$

and

$$\Omega_r = \left\{u \in K : \min_{a_0\le t\le b_0} u(t) < cr\right\}, \qquad K_r := \{u \in K : \|u\| < r\}.$$

THEOREM 2.1. *(See [7].) Let (A1)–(A3) hold and assume that one of the following conditions holds:*

(h1) $0 \le f^0 < m$ *and* $M < f_\infty \le \infty$;
(h2) $0 \le f^\infty < m$ *and* $M < f_0 \le \infty$.

Then equation (2.1) has a nonzero solution in K.

THEOREM 2.2. *(See [6].) Let (A1)–(A3) hold and there exists* $r > 0$ *such that one of the following conditions holds:*

(E1) $0 \le f^0 < m$, $f_{cr}^r \ge cM$, $x \ne Tx$ *for* $x \in \partial\Omega_r$, $0 \le f^\infty < m$;
(E2) $M < f^0 \le \infty$, $f_0^r \le m$, $x \ne Tx$ *for* $x \in \partial\Omega_r$, $M < f^\infty \le \infty$.

Then equation (2.1) has two nonzero solutions in K.

3. A NEW REPRESENTATION

In this section, we will give a new equivalent integral equation to the m-point boundary value problem (1.3). The following lemma is a generalization of [8, Lemma 2.1].

LEMMA 3.1. *Assume that (C2) holds. Let* ϕ_1 *and* ϕ_2 *be the solutions of*

$$\begin{gathered}\phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t) = 0,\\ \phi_1(0) = 0, \qquad \phi_1(1) = 1,\end{gathered} \tag{3.1}$$

and

$$\begin{gathered}\phi_2''(t) + a(t)\phi_2'(t) + b(t)\phi_2(t) = 0,\\ \phi_2(0) = 1, \qquad \phi_2(1) = 0.\end{gathered} \tag{3.2}$$

Then

(i) ϕ_1 *is strictly increasing on* $[0,1]$;
(ii) ϕ_2 *is strictly decreasing on* $[0,1]$.

PROOF. We will give a proof for (i). The proof of (ii) follows in a similar manner.

First we claim that $\phi_1(t) \ge 0$ on $[0,1]$.

Suppose the contrary and let $\phi_1(\tau_0) < 0$ for some $\tau_0 \in (0,1)$. Then there exists $\tau_1 \in (0,1)$ such that

$$\phi_1(\tau_1) = \min\{\phi_1(t) \mid t \in [0,1]\} < 0$$

so that $\phi_1'(\tau_1) = 0$ and $\phi_1''(\tau_1) \ge 0$. On the other hand, $\phi_1''(\tau_1) = -b(\tau_1)\phi_1(\tau_1) < 0$, a contradiction!

Next we show that $\phi_1'(t) \neq 0$ on $(0,1)$.

Suppose the contrary and let t_0 be the first point in $(0,1]$ such that $\phi_1'(t_0) = 0$ (we note that $\phi_1'(0) > 0$ since $\phi_1(0) = 0$). Then $\phi_1''(t_0) = -b(t_0)\phi_1(t_0) > 0$. This implies that t_0 is a minimum point. Since $\phi_1(0) = 0 < \phi_1(t_0)$ (we note that $\phi_1(t_0) > 0$ since $\phi_1'(t_0) = 0$), we conclude that $\phi_1(t)$ has a maximum at a point $t_1 \in (0, t_0)$, a contradiction!

Since $\phi_1(0) = 0$, it follows that $\phi_1'(0) > 0$, and moreover, $\phi_1'(t) > 0$ on $(0,1)$, completing the proof of (i).

LEMMA 3.2. *Assume that (C2) holds. Then (3.1) and (3.2) have unique solutions, respectively.*

PROOF. Let u_1 and u_2 be two distinct solutions of (3.1). Set $z = u_1 - u_2$ and $w = -z$. Then

$$\begin{gathered} z''(t) + a(t)z'(t) + b(t)z(t) = 0, \qquad t \in (0,1), \\ z(0) = 0, \quad z(1) = 0, \end{gathered} \tag{3.3}$$

$$\begin{gathered} w''(t) + a(t)w'(t) + b(t)w(t) = 0, \qquad t \in (0,1), \\ w(0) = 0, \quad w(1) = 0. \end{gathered} \tag{3.4}$$

Using a similar method to prove (i) of Lemma 3.1, we get that

$$z(t) \geq 0, \qquad w(t) \geq 0,$$

for all $t \in (0,1)$. Therefore $z(t) = w(t) \equiv 0$ for all $t \in (0,1)$, a contradiction!

Similarly, we can show that (3.2) has a unique solution.

In the rest of the paper, we need the following:

(C3) $0 < \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i) < 1$.

LEMMA 3.3. *Assume that (C1)–(C3) hold. Let $y \in C[0,1]$. Then the problem*

$$\begin{gathered} u''(t) + a(t)u'(t) + b(t)u(t) + y(t) = 0, \qquad t \in (0,1), \\ u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} \alpha_i u(\xi_i) = 0, \end{gathered} \tag{3.5}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)p(s)y(s)\,ds + A\phi_1(t), \tag{3.6}$$

here

$$A = \frac{\sum\limits_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i, s)p(s)y(s)\,ds}{1 - \sum\limits_{i=1}^{m-2} \alpha_i \phi_1(\xi_i)}, \tag{3.7}$$

$$p(t) = \exp\left(\int_0^t a(s)\,ds\right), \tag{3.8}$$

$$G(t,s) = \frac{1}{\rho}\begin{cases} \phi_1(t)\phi_2(s), & \text{if } s \geq t, \\ \phi_1(s)\phi_2(t), & \text{if } t \geq s, \end{cases} \tag{3.9}$$

and

$$\rho := \phi_1'(0). \tag{3.10}$$

Moreover, $u(t) \geq 0$ on $[0,1]$ provided $y \geq 0$.

PROOF. First we show that the unique solution of (3.5) can be represented by (3.6).

In fact, we know from Lemma 3.1 that the equation

$$u'' + a(t)u' + b(t)u = 0$$

has the two linearly independent solutions ϕ_1 and ϕ_2 since

$$\begin{vmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{vmatrix} = -\phi_1'(0) \neq 0.$$

Now by the method of variation of constants, we can obtain that the unique solution of problem (3.5) can be represented by

$$u(t) = \int_0^1 G(t,s)p(s)y(s)\,ds + A\phi_1(t),$$

where G, A, and p are as in (3.9), (3.7), and (3.8), respectively.

Next we check that the function defined by (3.6) is a solution of (3.5).

From (3.9), we know that

$$u(t) = \int_0^t \frac{1}{\rho}\phi_1(s)\phi_2(t)p(s)y(s)\,ds + \int_t^1 \frac{1}{\rho}\phi_2(s)\phi_1(t)p(s)y(s)\,ds + A\phi_1(t), \tag{3.11}$$

$$u'(t) = \phi_2'(t)\int_0^t \frac{1}{\rho}\phi_1(s)p(s)y(s)\,ds + \phi_1'(t)\int_t^1 \frac{1}{\rho}\phi_2(s)p(s)y(s)\,ds + A\phi_1'(t), \tag{3.12}$$

and

$$\begin{aligned} u''(t) &= \phi_2''(t)\int_0^t \frac{1}{\rho}\phi_1(s)p(s)y(s)\,ds + \phi_2'(t)\frac{1}{\rho}\phi_1(t)p(t)y(t) \\ &\quad + \phi_1''(t)\int_t^1 \frac{1}{\rho}\phi_2(s)p(s)y(s)\,ds - \phi_1'(t)\frac{1}{\rho}\phi_2(t)p(t)y(t) + A\phi_1''(t) \end{aligned} \tag{3.13}$$

so that

$$\begin{aligned} &u''(t) + a(t)u'(t) + b(t)u(t) \\ &\quad = \frac{1}{\rho}\begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix} p(t)y(t) + A\left[\phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t)\right] \\ &\quad = \frac{1}{\rho}\begin{vmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{vmatrix} \exp\left(-\int_0^t a(s)\,ds\right) p(t)y(t) + A\left[\phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t)\right] \\ &\quad = -y(t) + A\left[\phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t)\right] \\ &\quad = -y(t). \end{aligned}$$

Since

$$u(1) = A, \tag{3.14}$$

$$u(\xi_i) = \int_0^1 G(\xi_i,s)p(s)y(s)\,ds + A\phi_1(\xi_i), \qquad i \in \{1,\dots,m-2\}. \tag{3.15}$$

By combining (3.14) and (3.15) and (3.7), we can verify that $u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i)$. From (3.1) and (3.6), we know that $u(0) = 0$. The proof is completed.

REMARK 3.4. If $\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i) > 1$, then $y \in C[0,1]$ with $y(t) \geq 0$ for $t \in (0,1)$ does not imply that (3.5) has a positive solution.

In fact, in [5] we studied the problem

$$\begin{aligned} &u''(t) + y(t) = 0, \qquad t \in (0,1), \\ &u(0) = 0, \quad u(1) - \alpha u(\eta) = 0, \end{aligned} \tag{3.16}$$

which is a special case of (3.5) when $a(t) = b(t) = 0$. We have proved the following proposition.

PROPOSITION. *(See [5, Lemma 3].) If $\alpha\eta > 1$, and if $y \in C[0,1]$ with $y(t) \geq 0$ for $t \in (0,1)$, then problem (3.16) has no positive solution.*

4. POSITIVE SOLUTIONS OF m-POINT BVPS

Choose $\delta \in (0, 1/2)$ such that $[a_0, b_0] \subset (\delta, 1-\delta)$.

We now consider the m-point boundary value problem

$$\begin{gathered} u''(t) + a(t)u'(t) + b(t)u(t) + g(t)f(u(t)) = 0, \qquad t \in [0,1], \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i). \end{gathered} \tag{4.1}$$

In view of Lemma 3.3, (4.1) is equivalent to

$$u(t) = \int_0^1 K(t,s)g(s)f(u(s))\,ds, \tag{4.2}$$

where $K : [0,1] \times [0,1] \to [0, \infty)$ is defined by

$$K(t,s) := G(t,s)p(s) + \frac{\sum_{i=1}^{m-2} \alpha_i G(\xi_i, s)\phi_1(t)p(s)}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i)}; \tag{4.3}$$

here p and G are defined by (3.8) and (3.9).

To apply the results in Section 2, we verify that (A3) holds.

Set

$$q(t) = \frac{1}{\rho}\phi_1(s)\phi_2(s) \tag{4.4}$$

and

$$\Phi(s) = q(s)p(s)\left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i)}\right). \tag{4.5}$$

It is easy to check that

$$G(t,s) \le G(s,s) = q(s), \qquad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{4.6}$$

This implies

$$G(\xi_i, s) \le G(s,s) = q(s), \qquad \text{for } (t,s) \in [0,1] \times [0,1], \quad i = 1, 2, \dots, m-2. \tag{4.7}$$

This together with (4.6) and (4.5) and (4.3) implies that

$$K(t,s) \le \Phi(s), \qquad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{4.8}$$

Lower Bounds

For $(t,s) \in [\delta, 1-\delta] \times [0,1]$, we have that

$$\begin{aligned} K(t,s) &\ge G(t,s)p(s) \\ &\ge \frac{p(s)}{\rho}\begin{cases} \phi_1(\delta)\phi_2(s), & \text{if } s \ge t, \\ \phi_1(s)\phi_2(1-\delta), & \text{if } t \ge s, \end{cases} \\ &\ge \frac{p(s)}{\rho}\min\{\phi_1(\delta), \phi_2(1-\delta)\}\min\{\phi_1(s), \phi_2(s)\} \\ &\ge p(s)\min\{\phi_1(\delta), \phi_2(1-\delta)\}q(s) \\ &= \Gamma\Phi(s), \end{aligned} \tag{4.9}$$

where

$$\Gamma = \min\{\phi_1(\delta), \phi_2(1-\delta)\}\left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i)}\right)^{-1}. \tag{4.10}$$

As a subsequence of Theorems 2.1 and 2.2, we have the following theorem.

THEOREM 4.1. *Let (C1)–(C3) and (A1),(A2) hold. Then (4.1) has at least one positive solution if either*

(h1) $0 \le f^0 < m_1$ *and* $M_1 < f_\infty \le \infty$, *or*

(h2) $0 \le f^\infty < m_1$ *and* $M_1 < f_0 \le \infty$,

and has two positive solutions if there is $r > 0$ *such that either*

(E1) $0 \le f^0 < m_1$, $f_{cr}^r \ge cM_1$, $x \ne Tx$ *for* $x \in \partial\Omega_r$, $0 \le f^\infty < m_1$, *or*

(E2) $M < f^0 \le \infty$, $f_0^r \le m_1$, $x \ne Tx$ *for* $x \in \partial\Omega_r$, $M_1 < f^\infty \le \infty$.

Here

$$m_1 = \left(\max_{0\le t\le 1} \int_0^1 K(t,s)g(s)\,ds\right)^{-1}, \qquad M_1 = \left(\min_{\delta\le t\le 1-\delta} \int_\delta^{1-\delta} K(t,s)g(s)\,ds\right)^{-1}.$$

REMARK 4.2. Theorem 4.1 generalizes the main results of [5] in four main directions.

(a) More general linear differential operators $Lu = u'' + a(t)u' + b(t)u$ are considered.

(b) If $a(t) = b(t) \equiv 0$ on $[0,1]$ and $\alpha_2 = \cdots = \alpha_{m-2} = 0$, then (C1): $0 < \sum_{i=1}^{m-2} \alpha_1\phi_1(\xi_i) < 1$ reduces to $0 < \alpha\eta < 1$ which is a key condition (A1) of [5, Theorem 1]. Also (A2) in Theorem 4.1 is weaker than the condition

$$g \in C([0,1],[0,\infty)) \text{ and there exists } t_0 \in [\eta, 1] \text{ such that } x(t_0) > 0. \tag{4.11}$$

(4.11) is a key condition in [5, Theorem 1].

(c) We allow that f^∞, f^0, f_∞, and $f_0 \in [0,\infty)$, but in [5], only the superlinear case and sublinear case were considered.

(d) In Theorem 4.1, we give sufficient conditions to ensure the existence of at least two positive solutions. But in [5], only the existence of positive solutions was proved.

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