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Endomorphisms of $\mathcal{B}(\mathcal{H})$

II. Finitely Correlated States on $\mathcal{O}_n$

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We identify sets of conjugacy classes of ergodic endomorphisms of $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a fixed separable Hilbert space. They correspond to certain equivalence classes of pure states on the Cuntz algebras $\mathcal{O}_n$ where n is the Powers index. These states, called finitely correlated states, and strongly asymptotically shift invariant states, are defined and characterized. The subsets of these states defining shifts will in general be identified in a later work, but here an interesting cross section for the conjugacy classes of shifts called diagonalizable shifts is introduced and studied.

1. INTRODUCTION

Let $\mathcal{H}$ be a given separable infinite-dimensional Hilbert space. If α is a unital endomorphism of $\mathcal{B}(\mathcal{H})$, the (Powers) index of α is defined as the $n \in \{1, 2, ..., \infty\}$ such that the commutant of $\alpha(\mathcal{B}(\mathcal{H}))$ is isomorphic to the factor of type I_n, [Pow2]. Throughout this paper, we will always let "endomorphism" mean unital *-endomorphism. It is well known (see [Arv], [Lac1, Theorem 2.1, Proposition 2.2] and [BJP, Theorem 3.1]) that there is a one-one correspondence between endomorphisms of $\mathcal{B}(\mathcal{H})$ of index n, and non-degenerate *-representations (henceforth called representations) of $\mathcal{O}_n$ on $\mathcal{H}$, up to the canonical action of $U(n)$ on $\mathcal{O}_n$, where $\mathcal{O}_n$ is the Cuntz algebra of order n. We say that two endomorphisms α, β in $\mathrm{End}(\mathcal{B}(\mathcal{H}))$ are *conjugate* if there is an automorphism γ of $\mathcal{B}(\mathcal{H})$ such that $\alpha \circ \gamma = \gamma \circ \beta$; and this means that they have the same index n, and

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that the corresponding representations of $\mathcal{O}_n$ are unitarily equivalent up to the action of $U(n)$, see [Lac1, Proposition 2.4] and [BJP, Theorem 3.3]. We are interested in two subclasses of the class of endomorphisms of $\mathcal{B}(\mathcal{H})$, namely the class of *ergodic* endomorphisms (i.e., those such that $\mathbb{C}1$ are the only invariant elements) and the even smaller class of *shifts* (i.e., those endomorphisms α such that $\bigcap_{n=1}^{\infty} \alpha^n(\mathcal{B}(\mathcal{H})) = \mathbb{C}1$). The first of these families corresponds to irreducible representations of $\mathcal{O}_n$, and the classification of their conjugacy classes thus amounts to the classification of pure states of $\mathcal{O}_n$, up to the action of $U(n)$ and unitary equivalence. Since $\mathcal{O}_n$ is an antiliminal C^*-algebra, this classification is therefore non-smooth, [BJP, Theorem 1.1], [Dix], [Gli]. We show here in Sections 3–6 that the smaller set of finitely correlated states (definition below) on $\mathcal{O}_n$ gives both a "rich" set of conjugacy classes of ergodic endomorphisms, and at the same time these states lend themselves to explicit calculations. They form a union of finite-dimensional manifolds. The conjugacy classes can be calculated. Using recent concepts and results of Fannes *et al.* [FNW2] we will in a forthcoming paper, [BJW], identify those finitely correlated states on $\mathcal{O}_n$ which correspond to shifts on $\mathcal{B}(\mathcal{H})$.

Although our main concern is with pure states of $\mathcal{O}_n$ which give rise to shifts, i.e., pure states such that the canonical UHF-subalgebra UHF_n is weakly dense in the operators on the representation Hilbert space, a generic pure state of $\mathcal{O}_n$ will of course not have this property. In fact, UHF_n is the fixed point algebra of the gauge action of $\mathbb{T}$ of $\mathcal{O}_n$, and this is a quasi-product action by condition 11 of the main theorem in [BEEK]. By condition 9 of that theorem, or, more explicitly by [Eva], $\mathcal{O}_n$ has gauge invariant pure states ω, and then $\omega|_{\mathrm{UHF}_n}$ is pure, but UHF_n is not dense, so these define ergodic endomorphisms which are not shifts. For the case $n = \infty$, see [Lac2, Theorem 4.3].

Let S be an isometry on a Hilbert space $\mathcal{H}$, and let $n := \dim N(S^*)$. Then for every k, we have a canonical decomposition

$$\mathcal{H} = \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{k \text{ times}} \oplus S^k \mathcal{H}.$$

If S is a shift, i.e., $\bigcap S^m \mathcal{H} = \{0\}$, we say that n is the multiplicity of the shift. It is known that n is a complete unitary invariant for the shifts. For an endomorphism α of $\mathcal{B}(\mathcal{H})$ of finite index n we similarly have a canonical decomposition

$$\mathcal{B}(\mathcal{H}) = \underbrace{M_n \otimes \cdots \otimes M_n}_{k \text{ times}} \otimes \alpha^k(\mathcal{B}(\mathcal{H}))$$

where n denotes the Powers index. But now as noted, even when α is a shift on $\mathscr{B}(\mathscr{H})$, n is not a complete conjugacy invariant. In fact, in [BJP], we display a nonsmooth continuum of nonconjugate $\mathscr{B}(\mathscr{H})$ shifts for each value of the Powers index $n \geqslant 2$.

In Section 6, we characterize the pure states ω on $\mathcal{O}_n$ with the property $\omega \circ \sigma^{k+1} = \omega \circ \sigma^k$ for some $k \in \mathbb{N}$, where σ is the canonical shift on $\mathcal{O}_n$, see (3.1). The set S_k of these states has a natural structure as a finite-dimensional differentiable manifold, and as a manifold it is diffeomorphic to the manifold $\mathscr{L}_{n,k}$ consisting of all pairs (L, R), where

$$L \in \mathscr{L}(\mathbb{C}^n, \mathscr{B}(\mathbb{C}^{n^k})),$$
$$R \in \mathscr{B}(\mathbb{C}^{n^k}),$$

and, with

$$L_i = L(|i\rangle),$$

we have the following properties:

$$R \geqslant 0 \qquad \text{and} \qquad \operatorname{Tr}(R) = 1,$$

$$\sum_{i=1}^{n} L_i L_i^* = P$$

where P is a projection in $\mathscr{B}(\mathbb{C}^{n^k})$,

$$PL_i = L_i P = L_i, \qquad PR = RP = R,$$

$RP \geqslant \lambda P$ for some $\lambda > 0$.

$$\sum_{i=1}^{n} L_i^* R L_i = R,$$

and, up to a scalar, R is the unique solution of this equation. See Theorem 6.1 for other versions of the latter conditions.

In Section 7, we show that the action $\omega \to \omega \circ \tau_{g^{-1}}$ of $U(n)$ on the state space of $\mathcal{O}_n$ gives rise to an action R_n of $U(n)$ on the manifold $\mathscr{L}_{n,k}$ by

$$(R_n(g)\, L)(x) = \operatorname{Ad}_k(g)\, L(g^{-1}x)$$
$$(R_n(g)\, R) = \operatorname{Ad}_k(g)\, R$$

for $x \in \mathbb{C}^n$, $g \in U(n)$, where

$$\operatorname{Ad}_k(g) = \underbrace{\operatorname{Ad}(g) \otimes \cdots \otimes \operatorname{Ad}(g)}_{k \text{ times}}$$

and τ is the canonical action of $U(n)$ on $\mathcal{O}_n$, see end of Section 2. The associated orbits correspond 1–1 to conjugacy classes of shifts with Powers index n. (In Section 7, the action R_n will actually be replaced by the coaction $g \to R_n(g^{-1})$.)

Of course, by linearization, we may embed $\mathcal{L}_{n,k}$ as a closed submanifold of a Hilbert space with inner product

$$\langle (L, R) \mid (L', R') \rangle = \mathrm{Trace}_{M_{n^k}}\left(\sum_{j=1}^{n} L_j^* L_j' \right) + \mathrm{Trace}_{M_n}(R^* R')$$

and the action of $U(n)$ then extends to a unitary representation.

We are concerned in Section 7 with elements in a closed subset of $\bigcup_{k=1}^{\infty} P_k$, where P_k is defined in the introduction to Section 3. Section 8 is about the complement of the closure of $\bigcup_k P_k$. Suppose $\omega \in P_k$, then $\omega \circ \sigma^{k+1} = \omega \circ \sigma^k$, and so $\omega \circ \sigma^k$ is σ-invariant. This state therefore extends canonically to a shift invariant state on the UHF-algebra

$$\bigotimes_{\mathbb{Z}} M_n = \bigotimes_{-\infty}^{\infty} M_n$$

which will be denoted ω_∞. The space $\mathcal{L}_n$ will be defined in Section 7 such that the mapping $(\mathcal{L}_n \ni (L, R)) \to \underbrace{\omega_\infty(L, R)}_{1}$ is 1–1. If τ_g^∞ denotes the $U(n)$-action

$$\tau_g^\infty := \bigotimes_{-\infty}^{\infty} \mathrm{Ad}(g)$$

on $\bigotimes_{-\infty}^{\infty} M_n$, then the representation $R_n(g)$ is given by

$$\omega_\infty(L, R) \circ \tau_{g^{-1}}^\infty = \omega_\infty(R_n(g)(L, R)).$$

Also the assignment $\omega \to \omega_\infty$ is such that the two shifts α_{π_ω} and $\alpha_{\pi_{\omega'}}$ (for given $\omega, \omega' \in P$) are conjugate iff there is a $g \in U(n)$ such that

$$\omega'_\infty = \omega_\infty \circ \tau_g^\infty,$$

or equivalently, for the corresponding elements $L, L' \in \mathcal{L}_n$, we have $L' = R_n(g)\, L$.

To identify these infinite families of nonconjugate shifts we introduce in Section 7 a class of elements $\omega \in P$ which we call diagonalizable. If π_0 denotes the Haar representation (see [BJP]) of $\mathcal{O}_n$ acting on $\mathcal{H}_0 = L^2(X, \mu_0)$, where $X = \mathbb{Z}_n^{\mathbb{N}}$, and μ_0 denotes the corresponding Haar measure on X, then we say that π is *diagonalizable* if there is a measurable function $u: X \to \mathbb{T}^1$ such that $\pi(s_i) = M_u \pi_0(s_i)$ where M_u is the multiplication operator defined from u. The diagonalizable elements will be denoted by $P_{\mathcal{D}}$. The result in Section 7 is the assertion that $P_{\mathcal{D}}$ is a "section" for the $U(n)$-orbits under the representation R_n described above: Specifically, $P_{\mathcal{D}}$ intersects a generic set of $U(n)$-orbits in a finite dimensional manifold diffeomorphic to a disjoint union of $n!$ copies of $\mathbb{T}^n$. This means that by just varying the functions $u: X \to \mathbb{T}$ we get a set of distinct conjugacy classes in P.

2. PRELIMINARIES AND NOTATION

Let $H = H_n \simeq \mathbb{C}^n$ be a finite-dimensional complex Hilbert space. The dimension n will be fixed throughout, and the inner product on H will be the usual one

$$\langle x \mid y \rangle = \sum_{i=1}^{n} \bar{x}_i y_i \tag{2.1}$$

for elements $x, y \in H$ with coordinate representation $x = (x_1, ..., x_n)$; and the norm $\|\cdot\|$ is given by

$$\|x\|^2 = \langle x \mid x \rangle = \sum_1^n |x_i|^2.$$

Consider the free unital *-algebra generated by H, i.e., the *-algebra of all polynomials of $h \in H$ and $h^* \in \bar{H}$, where $\bar{H}$ is the conjugate Hilbert space of H. If one adds the relation

$$h^* k = \langle h, k \rangle \, 1 \tag{2.2}$$

then the C^*-envelope of the resulting *-algebra is the familiar Cuntz–Toeplitz C^*-algebra, [Eva], [JSW]. If $\{e_i\}_{i=1}^n$ is a basis for H, e.g.,

$$e_i = (\underbrace{0, ..., 0,}_{i-1 \text{ places}}\ 1, 0, ..., 0), \tag{2.3}$$

and one adds the relation

$$\sum_{i=1}^{n} e_i e_i^* = 1 \tag{2.4}$$

then the resulting C^*-algebra $\mathcal{O}_n$ is the *Cuntz-algebra*. It is well known [Cun] to be simple, and it plays a crucial role (see [Lac1], [Lac2], [Arv], and [BJP]) in the study of the endomorphisms of $\mathcal{B}(\mathcal{H})$.

To stress the distinction between elements in H, and elements in one of the involutive algebras generated by H and $\bar{H}$, we adopt the notation s_h and s_h^* for the corresponding elements in the algebra. With the specific choice of basis, we write s_i for s_{e_i}. The relation (2.2) may then be written in the familiar form

$$s_i^* s_j = \delta_{ij} 1, \tag{2.5}$$

or in a basis free form

$$s_h^* s_k = \langle h, k \rangle 1. \tag{2.6}$$

The second relation (2.4) becomes

$$\sum_{i=1}^{n} s_i s_i^* = 1.$$

Let $\mathcal{K}$ be the C^*-algebra of the compact operators (on a separable Hilbert space). Then we have the familiar short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}_n \to \mathcal{O}_n \to 0$$

where $\mathcal{T}_n$ denotes the Cuntz–Toeplitz algebra. See [Eva] and [BEGJ] for details. In fact $\mathcal{K}$ is isomorphic to the two-sided ideal in $\mathcal{T}_n$ generated by $1 - \sum_{i=1}^{n} s_i s_i^*$.

Let π be a representation of $\mathcal{O}_n$ on a Hilbert space $\mathcal{H}$, and set $S_i = \pi(s_i)$. Then the formula

$$\alpha(A) = \sum_{i=1}^{n} S_i A S_i^* \tag{2.7}$$

for $\forall A \in \mathcal{B}(\mathcal{H})$ defines an endomorphism of $\mathcal{B}(\mathcal{H})$, of Powers index n (see [Pow2] and [BJP]). As mentioned in the introduction, every endomorphism of $\mathcal{B}(\mathcal{H})$ arises this way. (The result (see [Lac2]) may be modified to apply also to the case when the Powers index is infinite.)

Recall that $\alpha \in \operatorname{End}(\mathcal{B}(\mathcal{H}))$ is *ergodic* if the subalgebra

$$\{A \in \mathcal{B}(\mathcal{H})) : \alpha(A) = A\}$$

is one-dimensional; and that α is a *shift* if

$$\bigcap_{k=1}^{\infty} \alpha^k(\mathcal{B}(\mathcal{H}))$$

is one-dimensional, i.e., if the intersection is of the form $\mathbb{C}1$ where 1 is the identity in $\mathscr{B}(\mathscr{H})$.

There is an action τ by automorphisms of the group $\mathbb{T}$ on $\mathcal{O}_n$, given by $\tau_z(s_h) = zs_h$, for $z \in \mathbb{T}$ and $h \in H$. The corresponding subalgebra

$$\mathcal{O}_n^\tau = \{a \in \mathcal{O}_n : \tau_z(a) = a, \forall z \in \mathbb{T}\} \tag{2.8}$$

is denoted by UHF_n, and has the form

$$\underbrace{M_n \otimes M_n \otimes \cdots}_{1 \text{ to } \infty}. \tag{2.9}$$

Recall from [Cun] that UHF_n is generated linearly by the following elements

$$s_{i_1} s_{i_2} \cdots s_{i_m} s_{j_m}^* \cdots s_{j_2}^* s_{j_1}^*. \tag{2.10}$$

In fact, the isomorphism between (2.8) and (2.9) is given by letting the element (2.10) correspond to

$$e_{i_1 j_1}^{(1)} \otimes e_{i_2 j_2}^{(2)} \otimes \cdots \otimes e_{i_m j_m}^{(m)} \tag{2.11}$$

where e_{ij} denote the usual matrix units in M_n. We will sometimes use the Dirac notation

$$e_{ij} = |e_i\rangle\langle e_j|. \tag{2.12}$$

As mentioned in the introduction, the action τ_z of $\mathbb{T}$ naturally extends to an action τ of the unitary group $U(n)$ of $\mathbb{C}^n$. For $g \in U(n)$, the automorphism τ_g on $\mathcal{O}_n$ is determined by

$$\tau_g(s_x) := s_{gx} \qquad \text{for} \quad \forall x \in \mathbb{C}^n.$$

The restriction of τ_g to the subalgebra UHF_n is just the product action

$$\underbrace{\mathrm{Ad}(g) \otimes \mathrm{Ad}(g) \otimes \cdots}_{1 \text{ to } \infty} \qquad \text{on} \quad \bigotimes_1^\infty M_n.$$

As we pointed out in the introduction, a given $\alpha \in \mathrm{End}(\mathscr{B}(\mathscr{H}))$ is *ergodic* iff the corresponding $\pi \in \mathrm{Rep}(\mathcal{O}_n, \mathscr{H})$ is irreducible. We also showed in [BJP] that α is a *shift* iff the restriction $\pi|_{\mathrm{UHF}_n}$ is already irreducible. As a consequence, we found, in [BJP], that a classification of the shifts up to conjugacy is given by *equivalence classes* in the set P of all pure states ω on UHF_n such that ω is quasi-equivalent to the shifted state, given by $x \mapsto \omega(1 \otimes x)$, for $\forall x \in \mathrm{UHF}_n$. This equivalence relation is quasi-equivalence up to the action of $U(n)$. But the classification problem is difficult in the sense that the classifiers $P/\sim$ form a non-smooth space.

3. STRONGLY ASYMPTOTICALLY SHIFT INVARIANT STATES AND FINITELY CORRELATED STATES

The present paper deals with a smaller problem. Let σ denote the canonical shift on $\mathcal{O}_n$, defined by

$$\sigma(x) = \sum_{i=1}^{n} s_i x s_i^*, \qquad \forall x \in \mathcal{O}_n. \tag{3.1}$$

We will be considering pure states ω on $\mathcal{O}_n$ such that, for some k,

$$\omega \circ \sigma^{k+1} = \omega \circ \sigma^k. \tag{3.2}$$

These states are said to be *strongly asymptotically shift invariant* (of order k). If k is given, the corresponding set of pure states will be denoted S_k. If ω is a pure state on the subalgebra UHF_n with the invariance property (3.2), we say that $\omega \in P_k$. In the latter case, it follows from [BJP, Lemma 5.2] that $\omega \sim_q \omega \circ \sigma$ on UHF_n, and ω corresponds to a shift on $\mathcal{B}(\mathcal{H})$. Note that if $\omega \in S_k$ restricts to a pure state on UHF_n, then the restriction is contained in P_k. If then $\rho = \omega|_{\mathrm{UHF}_n}$, we proved in [BJP, Lemma 5.2] that ρ extends to a pure state φ on $\mathcal{O}_n$ such that $\pi_\varphi(\mathrm{UHF}_n)$ is weakly dense in $\mathcal{B}(\mathcal{H}_\varphi)$, and it is easily checked that the extension has the invariance property (3.2). It is also clear from the construction in [BJP, Lemma 5.2] that the extension φ is unique up to the gauge action τ of $\mathbb{T}$ (see [Lac1, Theorem 4.3] for the corresponding result when $n = \infty$), and it follows from [BEEK] that the extensions $\varphi \circ \tau_z$, $z \in \mathbb{T}$, are mutually disjoint in the strong sense that

$$\left(\int_{\mathbb{T}}^{\oplus} \pi \circ \tau_z \, dz \right) (\mathcal{O}_n)'' = \mathcal{B}(\mathcal{H}_\varphi) \otimes L^\infty(\mathbb{T}).$$

In fact, this is equivalent to $\pi_\varphi(\mathrm{UHF}_n)$ being dense in $\mathcal{B}(\mathcal{H}_\varphi)$ (see [BEEK] for details). We will show in [BJW] that ω is one of *these* extensions, when $\omega \in S_k$ and $\omega|_{\mathrm{UHF}_n}$ is pure.

We will now introduce a class of states on $\mathcal{O}_n$ which will be called *finitely correlated states*, and in Section 4 we will show that $\bigcup_k S_k$ is contained in these states.

For a given state ω on $\mathcal{O}_n$, the GNS-representation will be denoted by $(\pi_\omega, \mathscr{H}_\omega, \Omega_\omega)$ or simply $(\pi, \mathscr{H}, \Omega)$, i.e., π is the cyclic representation of $\mathcal{O}_n$ on $\mathscr{H}$, with cyclic vector Ω, such that

$$\omega(x) = \langle \Omega \mid \pi(x)\, \Omega \rangle \qquad \text{for} \quad \forall x \in \mathcal{O}_n. \tag{3.3}$$

Extending a definition in [FNW1, FNW2], we say that the state ω is *finitely correlated* if the subspace $\mathscr{V} \subset \mathscr{H}$ generated linearly by Ω and the vectors

$$\pi(s^*_{h_1} s^*_{h_2} \cdots s^*_{h_m})\, \Omega \tag{3.4}$$

for $h_i \in H$, and $m = 1, 2, ...,$ is finite-dimensional.

The space generated linearly by the vectors (3.4) with a fixed m will be denoted by $\mathscr{V}_m$, and $\mathscr{V}_0 = \mathbb{C}\Omega$. If ω is finitely correlated, there is a smallest k such that $\mathscr{V} = \sum_{i=0}^{k} \mathscr{V}_i$. If then $\mathscr{V}_k$ is left invariant by all S_i^*, we say that $\omega \in \mathrm{FC}_k$. (We say this whenever $\mathscr{V}_k$ is left invariant, even if k is not the minimal such k.) Note that FC_k is not necessarily increasing in k, and the union of the FC_k's is not necessarily the set of all finitely correlated states. The set of pure states in FC_k will be denoted by PFC_k.

The definition above is new, as [FNW2] is concerned with a different C^*-algebra, viz., the two-sided infinite tensor product $\bigotimes_{-\infty}^{\infty} M_n$ (see details in Section 8 below). Our present definition for $\mathcal{O}_n$ is on the face of it unrelated, but a main point in our paper is to show that our states may in fact be described with a set of labels which is directly related to those used in [FNW2] for $\bigotimes_{-\infty}^{\infty} M_n$.

The case when $\mathscr{V}$ from above is *one-dimensional*, yields the identity

$$\pi(s_h^*)\, \Omega = \langle h, \varphi \rangle\, \Omega \qquad \text{for} \quad \forall h \in H \tag{3.5}$$

where φ is some fixed vector in H such that $\|\varphi\| = 1$. The corresponding states are called *Cuntz states*. When $\omega = \omega_\varphi$ is a Cuntz-state, its restriction to UHF_n is the pure product state

$$\underbrace{\varphi \otimes \varphi \otimes \cdots}_{1 \text{ to } \infty} \tag{3.6}$$

corresponding to the representation (3.3) of UHF_n, so it follows that the Cuntz states are in $P_0 \simeq S_0$. We showed conversely in [BJP, Theorem 4.1] that every element in P_0 is a Cuntz-state.

Hence the set P_0 is parameterized by the unit-ball in the Hilbert space $H = \mathbb{C}^n$, and we shall show that a corresponding result is also true for S_k. Since clearly $P_k \subset S_k|_{\mathrm{UHF}_n}$, the results in [BJW] then give a parameterization of P_k.

For states ω on $\mathcal{O}_n$, the condition (3.2) is important because of a result which we now proceed to describe. We showed in [BJP, Lemma 5.2] that the pure states ω on UHF_n define shifts on $\mathcal{B}(\mathcal{H})$ iff $\omega \circ \sigma \sim_q \omega$ where $\sim_q$ denotes quasi-equivalence, [Dix]. If ω is given, and $(\pi_\omega, \mathcal{H}_\omega)$ is the GNS-representation (extended to $\mathcal{O}_n$ on the same Hilbert space as in [BJP]) then the corresponding shift α_ω on $\mathcal{B}(\mathcal{H}_\omega)$ is given by

$$\alpha_\omega(A) = \sum_{i=1}^{n} \pi_\omega(s_i)\, A \pi_\omega(s_i)^* \tag{3.7}$$

for $\forall A \in \mathcal{B}(\mathcal{H}_\omega)$. If ω and ω' are two such pure states, we showed ([BJP, Lemma 5.4]) that the corresponding shifts α_ω and $\alpha_{\omega'}$ are *conjugate*, i.e., that $\alpha_{\omega'} = \beta \circ \alpha_\omega \circ \beta^{-1}$ for some $\beta \in \operatorname{Aut} \mathcal{B}(\mathcal{H})$, iff $\exists g \in U(n)$ such that

$$\lim_{m \to \infty} \|\omega' \circ \sigma^m - \omega \circ \tau_g \circ \sigma^m\| = 0. \tag{3.8}$$

The following result is immediate from this:

PROPOSITION 3.1. *If $\omega, \omega' \in \bigcup_{k=1}^{\infty} P_k$, then the corresponding shifts α_ω and $\alpha_{\omega'}$ are conjugate iff $\exists m \in \mathbb{N}$ and $g \in U(n)$ such that*

$$\omega' \circ \sigma^m = \omega \circ \tau_g \circ \sigma^m. \tag{3.9}$$

Remark. Our main use of the more restricted family of states is the fact that the condition (3.9) in Proposition 3.1 above is easier to verify than the corresponding asymptotic property (3.8) for the general case. We also show in Section 6 below that (3.9) lends itself to explicit computations for the examples of conjugacy classes of shifts which we studied in the precursor [BJP].

Proof. The proof is the assertion that if the limit of an eventually constant sequence is zero, then the terms in the sequence must be identically zero from a step on.

4. STRONGLY ASYMPTOTICALLY SHIFT INVARIANT STATES ARE FINITELY CORRELATED

One main object of the present paper is the set of *shifts* on $\mathcal{B}(\mathcal{H})$, and the corresponding *conjugacy classes*. More generally, we shall consider endomorphisms which are not necessarily shifts; but we will also be more specific in that we look at those states ω on $\mathcal{O}_n$ which are invariant from a certain step on, i.e., satisfying (3.2) above. For each k, we show that these states form a finite-dimensional manifold, thus simplifying considerably the classification problem for the corresponding subclass of ergodic endomorphisms of $\mathscr{B}(\mathscr{H})$.

THEOREM 4.1. *Let k and n be positive integers, and let ω be a pure state on $\mathcal{O}_n$ such that $\omega \in S_k$. It follows that ω is finitely correlated and, moreover, the space $\mathscr{V}_k$ spanned by the vectors $\pi_\omega(s_{h_1}^* \cdots s_{h_k}^*)\,\Omega, h_1, ..., h_k \in \mathbb{C}^n$, is invariant under each of the operators $S_i^* = \pi_\omega(s_i^*)$.*

Proof. Since ω is a pure state on $\mathcal{O}_n$ the corresponding GNS-representation π is irreducible. If $S_i := \pi(s_i)$, then

$$\omega(\sigma(x)) = \sum_{i=1}^{n} \langle S_i^* \Omega \mid \pi(x)\, S_i^* \Omega \rangle$$

for all $x \in \mathcal{O}_n$. More generally, set

$$\Omega_{i_1 i_2 \cdots i_m} := S_{i_m}^* \cdots S_{i_2}^* S_{i_1}^* \Omega. \tag{4.1}$$

Then

$$\omega(\sigma^m(x)) = \sum_{i_1} \cdots \sum_{i_m} \langle \Omega_{i_1 \cdots i_m} \mid \pi(x)\, \Omega_{i_1 \cdots i_m} \rangle \qquad \text{for} \quad \forall x \in \mathcal{O}_n. \tag{4.2}$$

It follows that the GNS-representation of $\omega \circ \sigma$ identifies with the subrepresentation of the n-fold direct sum $\pi \oplus \pi \oplus \cdots \oplus \pi$ defined by the cyclic subspace generated by the free direct sum of the Ω_i vectors, i.e., $\Omega_1 \oplus \cdots \oplus \Omega_n$, and that of $\omega \circ \sigma^m$ is unitarily equivalent to the subrepresentation of the n^m-fold sum with cyclic vector

$$\sum_{i_1} \cdots \sum_{i_m} \oplus\, \Omega_{i_1 \cdots i_m}$$

where each index i_j runs over $\{1, ..., n\}$. Since π is irreducible, it follows that the commutant $\pi_{\omega \circ \sigma}(\mathcal{O}_n)'$ is naturally embedded in M_n. This is because the commutant of the representation

$$A \mapsto \underbrace{A \oplus A \oplus \cdots \oplus A}_{n \text{ times}}$$

consists of the operator matrices on $\bigoplus_1^n \mathscr{H}$ of the form $\sum_{ij} z_{ij} E_{ij}$, with scalar indices $z_{ij} \in \mathbb{C}$. The same result holds for $\omega \circ \sigma^m$ with the obvious modification coming from consideration of multi-indices.

Using (4.2) and (3.2), we now conclude that each of the states

$$\omega_{i_1 \cdots i_k i_{k+1}} = \langle \Omega_{i_1 \cdots i_k i_{k+1}} \mid \cdot\, \Omega_{i_1 \cdots i_k i_{k+1}} \rangle \tag{4.4}$$

is dominated by $\omega \circ \sigma^k$; and so, by using Segal's Radon–Nikodym theorem [Seg2], [Br-Rob, Theorem 2.5.19], or [KR], we conclude that, for each $(i_1, ..., i_k, i_{k+1})$ there are positive operators $Z = Z_{i_1 \cdots i_{k+1}}$, in the commutant $\pi_{\omega \circ \sigma^k}(\mathcal{O}_n)'$ such that

$$\omega_{i_1 \cdots i_{k+1}}(A) = \omega \circ \sigma^k(AZ) \tag{4.5}$$

where, on the right hand side, we have extended $\omega \circ \sigma^k$ to $\mathcal{B}(\mathcal{H}_{\omega \circ \sigma^k})$ in the obvious manner. By the above argument, the representation $\pi_{\omega \circ \sigma^k}$ is a subrepresentation of the n^k-fold direct sum of π_ω, and the commutant of the latter representation is isomorphic to M_{n^k}. The subrepresentation corresponds to a projection E in M_{n^k}, and the operators Z live inside this projection. We may extend Z to operators in M_{n^k} by setting $(1-E)\,Z = Z(1-E) = 0$. The formula (4.5) may now be written in multi-index summation form, $p = (p_1, ..., p_k)$, $q = (q_1, ..., q_k)$, with p_j and q_j in $\{1, ..., n\}$. The matrix Z and its entries $z_{p,q}$ still depend on $(i_1, ..., i_{k+1})$, but the latter multi-index is fixed for the moment. We get

$$\omega_{i_1 \cdots i_{k+1}}(A) = \sum_p \sum_q z_{pq} \langle \Omega_p \mid A\Omega_p \rangle \qquad \text{for} \quad \forall A \in \mathcal{B}(\mathcal{H}). \tag{4.6}$$

But the matrix Z is positive, so of the form $Z = Y^*Y$ where $Y = [\,y_{rp}] \in M_{n^k}$, e.g., take $Y := \sqrt{Z}$. Now set

$$\xi_r := \sum_p y_{r,p} \Omega_p \in \mathcal{H} \tag{4.7}$$

where $r = (r_1, ..., r_k)$ is also a multi-index. Formula (4.6) then takes the form

$$\omega_{i_1 \cdots i_{k+1}}(A) = \sum_r \langle \xi_r \mid A\xi_r \rangle \tag{4.8}$$

for $A \in \pi_\omega(\mathcal{O}_n)$, and thus, by closure, for all $A \in \mathcal{B}(\mathcal{H})$. But $\omega_{i_1 \cdots i_{k+1}}$ is a vector functional on $\mathcal{B}(\mathcal{H})$ and thus proportional to a pure state, and it follows from (4.8) that each of the vector functionals $\langle \xi_r \mid \cdot\, \xi_r \rangle$ are proportional to $\omega_{i_1 \cdots i_{k+1}}$, and thus each of the ξ_r are a scalar multiple of $\Omega_{i_1 \cdots i_{k+1}}$. Thus $\Omega_{i_1 \cdots i_{k+1}}$ is a scalar multiple of some ξ_r. But the vectors ξ_r are linear combinations of the vectors $\Omega_p = S^*_{p_k} \cdots S^*_{p_2} S^*_{p_1} \Omega$, and thus $\Omega_{i_1 \cdots i_{k+1}}$ are so. This proves Theorem 4.1.

5. A RECONSTRUCTION THEOREM

In this section, we first, in Theorem 5.1, describe a map from the set of all finitely correlated states on $\mathcal{O}_n$ into a system consisting of a state on a matrix algebra and a partition of unity. The hypotheses of this theorem are in particular fulfilled for $\omega \in S_k$, by Theorem 4.1. Subsequently, we show in Theorem 5.2 that such a system defines a state on $\mathcal{O}_n$. Finally, in Theorem 5.3, we give necessary and sufficient conditions on the system for the state to be pure. In Section 6 we will specialize to the case $\omega \circ \sigma^{k+1} = \omega \circ \sigma^k$.

The first result is a corollary to our previous theorem. Let $\mathfrak{A}_k$ be the subalgebra of UHF_n spanned linearly by the elements $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*$, where $i_m, j_m = 1, ..., n$. As explained around (2.10), (2.11), $\mathfrak{A}_k$ is isomorphic to $\underbrace{M_n \otimes \cdots \otimes M_n}_{k \text{ times}} \simeq M_{n^k}$. If $x = (x_1, ..., x_k)$, where $x_i \in H$, we will use the notation $s_x = s_{x_1} s_{x_2} \cdots s_{x_k}$, and if $x, y \in H^k$, $e_{xy} = s_x s_y^* = |x\rangle\langle y|$.

Theorem 5.1. *Let k and n be positive integers, and let $\omega \in \mathrm{FC}_k$; i.e., ω is a finitely correlated state such that each S_i^* leaves the subspace $\mathscr{V}_k \subset \mathscr{H}_\omega$ invariant. Then there are elements $L_i \in \mathfrak{A}_k$ $(i = 1, ..., n)$ such that the state ω is given by*

$$\omega(s_x s_{i_1} \cdots s_{i_{m_1}} s_{j_{m_2}}^* \cdots s_{j_1}^* s_y^*) = \omega(L_{i_{m_1}} \cdots L_{i_1} e_{xy} L_{j_1}^* \cdots L_{j_{m_2}}^*) \tag{5.1}$$

for $x, y \in H^k$, $i_l, j_l \in \{1, ..., n\}$. In particular, restriction of ω to UHF_n is given by

$$\begin{aligned} &\omega(A \otimes e_{i_1 j_1}^{(k+1)} \otimes \cdots \otimes e_{i_m j_m}^{(k+m)}) \\ &\quad = \omega(L_{i_m} \cdots L_{i_2} L_{i_1} A L_{j_1}^* \cdots L_{j_m}^*) \qquad \text{for} \quad \forall A \in \mathfrak{A}_k. \end{aligned} \tag{5.2}$$

Hence ω is determined by its restriction to $\mathfrak{A}_k$ and the elements $\{L_i\}_{i=1}^n$ in $\mathfrak{A}_k$, and we have

$$\sum_{i=1}^n \omega(L_i A L_i^*) = \omega(A) \qquad \text{for all} \quad A \in \mathfrak{A}_k. \tag{5.3}$$

Furthermore, if $P \in \mathfrak{A}_k$ is the support projection of the restriction of the state ω to $\mathfrak{A}_k$, the elements $L_i \in \mathfrak{A}_k$ may be chosen such that $P L_i P = L_i$, and with this choice the L_i's are unique.

Remark. Since $\sum_i s_i s_i^* = 1$, the algebra $\mathcal{O}_n$ is the closed linear span of operators of the form $s_x s_{i_1} \cdots s_{i_{m_1}} s_{j_{m_2}}^* \cdots s_{j_1}^* s_y^*$, and so (5.1) defines ω uniquely from $\rho := \omega|_{\mathfrak{A}_k}$ and $\{L_i\}_{i=1}^n$.

The following useful formula follows immediately from (5.1):

$$\pi_\omega(\sigma^k(s_j^*))\, \pi_\rho(X)\, \Omega = \pi_\rho(X L_j^*)\, \Omega \tag{5.4}$$

for $X \in \mathfrak{A}_k$, $j = 1, ..., n$, as follows: By (5.1)

$$\omega(s_x s_j^* s_y^*) = \omega(e_{xy} s_j^*) = \omega(s_x s_y^* s_j^*)$$

But since $S_j^* \mathscr{V}_k \subseteq \mathscr{V}_k$ we obtain from here

$$S_j^* S_y^* \Omega = S_y^* \pi(L_j^*) \, \Omega.$$

Multiplying to the left by S_y and summing over y in an orthonormal basis for $\bar{H}^k$, we obtain

$$\pi(\sigma^k(s_j^*)) \, \Omega = \pi(L_j^*) \, \Omega$$

and since $\sigma^k(s_j^*) \in \mathfrak{A}_k^c$, the formula (5.4) follows.

This can also be used to give an alternative definition of $L_j^* \in P\mathfrak{A}_k P$, where P is the support projection of $\omega|_{\mathfrak{A}_k}$. One has

$$\pi_\omega(\sigma^k(s_j^*)) \, \omega = \sum_{i_1 \cdots i_k} S_{i_1} \cdots S_{i_k} S_j^* S_{i_k}^* \cdots S_{i_1}^* \Omega.$$

But by assumption, $S_j^* S_{i_k}^* \cdots S_{i_1}^* \Omega$ is in $\mathscr{V}_k$, and hence the sum above is a linear combination of elements of the form $S_{i_1} \cdots S_{i_k} S_{j_k}^* \cdots S_{j_1}^* \Omega$, i.e., of elements in $\pi(\mathfrak{A}_k) \, \Omega$. Hence there exists an $L_j^* \in \mathfrak{A}_k$ such that

$$\pi(\sigma^k(s_j^*)) \, \Omega = \pi(L_j^*) \, \Omega$$

and then (5.4) is valid for all $X \in \mathfrak{A}_k$.

Now, as P is the smallest projection in $\mathfrak{A}_k$ such that $\pi(P) \, \Omega = \Omega$, it follows that we may replace L_j^* by $L_j^* P$ in the last formula. Furthermore, as $\sigma^k(s_j^*) \in \mathfrak{A}_k'$, we have

$$\pi(P) \, \pi(\sigma^k(s_j^*)) \, \Omega = \pi(\sigma^k(s_j^*)) \, \pi(P) \, \Omega = \pi(\sigma^k(s_j^*)) \, \Omega$$

so the formula is unchanged if L_j^* is replaced by PL_j^*. Thus, we may assume $L_j = PL_j P$. But as Ω is separating for $\pi(P\mathfrak{A}_k P)$, the L_j is then uniquely determined by the formula.

Next iterating the formula

$$as_i^* = s_i^* \sigma(a),$$

valid for all $a \in \mathcal{O}_n$, one obtains

$$as_x^* = s_x^* \sigma^k(a)$$

for all $x \in H^k$. Combining this with an iteration of (5.4) gives

$$S^*_{i_{m_1}} \cdots S^*_{i_1} S^*_x \Omega = S^*_x \pi(L_{i_1}) \cdots \pi(L_{i_{m_1}})\, \Omega$$

and (5.1) follows. We now proceed to another proof.

Proof of Theorem 5.1. Since $\omega \in \mathrm{FC}_k$, the space $\mathscr{V}_k$ spanned by $\{S^*_{i_1} \cdots S^*_{i_k}\Omega\}$ is invariant under each of the operators $S^*_i := \pi(s^*_i)$ where π is the GNS-representation of ω. We also have an antilinear map from the k-fold tensor product $H \otimes \cdots \otimes H$ into $\mathscr{V}_k$ where $H \simeq \mathbb{C}^n$. This map is given by

$$\Omega(x_1 \otimes \cdots \otimes x_k) := S^*_{x_k} \cdots S^*_{x_2} S^*_{x_1} \Omega \tag{5.5}$$

for $x_i \in H, i = 1, ..., k$. The antilinearization of formula (5.5) may be abbreviated $\Omega(x) = S^*_x \Omega$, $x \in \mathbb{C}^{n^k}$; so we get $\mathscr{V}_k$ as a quotient space, $\mathbb{C}^{n^k}$ divided out with a linear subspace N consisting of vectors x such that $\|\Omega(x)\|^2 = 0$, i.e., $\mathbb{C}^{n^k}/N \simeq \mathscr{V}_k$.

Let L_i be some lifting to $\mathbb{C}^{n^k}$ of the induced operator on the quotient,

$$S^*_i \Omega(x) = \Omega(L_i x). \tag{5.6}$$

for all $x \in \mathbb{C}^{n^k}$. We conclude that N must be invariant for each L_i. Each L_i may be identified with an element in $\mathfrak{A}_k \simeq \mathscr{B}(\mathbb{C}^{n^k})$ in the following way: Once the basis e_i for H has been chosen as in (2.3), then the element $e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(k)}_{i_k j_k}$ in $\mathfrak{A}_k$ acts on $H^{\otimes k}$ in a canonical fashion, giving a *-isomorphism between $\mathfrak{A}_k$ and $\mathscr{B}(H^{\otimes k})$. Transporting L_i back with this *-isomorphism, L_i identifies with an element in $\mathfrak{A}_k$. Doing this, one verifies the formula

$$L_i e_{xy} L^*_j = e_{L_i x,\, L_j y} \tag{5.7}$$

for $x, y \in \mathbb{C}^{n^k}$, as follows: If $u, v \in \mathbb{C}^{n^k}$, then

$$\begin{aligned} \langle u \mid L_i e_{xy} L^*_j v \rangle &= \langle u \mid L_i x \rangle \langle y \mid L^*_j v \rangle \\ &= \langle u \mid L_i x \rangle \langle L_j y \mid v \rangle \\ &= \langle u \mid e_{L_i x,\, L_j y} \mid v \rangle. \end{aligned}$$

Let us now verify formula (5.2). We note that the element A in $\mathfrak{A}_k$ may be taken to be in the form $A = e_{xy}$ where $x, y \in \mathbb{C}^{n^k}$. Then

$$\begin{aligned}
\omega(e_{xy} \otimes e^{(k+1)}_{i_1 j_1} \otimes \cdots \otimes e^{(k+m)}_{i_m j_m}) &= \langle S^*_{i_m} \cdots S^*_{i_1} S^*_x \Omega \mid S^*_{j_m} \cdots S^*_{j_1} S^*_y \Omega \rangle \\
&= \langle S^*_{i_m} \cdots S^*_{i_1} \Omega(\bar{x}) \mid S^*_{j_n} \cdots S^*_{j_1} \Omega(\bar{y}) \rangle \\
&= \langle \Omega(L_{i_m} \cdots L_{i_1} \bar{x}) \mid \Omega(L_{j_m} \cdots L_{j_1} \bar{y}) \rangle \\
&= \omega(e_{L_{i_m} \cdots L_{i_1} x,\, L_{j_m} \cdots L_{j_1} \bar{y}}) \\
&= \omega(L_{i_m} \cdots L_{i_1} e_{xy} L^*_{j_1} \cdots L^*_{j_m}).
\end{aligned}$$

which is the desired formula.

Formula (5.3) follows by putting $m=1$ and $j_1 = i_1 = i$ in (5.2), and then summing over $i=1$ to n, using

$$\sum_{i=1}^{n} e_{ii} = 1.$$

Let us now prove the last statement of Theorem 5.1. So far, the L_i's are only unique up to their action on N. But note that

$$\begin{aligned}
N &= \{ x \in \mathbb{C}^{n^k} \mid \Omega(x) = 0 \} \\
&= \{ x \in \mathbb{C}^{n^k} \mid \langle \Omega(x), \Omega(x) \rangle = 0 \} \\
&= \{ x \in \mathbb{C}^{n^k} \mid \omega(e_{xx}) = 0 \}.
\end{aligned}$$

Moreover e_{xx} ranges over all multiples of one-dimensional projections in $\mathscr{B}(\mathbb{C}^{n^k})$ when x ranges over $\mathbb{C}^{n^k}$, and it follows from the above formula that

$$N = (1-P)\, \mathbb{C}^{n^k}$$

where P is the support projection of ω. But as $L_i N \subseteq N$, we have $L_i(1-P) = (1-P)\, L_i(1-P)$ and hence $PL_i(1-P) = 0$.

Now

$$\begin{aligned}
S^*_i \Omega(x) &= \Omega(L_i x) \\
&= \Omega(PL_i x + (1-P)\, L_i x) \\
&= \Omega(PL_i x)
\end{aligned}$$

since $(1-P)\, L_i x \in N = \ker \Omega(\cdot)$. Thus

$$\begin{aligned}
S^*_i \Omega(x) &= \Omega(PL_i x) \\
&= \Omega(PL_i Px) + \Omega(PL_i(1-P)\, x) \\
&= \Omega(PL_i Px)
\end{aligned}$$

since $PL_i(1-P)=0$. Thus, if L_i is replaced by PL_iP, one still has the formula $S_i^*\Omega(x)=\Omega(L_ix)$, and hence one derives (5.2) as before. Thus L_i may be chosen such that $L_i=PL_iP$. But since the map induced by $\Omega(\cdot)$ from $P\mathscr{B}(\mathbb{C}^{n^k})\,P$ to $\mathscr{B}(\mathscr{V}_k)$ is an isomorphism, this choice of L_i is unique.

We will now show conversely that if k and n are given, then every system $\{L_i\}_{i=1}^n$ of matrices in $\mathscr{B}(\mathbb{C}^{n^k})$, together with a positive matrix R in $\mathscr{B}(\mathbb{C}^{n^k})$ of trace 1, determine a state on $\mathcal{O}_n$ by the formula (5.1), if the pair $\{R, \{L_i\}\}$ satisfy a certain normalization condition (5.8).

The question becomes one of extending the fixed state $\rho=\mathrm{Tr}(R\cdot)$ on $\mathfrak{A}_k$ to $\mathcal{O}_n$ such that the extended state ω is given by (5.1). For $\forall x, y\in\mathbb{C}^{n^k}$, we then have

$$\rho(e_{xy})=\langle x|\ R\ |y\rangle.$$

We shall say that the operators $\{L_i\}_{i=1}^n$ are *normalized* if

$$\sum_i \rho(e_{L_ixL_iy})=\rho(e_{xy}) \qquad \forall x, y\in\mathbb{C}^{n^k},$$

or, equivalently,

$$\sum_i L_i^*RL_i=R. \tag{5.8}$$

This is again equivalent to (5.3).

The normalization is a condition on the combined system consisting of the L_i's and R, or equivalently the L_i's and ρ. We will see during the proof of the next theorem that normalization is a translation of the Cuntz property $\sum_{i=1}^n A_iA_i^*=I_{\mathscr{V}_k}$ to the L_i's.

THEOREM 5.2. *Let k and n be positive integers, and ρ be a state on the subalgebra $\mathfrak{A}_k\subset\mathcal{O}_n$. Let $\{L_i\}_{i=1}^n$ be a system of elements in $\mathfrak{A}_k$ which are normalized relative to ρ. Then the formula*

$$\omega(s_xs_{i_1}\cdots s_{i_{m_1}}s_{j_{m_2}}^*\cdots s_{j_1}^*s_y^*)=\rho(L_{i_{m_1}}\cdots L_{i_1}e_{xy}L_{j_1}^*\cdots L_{j_{m_2}}^*)$$

defines a state ω on $\mathcal{O}_n$ which extends ρ. Furthermore, $\omega\in\mathrm{FC}_k$.

Proof. If

$$e_{xy}\in M_{n^k}\simeq\mathfrak{A}_k\subset\mathrm{UHF}_n\subset\mathcal{O}_n$$

we have, with Ω the cyclic vector in the GNS representation π of $\mathfrak{A}_k$,

$$\begin{aligned}\rho(e_{xy})&=\langle\Omega\mid\pi(e_{xy})\,\Omega\rangle=\langle x\mid Ry\rangle\\&=\mathrm{trace}(|R^{1/2}y\rangle\langle xR^{1/2}|)=\mathrm{trace}(R^{1/2}e_{xy}R^{1/2}).\end{aligned} \tag{5.9}$$

Since the L_i operators are normalized relative to R, we have

$$\sum_i (R^{1/2}L_i)^* (R^{1/2}L_i) = (R^{1/2})^2$$

and hence $R^{1/2}x = 0 \Rightarrow R^{1/2}L_i x = 0$. Thus each operator L_i passes to the quotient space

$$\mathscr{V}_k := \mathbb{C}^{n^k} / \{x \in \mathbb{C}^{n^k} : R_k^{1/2} x = 0\}. \tag{5.10}$$

For each i, we denote the corresponding induced operator on $\mathscr{V}_k$ by A_i^*. Specifically

$$A_i^* (x + \ker(R_k^{1/2})) = (L_i x) + \ker(R_k^{1/2}). \tag{5.11}$$

Relative to the norm, $x \mapsto \|R_k^{1/2} x\|$ on

$$\mathbb{C}^{n^k} / \ker(R_k^{1/2}),$$

the normalization property (5.8) then translates into

$$\sum_{i=1}^{n} A_i A_i^* = I_{\mathscr{V}_k}. \tag{5.12}$$

Using [Pop1, Theorem 2.1] we conclude the existence of a representation $(\pi, \mathscr{H}_\pi)$ of $\mathcal{O}_n$ such that $\mathscr{V}_k$ is isometrically embedded in $\mathscr{H}_\pi$, and

$$\pi(s_i^*) |_{\mathscr{V}_k} = A_i^*. \tag{5.13}$$

(See the remarks before (6.3) for more details on this.) Let P_k denote the orthogonal projection of $\mathscr{H}_\pi$ onto $\mathscr{V}_k$, and consider the completely positive mapping

$$\varphi : \mathcal{O}_n \to \mathscr{B}(\mathscr{V}_k)$$

given by

$$\varphi(a) := P_k \pi(a) |_{\mathscr{V}_k} \qquad \text{for} \quad \forall a \in \mathcal{O}_n. \tag{5.14}$$

Viewing the A_i's as operators on $\mathscr{H}_\pi$ by setting them equal to zero on the orthogonal complement of $\mathscr{V}_k$, we have from (5.13):

$$S_i^* P_k = P_k S_i^* P_k = A_i^*$$

and we conclude that

$$\begin{aligned}\varphi(s_{i_1}\cdots s_{i_l}s^*_{j_m}\cdots s^*_{j_1}) &= P_k S_{i_1}\cdots S_{i_l}S^*_{j_m}\cdots S^*_{j_1}P_k\\ &= P_k S_{i_1}P_k\cdots P_k S_{i_l}P_k S^*_{j_m}P_k\cdots P_k S^*_{j_1}P_k\\ &= A_{i_1}\cdots A_{i_l}A^*_{j_m}\cdots A^*_{j_1} \qquad (5.15)\end{aligned}$$

for all $l, m\in\mathbb{N}$ and all corresponding multi-indices (see [BEGJ, Proposition 2.1] for a similar argument). We may define a state ω on $\mathcal{O}_n$ by the formula

$$\omega(a) := \langle\Omega|\,\pi(a)\,\Omega\rangle = \langle\Omega\mid\varphi(a)\,\Omega\rangle \qquad \text{for} \quad \forall a\in\mathcal{O}_n.$$

Specifically

$$\omega(s_{i_1}\cdots s_{i_l}s^*_{j_m}\cdots s^*_{j_1}) = \langle A^*_{i_l}\cdots A^*_{i_1}\Omega\mid A^*_{j_m}\cdots A^*_{j_1}\Omega\rangle, \qquad (5.16)$$

and it follows that ω on $\mathcal{O}_n$ does restrict to the given state ρ on $\mathfrak{A}_k$. Let us introduce the operator $V=\sum_{i=1}^n L_i^*\otimes e_i$ from $\mathbb{C}^{n^k}\otimes\mathbb{C}^n=\mathbb{C}^{n^{k+1}}$. A calculation yields

$$\omega(a\otimes e_{ij}) = \rho(V^*(a\otimes e_{ij})\,V) = \rho(L_i a L_j^*)$$

for $\forall a\in\mathfrak{A}_k$, $\forall i, j\in\{1, ..., n\}$, where as usual e_{ij} denotes the matrix entries in M_n. The notation $a\otimes e_{ij}$ is short for $a\otimes e_{ij}^{(k+1)}$, with the e_{ij}-term sitting in the tensor slot $k+1$ relative to the infinite tensor product representation (2.9). The asserted formula (5.1) now follows precisely as in the proof of Theorem 5.1 above. This formula immediately implies that $\omega\in\mathrm{FC}_k$.

Theorems 5.1 and 5.2 say that there is a one-one correspondence between states $\omega\in\mathrm{FC}_k$ and pairs $\rho(\cdot)=\mathrm{Tr}(R\cdot)$, $\{L_i\}_{i=1}^n$ consisting of a state ρ on $\mathfrak{A}_k$ (alias density matrix R) with support projection P (alias range projection of R), and n operators $L_i\in P\mathfrak{A}_k P$ satisfying the normalization condition $\sum_i L_i^* R L_i = R$. We now address the question on when $\omega\in\mathrm{PFC}_k$. The answer is:

THEOREM 5.3. *Let* $\omega\in\mathrm{FC}_k$, *and let* $L_i\in P\mathfrak{A}_k P$, $\rho(\cdot)=\mathrm{Tr}(R\cdot)$ *be the objects associated to* ω *by Theorems* 5.1 *and* 5.2. *The following conditions are equivalent*:

(i) ω *is pure.*

(ii) *The operator equation*

$$\sum_i L_i^* x L_i = x$$

has a unique positive solution $x\in\mathfrak{A}_k$ *with* $\mathrm{Tr}(x)=1$ (*namely*, $x=R$).

Remark. We defer a more detailed discussion of the condition (ii) until the Theorem 6.1, but note that the condition is at least as strong as irreducibility of the system $\{L_i, L_i^*\}$ of operators on $P\mathbb{C}^{n^k}$, given that the equation has a solution.

Proof. The state ω is pure if and only if any state φ for which there exists a $\lambda > 0$ with $\lambda\varphi \leqslant \omega$ is a multiple of ω, so we must characterize those φ. The starting point is the relation (5.4)

$$\pi_\omega(\sigma^k(s_i^*))\,\Omega_\omega = \pi_\omega(L_i^*)\,\Omega_\omega$$

which can be written

$$\omega((\sigma^k(s_i) - L_i)(\sigma^k(s_i) - L_i)^*) = 0.$$

Since $\lambda\varphi \leqslant \omega$, we obtain

$$\varphi((\sigma^k(s_i) - L_i)(\sigma^k(s_i) - L_i)^*) = 0$$

and thus

$$\pi_\varphi(\sigma^k(s_i^*))\,\Omega_\varphi = \pi_\varphi(L_i^*)\,\Omega_\varphi.$$

If $A \in \mathfrak{A}_k$, this implies

$$\begin{aligned}\pi_\varphi(\sigma^k(s_i^*))\,\pi_\varphi(A)\,\Omega_\varphi &= \pi_\varphi(A)\,\pi_\varphi(\sigma^k(s_i^*))\,\Omega_\varphi \\ &= \pi_\varphi(AL_i^*)\,\Omega_\varphi\end{aligned}$$

and iterating this, we obtain

$$\pi_\varphi(\sigma^k(s_{j_1}^*)\cdots\sigma^k(s_{j_m}^*)\,\pi_\varphi(A)\,\Omega_\varphi = \pi_\varphi(AL_{j_m}^*\cdots L_{j_1}^*)\,\Omega_\varphi.$$

Thus

$$\varphi(\sigma^k(s_{i_1}\cdots s_{i_{m_1}}s_{j_{m_2}}^*\cdots s_{j_1}^*)\,A) = \varphi(L_{i_{m_1}}\cdots L_{i_1}AL_{j_1}^*\cdots L_{j_{m_2}}^*)$$

for all $A \in \mathfrak{A}_k$, and hence $\varphi \in \mathrm{FC}_k$, and the L_i's associated to φ are the same as those associated to ω, and φ is determined by its restriction to $\mathfrak{A}_k$. This restriction is determined by the density matrix $x \in \mathfrak{A}_k$ of φ:

$$\varphi(A) = \mathrm{Tr}(xA)$$

for $A \in \mathfrak{A}_k$. But the Cuntz relation $\sum_i s_i s_i^* = 1$ implies as before the normalization condition

$$\sum_i L_i^* x L_i = x$$

and as φ is determined by x and $\{L_i\}_{i=1}^n$, the equivalence of (i) and (ii) is clear.

COROLLARY 5.4. *If $\omega \in \mathrm{FC}_k$ with associated objects R, $\{L_i\}$, then the face generated by ω in the state space of $\mathcal{O}_n$ is finite dimensional, and affinely isomorphic to the convex set of matrices $x \in \mathfrak{A}_k$ with the properties*

$$x \geqslant 0, \qquad \mathrm{Tr}(x) = 1, \qquad \text{and} \qquad \sum_i L_i^* x L_i = x.$$

Proof. We showed during the proof of Theorem 5.3 that if φ is a state dominated by a multiple of ω, then $\varphi \in \mathrm{FC}_k$ and has the same $\{L_i\}$ as ω, and the density matrix has the properties stated in the corollary. Conversely, if x has the properties in the corollary, then the support of x is contained in P, and if $\varphi \in \mathrm{FC}_k$ is the corresponding state, it follows from finite dimensionality that there exists a $\lambda > 0$ such that $\lambda\varphi|_{\mathfrak{A}_k} \leqslant \omega|_{\mathfrak{A}_k}$. But as the L_i's are the same for φ and ω, this inequality extends to $\mathcal{O}_n$.

6. ASYMPTOTICALLY SHIFT INVARIANT STATES

In this section we specialize the theorems in Section 5 to the case $\omega \in S_k$. We already noted in Theorem 4.1 that ω is finitely correlated and that $S_k \subset \mathrm{PFC}_k$; and we will now study which additional requirements the fact that $\omega \in S_k$ places on $\{L_i\}$ and ρ.

THEOREM 6.1. *Let $n, k \in \mathbb{N}$, let ρ be a state on $\mathfrak{A}_k \subset \mathcal{O}_n$, and let $\{L_i\}_{i=1}^n$ be elements in $\mathfrak{A}_k$ satisfying the normalization condition* (5.8). *Then the corresponding state ω on $\mathcal{O}_n$ from Theorem* 5.2 *satisfies*

$$\omega \circ \sigma^k = \omega \circ \sigma^{k+1} \tag{3.2}$$

if

$$\sum_{i=1}^n L_i L_i^* = 1 \qquad \text{on the support of } \rho. \tag{6.1}$$

Conversely, if $\omega \in S_k$, then the associated operators L_i (*which exist by Theorem* 4.1 *and Theorem* 5.1) *satisfy* (6.1).

Moreover, let ω be a state on $\mathcal{O}_n$ defined by ρ and $\{L_i\}$ as in Theorem 5.2, *such that both the normalization conditions* (5.8) *and* (6.1) *are satisfied, and $PL_i P = L_i$ where P is the support projection of ρ, so that*

$$\sum_{i=1}^n L_i L_i^* = P \qquad \text{and} \qquad \sum_{i=1}^n L_i^* R L_i = R$$

where R is the density matrix of ρ. Let P_k be the projection from $\mathscr{H}_\omega$ onto $\mathscr{V}_k$. The following conditions are equivalent:

(i) *ω is pure on $\mathcal{O}_n$.*

(ii) *$\{L_i, L_i^*\}$ acts irreducibly on $P\mathbb{C}^{n^k}$ (i.e., $S_i^*|_{\mathscr{V}_k}$ acts irreducibly on $\mathscr{V}_k$) and $P_k \in \pi_\omega(\mathcal{O}_n)''$.*

(iii) *The only positive solutions of the operator equation*

$$\sum_i L_i^* x L_i = x$$

are the positive scalar multiples of R.

(iv) *The operator $\mathfrak{A}_k \mapsto \mathfrak{A}_k : x \mapsto \sum_i L_i^* x L_i$ has 1 as eigenvalue of multiplicity one.*

(v) *The only positive solutions of the operator equation*

$$\sum_i L_i x L_i^* = x$$

are the positive scalar multiples of P.

(vi) *The operator $\mathfrak{A}_k \mapsto \mathfrak{A}_k : x \mapsto \sum_i L_i x L_i^*$ has 1 as eigenvalue of multiplicity one.*

Proof. From (5.1), we get

$$\omega \circ \sigma^k (e_{i_1 j_1} \otimes \cdots \otimes e_{i_m j_m}) = \rho(L_{i_m} \cdots L_{i_1} L_{j_1}^* \cdots L_{j_m}^*)$$

and

$$\omega \circ \sigma^{k+1} (e_{i_1 j_1} \otimes \cdots) = \sum_i \rho(L_{i_m} \cdots L_{i_1} L_i L_i^* L_{j_1}^* \cdots L_{j_m}^*).$$

It is clear from this that (3.2) holds if $\sum_I L_i L_i^* = 1$ on the support of ρ. But when the L_i operators act irreducibly on $P\mathbb{C}^{n^k}$, then this condition is also necessary, as follows from the respective formulas for $\omega \circ \sigma^k$ and $\omega \circ \sigma^{k+1}$.

We next show that the purity of ω, or equivalently the irreducibility of the representation π from (3.3), is equivalent to irreducibility of the $\{L_i\}$ system, together with the condition $P_k \in \pi_\omega(\mathcal{O}_n)''$. But this follows from the commutant lifting theorem (see [NaFo]) which is part of the conclusion of [Pop1, Theorem 2.1]; see also [BEGJ] for more details. Specifically, we need to use the formula (5.11) which relates the L_i's to the A_i's. When the

A_i's are given, and π is a representation of $\mathcal{O}_n$ which serves as a *minimal dilation*, i.e.,

$$[\pi(\mathcal{O}_n)\,\mathscr{V}_k] = \mathscr{H}_\pi \tag{6.2}$$

and (5.13), then we first observe by GNS representation techniques that the representation π is determined up to unitary equivalence by the system A_i in the sense that if A'_i is another system of operators on a finite dimensional Hilbert space $\mathscr{V}'_k$, and there is a unitary $U: \mathscr{V}_k \to \mathscr{V}'_k$ such that $A'_i U = UA_i$, then the associated minimal dilations π and π' are unitarily equivalent representations of $\mathcal{O}_n$. This is proved in the same way as one proves that the cyclic representation associated to a state is determined up to unitary equivalence.

More nontrivially, the commutant lifting theorem states that there is a canonical isomorphism between the commutant of the operator system $\{A_i\}$ and the commutant of the representation π. In view of the uniqueness of the minimal dilation, in order to prove this it suffices to prove it for a particular explicit construction of the minimal dilation which we are now going to describe. We emphasize that by the commutant of the operator system $\{A_i\}$ we mean those operators that commute both with A_i and A_i^* for $i = 1, ..., n$, i.e., the von Neumann algebra generated by those unitaries $U \in \mathscr{B}(\mathscr{V}_k)$ such that $UA_i U^* = A_i$.

Specifically, let the operator system $\{A_i\}_{i=1}^n$ on $\mathscr{V}_k$ be given. Let A be the operator-row matrix $[A_1, ..., A_n]$, and set $D_A := (I_n - A^*A)^{1/2}$, and $\mathscr{D} := D_A(\bigoplus_{i=1}^n \mathscr{V}_k)$. (Note that since $AA^* = 1$, we have that $\|A^*A\| = \|AA^*\| = 1$, and hence D_A is well defined.) Let $\mathscr{F}(\mathbb{C}^n) = \mathbb{C} \oplus \mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^n) \oplus \cdots$ be the unrestricted Fock space over $\mathbb{C}^n$, and define operators θ_i on $\mathscr{F}(\mathbb{C}^n)$ by

$$\theta_i(\xi_1 \otimes \cdots \otimes \xi_k) = e_i \otimes \xi_1 \otimes \cdots \otimes \xi_k$$

for $\xi_j \in \mathbb{C}^n$, where e_i is the standard basis. The θ_i then generate a representation of the Toeplitz–Cuntz algebra, [Eva], [BEGJ]. Let $\Omega_0 = (1 \oplus 0 \oplus (0 \otimes 0) \oplus \cdots)$ denote the vacuum vector in $\mathscr{F}(\mathbb{C}^n)$, and for $i \in \{1, ..., n\}$ define $\delta_i: \mathscr{V}_k \to \mathscr{D}$ by

$$\delta_i v = D_A(\underbrace{0, ..., 0}_{i-1 \text{ times}}, v, 0, ..., 0)$$

for $v \in \mathscr{V}_k$. Define also $T_i: \mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n) \to \mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n)$ by $T_i = 1 \otimes \theta_i$, and $D_i: \mathscr{V}_k \to \mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n)$ by

$$D_i v = \delta_i v \otimes \Omega_0.$$

Define S_i on $\mathscr{V}_k \oplus (\mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n))$ by

$$\begin{aligned} S_i(v+f) &= A_i v \oplus (\delta_i v \otimes \Omega_0 + (1 \otimes \theta_i) f) \\ &= A_i v \oplus (D_i v + T_i f) \\ &= \begin{pmatrix} A_i & 0 \\ D_i & T_i \end{pmatrix} \begin{pmatrix} v \\ f \end{pmatrix} \end{aligned} \tag{6.3}$$

for $\forall v \in \mathscr{V}_k$ and $\forall f \in \mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n)$. Then it can be checked (and follows from [Pop1] and [BEGJ]) that the S_i's satisfy the Cuntz relations,

$$S_i^* S_i = \delta_{ij} I \qquad \text{and} \qquad \sum_{i=1}^{n} S_i S_i^* = I \tag{6.4}$$

where I denotes the identity operator on $\mathscr{V}_k \oplus \mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n)$. Hence they define a representation π of $\mathcal{O}_n$ which is easily checked to be a minimal dilation.

To return to the proof of Theorem 6.1, note that the following version of the commutant lifting theorem is true. (For a general background on "commutant lifting" see e.g., [Pop2] and [DMP].)

Lemma 6.2. *Adopt the general assumptions of Theorem* 6.1 . *If U is a unitary on $\mathscr{V}_k$ commuting with the A_i's, then U has a unitary extension to $\mathscr{H}_\omega$ commuting with the S_i's. Moreover this extension is unique.*

Proof. As $A_i U = UA_i$, U commutes with all $A_i^* A_j$, and hence $U \otimes I_n$ commute with $(I_n - A^*A)^{1/2}$ on $\mathscr{V}_k \otimes \mathbb{C}^n$. In particular $U \otimes I_n$ leave the subspace $\mathscr{D}$ invariant, and if the restriction is called $U_{\mathscr{D}}$, then

$$U_{\mathscr{D}}(I_n - A^*A)^{1/2} = (I_n - A^*A)^{1/2}\, U_{\mathscr{D}}$$

and hence

$$(U_{\mathscr{D}} \otimes I_{\mathscr{F}(\mathbb{C}^n)})\, D_i = D_i\, U.$$

Thus, defining U' on $\mathscr{V}_k \oplus (\mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n))$ by

$$U' = U \oplus (U_{\mathscr{D}} \otimes I_{\mathscr{F}(\mathbb{C}^n)})$$

one has

$$U' S_i = S_i\, U'$$

so U' is the sought-after extension.

To prove uniqueness of the extension, note that any unitary extension of U must have the form

$$U' = \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix}$$

on $\mathscr{H}_\omega = \mathscr{V}_k \oplus (\mathscr{D} \otimes \mathscr{F}(\mathbb{C}^n))$, where W is unitary in $\mathscr{D} \otimes \mathscr{F}(\mathbb{C})$. That U' commute with

$$S_i = \begin{pmatrix} A_i & 0 \\ D_i & T_i \end{pmatrix}$$

means

$$UA_i = A_i U$$

$$WD_i = D_i U$$

$$WT_i = T_i W.$$

The first relation is fulfilled since $U \in \{A_i\}'$. Since the representation $i \to \theta_i$ of the Toeplitz algebra is irreducible, the last relation implies that W has the form

$$W = w \otimes 1_{\mathscr{F}(\mathbb{C}^n)}$$

where w is unitary on $\mathscr{D}$. Now, the second relation means

$$w\delta_i = \delta_i U.$$

But this means that w is uniquely defined on the sum of the ranges of the δ_i's by U, and since the sum of these ranges in $\mathscr{D}$, it follows that w is uniquely determined (in fact we computed earlier that $w = U_{\mathscr{D}}$). Thus the extension U' is unique, and Lemma 6.2 is proved.

Let us now continue the proof of Theorem 6.1 by establishing the equivalence of the two statements

(ii) $P_k \in \pi_\omega(\mathscr{O}_n)''$ and $\{A_i\}$ is irreducible

and

(i) π_ω is irreducible.

Clearly (i) $\Rightarrow$ (ii), since $A_i = P_k S_i^* P_k = S_i^* P_k$. Conversely, assume (ii) and let U be a unitary in $\pi_\omega(\mathscr{O}_n)'$. Then $UP_k = P_k U$, and $UP_k \in \{A_i\}'$ thus $UP_k = P_k U = P_k$ by irreducibility of $\{A_i\}$. But by the uniqueness part of Lemma 6.2 it follows that $U = 1$. This ends the proof of (i) $\Leftrightarrow$ (ii).

It remains to show that each of the conditions (iii)–(vi) are equivalent to (i):

(i) $\Leftrightarrow$ (iii): This follows from Theorem 5.3.

(iii) $\Leftrightarrow$ (iv): Clearly (iv) $\Rightarrow$ (iii). To prove the converse implication, assume that

$$\sum_i L_i^* x L_i = x$$

for some $x \in \mathfrak{A}_k$. Then

$$\sum_i L_i^* x^* L_i = x^*$$

and hence if $x_1 = \frac{1}{2}(x + x^*)$, $x_2 = (1/2i)(x - x^*)$ then x_1, x_2 are eigenelements of eigenvalue 1, $x = x_1 + ix_2$ and $x_1 = x_1^*$, $x_2 = x_2^*$. To show that x is a scalar multiple of R, it therefore suffices to assume that x is self-adjoint. But as $PL_i P = L_i$, it follows from $x = \sum_i L_i^* x L_i$ that $Px = xP = x$, and hence $-x \leqslant \lambda' P$ for some $\lambda' > 0$. But since P is the support projection of R it follows from finite dimensionality of $\mathfrak{A}_k$ that $P \leqslant \lambda'' R$, where λ'' is the inverse of the smallest nonzero eigenvalue of R. Hence

$$-x \leqslant \lambda' P \leqslant \lambda' \lambda'' R = \lambda R$$

where $\lambda > 0$. Thus $\lambda R + x \geqslant 0$, and since $\lambda R + x$ is an eigenelement of $y \mapsto \sum_i L_i^* y L_i$ of eigenvalue 1, it follows from (iii) that $\lambda R + x$ is a scalar multiple of R. Thus x is a scalar multiple of R, and (iv) is valid.

(v) $\Leftrightarrow$ (vi): This is proved as (iii) $\Leftrightarrow$ (iv), with P playing the role of R.

To finish the proof of Theorem 6.1 it remains to establish (iv) $\Leftrightarrow$ (vi), and this follows from the following lemma.

Lemma 6.3. *Let $\mathfrak{A}$ be a unital C^*-algebra with a faithful trace state* tr, *let $L_1, ..., L_n$ be elements in $\mathfrak{A}$ and let R, S be positive invertible elements in $\mathfrak{A}$ with*

$$\sum_{i=1}^{n} L_i^* R L_i = R \qquad \text{and} \qquad \sum_{i=1}^{n} L_i S L_i^* = S.$$

For any $x \in \mathfrak{A}$ and any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the following statements are equivalent:

(i) $\sum_{i=1}^{n} L_i S x L_i^* = \lambda S x$.

(ii) $\sum_{i=1}^{n} L_i^* x R L_i = \bar{\lambda} x R$.

Proof. Let us first consider the case $S=1$, and define

$$\Phi(x)=\sum_{i=1}^{n} L_i x L_i^*.$$

Then Φ is a completely positive map with $\Phi(1)=1$, and hence the generalized Cauchy–Schwarz inequality is valid

$$\Phi(x)^* \Phi(x) \leqslant \Phi(x^*x),$$

[Br-Rob, pp. 229–230]. We may assume that R is normalized such that $\operatorname{tr}(R)=1$ and then we may define a state ρ on $\mathfrak{A}$ by

$$\rho(x)=\operatorname{tr}(Rx).$$

Then

$$\rho(\Phi(x))=\sum_i \operatorname{tr}(RL_i xL_i^*)=\sum_i \operatorname{tr}(L_i^* RL_i x)=\operatorname{tr}(Rx)=\rho(x).$$

So ρ is Φ-invariant, and then

$$\rho(\Phi(x)^* \Phi(x)) \leqslant \rho(\Phi(x^*x))=\rho(x^*x)$$

by Cauchy–Schwarz. If $(\pi, \mathscr{H}, \Omega)$ is the GNS-representation associated to ρ, it follows that we may define a contraction W on $\mathscr{H}$ by

$$W\pi(x)\,\Omega=\pi(\Phi(x))\,\Omega.$$

Let us suppress the notation π from now on, and show that

$$W^*x\Omega=\sum_{i=1}^{n} L_i^* xRL_i R^{-1}\Omega$$

for all $x \in \mathfrak{A}$:

$$\begin{aligned}
\langle W^*x\Omega \mid y\Omega\rangle &= \langle x\Omega \mid Wy\Omega\rangle = \langle x\Omega \mid \Phi(y)\,\Omega\rangle = \rho(x^*\Phi(x))\\
&=\sum_i \operatorname{tr}(Rx^*L_i yL_i^*)=\sum_i \operatorname{tr}(L_i^* Rx^*L_i y)\\
&=\sum_i \operatorname{tr}(R(R^{-1}L_i^* R)\,x^*L_i y)\\
&=\sum_i \langle L_i^* xRL_i R^{-1}\Omega \mid y\Omega\rangle,
\end{aligned}$$

which shows the desired formula.

Now, choose a specific $x \in \mathfrak{A}$ such that

$$\Phi(x) = \lambda x$$

where $|\lambda| = 1$, and put $\xi = x\Omega$. Then $W\xi = \Phi(x)\,\Omega = \lambda x\Omega = \lambda\xi$. Now one computes

$$\| W^*\xi - \bar{\lambda}\xi \|^2 = \| W^*\xi \|^2 - |\lambda|^2 \, \|\xi\|^2$$

and as $\| W^* \| = \| W \| \leqslant 1$ and $|\lambda| = 1$ one deduces

$$W^*\xi = \bar{\lambda}\xi.$$

Using the explicit formula for W^*, one thus has the equivalences

$$\Phi(x) = \lambda x$$

$$\Updownarrow$$

$$Wx\Omega = \lambda x\Omega$$

$$\Updownarrow$$

$$W^*x\Omega = \bar{\lambda}x\Omega$$

$$\Updownarrow$$

$$\sum_{i=1}^{n} L_i^* \, xRL_i \, R^{-1}\Omega = \bar{\lambda}x\Omega$$

$$\Updownarrow$$

$$\sum_{i=1}^{n} L_i^* \, xRL_i \, R^{-1} = \bar{\lambda}x$$

$$\Updownarrow$$

$$\sum_{i=1}^{n} L_i^* \, xRL_i = \bar{\lambda}xR$$

where the next to last equivalence follows from faithfulness of tr, and thus of ρ. This proves Lemma 6.3 in the case $S = 1$.

For a general S, introduce

$$l_i = S^{-1/2} L_i \, S^{1/2}$$

and

$$R' = S^{1/2} R S^{1/2}.$$

Then

$$\sum_i l_i l_i^* = 1$$

and

$$\sum_i l_i^* R' l_i = R'.$$

Using the lemma with $S = 1$, we thus have the equivalence, for $|\lambda| = 1$;

$$\sum_i l_i y l_i^* = \lambda y$$

$$\Updownarrow$$

$$\sum_i l_i^* y S^{1/2} R S^{1/2} l_i = \bar{\lambda} y S^{1/2} R S^{1/2}$$

or

$$\sum_i L_i S^{1/2} y S^{1/2} L_i^* = \lambda S^{1/2} y S^{1/2}$$

$$\Updownarrow$$

$$\sum_i L_i^* S^{-1/2} y S^{1/2} R L_i = \bar{\lambda} S^{-1/2} y S^{1/2} R.$$

Introducing $x = S^{-1/2} y S^{1/2}$, this says

$$\sum_i L_i S x L_i^* = \lambda S x$$

$$\Updownarrow$$

$$\sum_i L_i^* x R L_i = \bar{\lambda} x R$$

and Lemma 6.3 is proved.

To prove the final equivalence (iv) $\Leftrightarrow$ (vi) of Theorem 6.1 we just apply Lemma 6.3 on $\mathfrak{A} = P \mathfrak{A}_k P$ and with $S = P$ and $\lambda = 1$, to deduce that the dimensions of the eigensubspaces of $x \mapsto \sum_i L_i^* x L_i$ and $x \mapsto \sum_i L_i x L_i^*$ corresponding to eigenvalue 1 must be the same. This ends the proof of Theorem 6.1.

Remark. Let $\Phi(x)=\sum_{i=1}^{n} L_i x L_i^*$, $Wx\Omega=\Phi(x)\,\Omega$, $x\in P\mathfrak{A}_k P$, be the operators introduced in the proof of Lemma 6.3. From [Al-HK], we know that

$$\sigma(W)\cap\mathbb{T}=\sigma(\Phi)\cap\mathbb{T}$$

is a subgroup of $\mathbb{T}$, in the present case a finite group, called the *Frobenius Group* G_Φ.

For the decomposition $W=U\oplus V$ on $L^2(\rho)$, with U unitary, and V completely nonunitary (see [NaFo]), we have $\sigma(U)=G_\Phi$ and the spectrum of V is contained in the interior of $\{\lambda\in\mathbb{C}: |\lambda|\leqslant 1\}$. This means that we have the following clustering iff $G_\Phi=\{1\}$: $\forall m\in\mathbb{N}$, $\forall A\in M_{n^m}$, $\forall B\in\mathcal{O}_n$:

$$\lim_{r\to\infty}\omega(A\sigma^{m+r}(B))=\omega(A)\,\omega(B)$$

and the convergence is exponential.

In [BJW] we will establish that a state $\omega\in S_k$ will actually define a state in P_k if and only if (in addition to the properties (i)–(vi) of Theorem 6.1) the peripheral spectrum of Φ consists of a 1 alone, i.e., $G_\Phi=\{1\}$. In general, if $G_\Phi\approx\mathbb{Z}_m$, the state $\omega|_{\mathrm{UHF}_n}$ has a decomposition into pure states "over $\mathbb{Z}_m$." We will illustrate this with an example in Example 6.2, where

$$\omega|_{\mathrm{UHF}_n}=\omega^\infty|_{\mathrm{UHF}_n}=\sum_{i=1}^{m}\frac{1}{m}\,\varphi\circ\sigma^i|_{\mathrm{UHF}_n}$$

and φ is a pure state on M_{n^∞} which is periodic with period m under the *two-sided* shift. The fact that $\omega|_{\mathrm{UHF}_n}=\omega^\infty|_{\mathrm{UHF}_n}$ is of course very special for this example. We defer the general discussion to [BJW].

The following example is a preamble to the class of examples analyzed in Section 7.

EXAMPLE 6.1. We consider the setting in Theorem 5.2 and Corollary 6.1 above. We have $n\in\mathbb{N}$, but set $k=1$. In [BJP, Theorem 8.1] we gave a concrete example of a state ω in P_1, i.e., a state ω on $\mathcal{O}_n$ such that $\omega\circ\sigma=\omega\circ\sigma^2$, and the restriction $\omega|_{\mathrm{UHF}_n}$ is pure. The corresponding shift on $\mathcal{B}(\mathcal{H})$ we showed was *not* conjugate to any shift defined from a product state on UHF_n. Note that the algebra $\mathfrak{A}_1$ is now just a copy M_n of the n by n complex matrices and the space $\mathcal{V}_1$ from (3.4) has dimension n. Using Theorems 4.1 and 5.2 we note that the state ω, and therefore, the corresponding shift on $\mathcal{B}(\mathcal{H})$, may be calculated directly from the elements

$\{L_i\}_{i=1}^n$ in $\mathfrak{A}_1 \simeq M_n$, and a simple calculation, using [BJP, Chapter 8] yields the formula

$$A_i = n^{-1/2} \sum_{j=1}^{n} \overline{\langle i, j \rangle}\, e_{ji} \tag{6.5}$$

where

$$\langle i, j \rangle := \exp(2\pi \sqrt{-1}\, ij/n) \qquad \text{for} \quad \forall i, j \in \{1, ..., n\}, \tag{6.6}$$

and $e_{ij}^{(1)}$ denote the usual matrix units for M_n (see (2.12) above). As a result, we note that there are vectors $h_i \in \mathbb{C}^n$, $\|h_i\| = 1\ \forall i$, such that

$$\begin{aligned} L_i &= |e_i\rangle \langle h_i|, \\ h_i(j) &:= n^{-1/2} \langle i, j \rangle. \end{aligned} \tag{6.7}$$

It is easy to check from (6.5) that

$$\sum_{i=1}^{n} L_i^* L_i = \sum_{i=1}^{n} L_i L_i^* = I_n$$

here I_n is the unit-matrix in M_n. Note also, in this case, that the set $\{\Omega, \Omega_1, ..., \Omega_n\}$ is *orthogonal*, where $\Omega_i = S_i^* \Omega$.

For this example, it is also easy to check the *minimality condition* from [FNW2, Definition 1.2]. It amounts to the assertion that there is no proper subalgebra of $\mathfrak{A}_1 \simeq M_n$ which contains the unit, and is invariant under all the operators

$$A \mapsto L_i A L_j^* \qquad \text{on} \quad \mathfrak{A}_1. \tag{6.8}$$

Let us discuss this condition a bit further in the present context, where we have normalization

$$\sum_i L_i^* R L_i = R$$

and strong asymptotic invariance

$$\sum_i L_i L_i^* = P.$$

By [FNW2, Theorem 1.5] minimality then means that the only eigenvalue of the operator $x \mapsto \sum_i L_i x L_i^*$ of absolute value 1 is 1, and the corresponding eigenspace is one-dimensional, i.e., the only eigenvector in $P\mathfrak{A}_k P$ of this operator with eigenvalue of modulus 1 is P. But then a simple argument (see the proof of Lemma 7.8) shows that the only solutions of

$$\sum_i L_i^* x L_i = x$$

are the scalar multiples of R, and hence Theorem 5.3 implies that minimality of the $\{L_i\}$ system implies purity of ω.

It can be shown that minimality of the $\{L_i\}$-system on $P\mathfrak{A}_k P$ is equivalent with irreducibility of the corresponding system

$$\{L_{i_m} \cdots L_{i_1} L_{j_1}^* \cdots L_{j_m}^*\}$$

$m = 1, 2, ..., i_1, ..., j_l = 1, ..., n$, on $P\mathbb{C}^{n^k}$, [FNW2].

EXAMPLE 6.2. Let us end by exhibiting a state in S_1 on $\mathcal{O}_3$ where $\{L_i\}$ is irreducible, but not minimal. Here $P = 1$, $\mathfrak{A}_1 \simeq M_3$ and

$$L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$\sum_i L_i L_i^* = \sum_i L_i^* L_i = 1$$

and

$$L_{i_m} \cdots L_{i_1} L_{j_1}^* \cdots L_{j_m}^* = \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_m j_m} e_{i_m - 1,\, j_m - 1}$$

so the linear span of these consists of all diagonal 3×3 matrices. Hence $\{L_i\}$ is not minimal, albeit irreducible.

Now, if

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

one computes that

$$\sum_i L_i^* x L_i = \begin{pmatrix} x_{22} & 0 & 0 \\ 0 & x_{33} & 0 \\ 0 & 0 & x_{11} \end{pmatrix}.$$

Thus the operator $x \to \sum_i L_i^* x L_i$ has 0 as an eigenvalue of multiplicity 6, and the three cube roots $\rho^m = 1, \rho, \rho^2$ of 1 as simple eigenvalues with corresponding eigenvectors

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^{2m} & 0 \\ 0 & 0 & \rho^m \end{pmatrix}.$$

In particular, the only possible choice of R is $R = \frac{1}{3}1$, and it follows from Theorem 5.3 that the corresponding state ω is pure, i.e., $\omega \in S_1$.

Let us compute the restriction of ω to UHF_3. If

$$I = (i_1, i_2, ..., i_m)$$

where $i_l \in \mathbb{Z}_3$, put

$$\delta(I) = \begin{cases} 1 & \text{if} \quad i_{p+1} = i_p + 1 \bmod 3,\ p = 1, ..., m-1 \\ 0 & \text{otherwise.} \end{cases}$$

Then a calculation using $L_i = e_{i,\, i-1}$ shows that

$$\begin{aligned} \omega(e_{kl} \otimes e^{(2)}_{i_1 j_1} \otimes \cdots \otimes e^{(m+1)}_{i_m j_m}) &= 3^{-1}\, \delta(I)\, \delta(J)\, \delta_{k,\, i_1 - 1}\, \delta_{l,\, j_1 - 1}\, \delta_{i_m j_m} \\ &= 3^{-1}\, \delta(I, J)\, \delta(I) \end{aligned}$$

where

$$\delta(I, J) = \begin{cases} 1 & \text{if} \quad I = J \\ 0 & \text{otherwise.} \end{cases}$$

Thus ω restricted to UHF_3 is a convex combination of three pure states

$$\omega = \tfrac{1}{3}\,(\omega_1 + \omega_2 + \omega_3)$$

where ω_i is the pure product state on $\mathrm{UHF}_3 = \bigotimes_{m=1}^{\infty} M_3$ defined by the infinite product vector

$$e_i \otimes e_{i+1} \otimes e_{i+2} \otimes \cdots \qquad (\text{cyclic notation from } \mathbb{Z}_3)$$

where $\{e_i\}_{i \in \mathbb{Z}_3}$ is the canonical basis of $\mathbb{C}^3$. In particular, this shows that if $\omega \in S_1$, then $\omega|_{\mathrm{UHF}_3}$ is not necessarily pure. Note that in the example $\omega|_{\mathrm{UHF}_3}$ is actually σ-invariant, it is a convex combination of 3 pure states of period 3 under σ, which form an orbit of length 3 under the action of σ^* on UHF_3^*. That the σ-invariant state ω is not pure then also follows from the fact that the peripheral spectrum of $x \mapsto \sum_i L_i x L_i^*$ consists of more than the point 1, namely the three cube roots of 1.

7. $U(n)$-ORBITS AND A CROSS SECTION

Let $n \in \mathbb{N}$ be fixed. From Proposition 3.1, we know that given states ω and ω' on UHF_n, both in $\bigcup_{k=0}^{\infty} P_k$, determine conjugate shifts on $\mathscr{B}(\mathscr{H})$ iff there is a $g \in U(n)$ such that $\omega'_\infty = \omega_\infty \circ \tau_g$, where ω_∞ and ω'_∞ are the associated translationally invariant pure states on $\bigotimes_{-\infty}^{\infty} M_n$. (For more details on the state ω_∞, see Section 1.) Each ω (and ω_∞) is associated with elements $L \in \mathscr{L}(\mathbb{C}^n, M_{n^k})$ for some k. We will now show that these elements L span a Hilbert space which in turn carries a unitary corepresentation of $U(n)$, $g \mapsto L^g$, such that L^g is associated with the state $\omega \circ \tau_g$ for $g \in U(n)$. We thus get the conjugacy classes of shifts on $\mathscr{B}(\mathscr{H})$ labeled by orbits for this unitary corepresentation. The examples we give below are a set of shifts (for fixed Powers index n) which are labeled by functions

$$u \colon \prod_{1}^{\infty} \mathbb{Z}_n \to \mathbb{T} \tag{7.1}$$

depending only on a finite number of variables. When $k > 0$, and u is a nonconstant function, then the corresponding shift α_u is *not* conjugate to any of the shifts which correspond to a product state on UHF_n, and which were considered in [Lac1], [BJP].

We will show in Theorem 7.5 that generically our u-function examples form a cross section for the $U(n)$-orbits in the L space in the sense that each $U(n)$-orbit intersects the set of u-function examples in at most a manifold homeomorphic to a disjoint union of $n!$ copies of $\mathbb{T}^n$: that is, when the conjugacy class is given then there is only at most this manifold of functions u which represent the shifts from the conjugacy class.

Now to details: Let $k, n \in \mathbb{N}$, and let

$$u \colon \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{(k+1) \text{ times}} \to \mathbb{T}$$

be a given function. Let $X = \prod_1^{\infty} \mathbb{Z}_n$ with Haar measure, and let $\mathscr{H} = L^2(X)$ be the corresponding Hilbert space. Then in [BJP] we have considered the $\pi \in \mathrm{Rep}(\mathcal{O}_n, \mathscr{H})$ given by

$$(\pi(s_i)\,\xi)(x_1, x_2, \ldots) = n^{1/2} u(x_1, \ldots, x_{k+1})\, \xi_{x_1 i} \xi(x_2, x_3, \ldots) \tag{7.2}$$

$$(\pi(s_i^*)\,\xi)(x_1, x_2, \ldots) = n^{-1/2} \bar{u}(i, x_1, \ldots, x_k)\, \xi(i, x_1, x_2, \ldots) \tag{7.3}$$

Now, let ω be the state corresponding to the vector $\Omega = 1 \in L^2(X)$. A calculation, using the formula for $\pi(s_i^*)$, now shows that

$$\begin{aligned}\omega(e^{(1)}_{i_1 j_1} \otimes e^{(2)}_{i_2 j_2} \otimes \cdots \otimes e^{(m)}_{i_m j_m}) &= \omega(s_{i_1} \cdots s_{i_m} s^*_{j_m} \cdots s^*_{j_m}) \\ &= \langle S^*_{i_m} \cdots S^*_{i_1} \Omega \mid S^*_{j_m} \cdots S^*_{j_1} \Omega \rangle \\ &= n^{-k-m} \mathscr{F}_{k,m}(i_1, ..., i_m; j_1, ..., j_m)\end{aligned}$$

where

$$\mathscr{F}_{k,m}(i_1, ..., i_m; j_1, ..., j_m) = \sum_{x_1, ..., x_k} \zeta_{k,m}(i, x)\, \overline{\zeta_{k,m}(j, x)} \tag{7.4}$$

and

$$\begin{aligned}\zeta_{k,m}(i, x) &= \zeta_{k,m}(i_1, ..., i_m, x_1, ..., x_k) \\ &= u(i_m, x_1, ..., x_k)\, u(i_{m-1}, i_m, x_1, ..., x_{k-1}) \\ &\quad \times \cdots u(i_{m-k}, ..., i_m, x_1)\, u(i_{m-k-1}, ..., i_m) \cdots u(i_1, ..., i_{k+1}).\end{aligned}$$

This means that $\zeta_{k,m}$ is given by the expression above if $m \geqslant k+2$, but if $m < k+1$ the product defining $\zeta_{k,m}$ just truncates after the factor

$$u(i_1, ..., i_m, x_1, ..., x_{k+1-m}).$$

Using

$$\sigma(e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(m)}_{i_m j_m}) = \sum_{i=1}^{n} e^{(1)}_{ii} \otimes e^{(2)}_{i_1 j_1} \otimes \cdots \otimes e^{(m+1)}_{i_m j_m},$$

and the formulae above, one now calculates (for $m > k+1$)

$$\begin{aligned}&\omega \circ \sigma(e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(m)}_{i_m j_m}) \\ &\quad = \frac{1}{n} \sum_{i=1}^{n} u(i, i_1, ..., i_k)\, \overline{u(i, j_1, ..., j_k)}\, \omega(e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(m)}_{i_m j_m}).\end{aligned}$$

Thus if $i_1 = j_1, ..., i_k = j_k$, then $\omega \circ \sigma$ is equal to ω, i.e.,

$$\omega \circ \sigma|_{A_k^c} = \sigma|_{A_k^c},$$

which amounts to the invariance $\omega \circ \sigma^{k+1} = \omega \circ \sigma^k$. To show $\omega \in P_k$, we must check that ω is pure on UHF_n.

We denote the state defined by Ω on $\mathcal{O}_n$ by $\bar{\omega}$ when it becomes important to distinguish it from the corresponding state ω on UHF_n.

PROPOSITION 7.1. *The restricted state* $\bar{\omega}|_{\mathrm{UHF}_n}$ *is pure on* UHF_n.

The proof will be based on a lemma (below) and some calculations which we proceed to describe.

Remark. It follows from [BJP, Lemma 5.2] that

$$\alpha_{\omega}(A) := \sum_{i=1}^{n} \pi_{\bar{\omega}}(s_i)\, A \pi_{\bar{\omega}}(s_i)^*$$

is a shift on $\mathscr{B}(\mathscr{H})$.

Proof. Set $S_i := \pi_{\bar{\omega}}(s_i)$ and $\Omega_{i_1 \cdots i_k} := S_{i_k}^* \cdots S_{i_1}^* \Omega$. We then have

$$\Omega_{i_1 i_2 \cdots i_k}(x_1, x_2, ...) = n^{-k/2} \overline{\zeta(i_1, i_2, ..., i_k, x_1, x_2, ...)}$$

and therefore

$$S_j^* \Omega_{i_1, ..., i_k} = n^{-1/2}\, \overline{u(i_1, i_2, ..., i_k, j)}\, \Omega_{i_2, ..., i_k, j}.$$

Let $\{L_j\}_{j=1}^n$ be the associated elements in $\mathfrak{A}_k \simeq M_{n^k}$. Let $e_{i_1 \cdots i_k} := e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_k}^{(k)}$ denote the canonical basis vectors in $\mathbb{C}^{n^k} = \underbrace{\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k}$.

The operators L_j may be expanded in the vectors $e_{i_1 \cdots i_k}$ as follows: Let $h_j^{(i_1 \cdots i_{k-1})} \in \mathbb{C}^{n^k}$ be given by

$$\begin{aligned} h_j^{(i_1 \cdots i_{k-1})}(\alpha_1, ..., \alpha_k) &= \langle e_{\alpha_1 \cdots \alpha_k} \mid h_j^{(i_1 \cdots i_{k-1})} \rangle_{\mathbb{C}^{n^k}} \\ &= \delta_{i_1 \alpha_2}\, \delta_{i_2 \alpha_3} \cdots \delta_{i_{k-1} \alpha_k} n^{-1/2} u(\alpha_1, i_1, ..., i_{k-1}, j). \end{aligned}$$

Then a small calculation, using the defining relation

$$\Omega(L_i x) = S_j^* \Omega(x)$$

for L_i, where $x = x_1 \otimes \cdots \otimes x_k \in \mathbb{C}^{n^k}$, and the expansion

$$\Omega(x) = \sum_{i_1 \cdots i_k} \bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k} \Omega_{i_1 \cdots i_k}$$

and the formula for $S_j^* \Omega_{i_1, ..., i_k}$, above, shows that

$$L_j = \sum_{i_1 \cdots i_{k-1}} |e_{i_1 \cdots i_{k-1} j} \rangle \langle h_j^{(i_1 \cdots i_{k-1})}| \tag{7.5}$$

where $|\ \rangle\langle\ |$ is the Dirac notation.

In this example, the general formulas from Theorem 6.1 can be verified directly:

LEMMA 7.2. *Let* (L_j) *and* $(h_j^{i_1 \cdots i_{k-1}})$ *be as above and define*

$$R = \sum_{i_1 \cdots i_k} |h_{i_k}^{i_1 \cdots i_{k-1}}\rangle\langle h_{i_k}^{i_1 \cdots i_{k-1}}|$$
$$= n^{-1} \sum_{i_1 \cdots i_k} \sum_{\alpha_1} \sum_{\beta_1} u(\alpha_1, i_1, ..., i_k)\, \bar{u}(\beta_1, i_1, ..., i_k)$$
$$\times e^{(1)}_{\alpha_1 \beta_1} \otimes e^{(2)}_{i_1 i_1} \otimes \cdots \otimes e^{(k)}_{i_{k-1} i_{k-1}}.$$

We have

$$\sum_{j=1}^{n} L_j L_j^* = 1$$

and

$$\sum_{j=1}^{n} L_j^* R L_j = R$$

when 1 *is the identity element in* $\mathfrak{A}_k \simeq M_{n^k}$.

Proof. The adjoints of $L_j \in \mathfrak{A}_k$ are

$$L_j^* = \sum_{i_1 \cdots i_{k-1}} |h_j^{(i_1 \cdots i_{k-1})}\rangle\langle e_{i_1 \ldots i_{k-1} j}|$$

and it follows that

$$\sum_{j=1}^{n} L_j L_j^* = \sum_j (|e_{i_1 \ldots i_{k-1} j}\rangle\langle h_j^{(i \cdots)}|)(|h_j^{(i' \cdots)}\rangle\langle e_{i'_1 \ldots j}|)$$
$$= \sum_i \sum_{i'} \sum_j \langle h_j^{i_1 \cdots} \mid h_j^{i'_1 \cdots} \rangle\, \delta_{i_1 i'_1} \cdots \delta_{i_{k-1} i'_{k-1}}\, e^{(1)}_{i_1 i'_1}$$
$$\otimes \cdots \otimes e^{(k-1)}_{i_{k-1} i'_{k-1}} \otimes e^{(k)}_{jj}$$
$$= n^{-1} \sum_{\alpha_1 i_1 \cdots i_{k-1} j} |u(\alpha_1, i_1, ..., i_{k-1}, j)|^2\, e^{(1)}_{i_1 i_1}$$
$$\otimes \cdots \otimes e^{(k-1)}_{i_{k-1} i_{k-1}} \otimes e^{(k)}_{jj}$$
$$= \sum_{i_1 \cdots i_{k-1} j} e^{(1)}_{i_1 i_1} \otimes \cdots \otimes e^{(k)}_{jj} = I.$$

The two systems L_j and L_j^* represent shift operators as follows:

$$L_j \,|i_1 \cdots i_k\rangle = n^{-1/2} u(i_1, ..., i_k, j) \,|i_2 \cdots i_k j\rangle \tag{7.6}$$

and

$$L_j^* \,|i_1 \cdots i_k\rangle = n^{-1/2} \sum_p \bar{u}(p, i_1, ..., i_{k-1}, i_k)\, \delta_{i_k, j} \,|p i_1 \cdots i_{k-1}\rangle \tag{7.7}$$

The density matrix $R \in \mathfrak{A}_k$,

$$R = \sum_{i_1 \cdots i_k} |h_{i_k}^{(i_1 \cdots i_{k-1})}\rangle \langle h_{i_k}^{(i_1 \cdots i_{k-1})}|$$

then satisfies

$$\begin{aligned}
\sum_j L_j^* R L_j &= \sum_j \sum_{i_1 \cdots i_k} \sum_\alpha \sum_\beta \left(|h_j^{(\alpha_1 \cdots \alpha_{k-1})}\rangle \langle e_{\alpha_1 \cdots \alpha_{k-1} j}|\right) \\
&\quad \times \left(|h_{i_k}^{(i_1 \cdots i_{k-1})}\rangle \langle h_{i_k}^{(i_1 \cdots i_{k-1})}|\right)\left(|e_{\beta_1 \cdots \beta_{k-1} j}\rangle \langle h_j^{(\beta_1 \cdots \beta_{k-1})}|\right) \\
&= \sum \cdots \sum \delta_{\alpha_1 \beta_1} \cdots \delta_{\alpha_{k-1} \beta_{k-1}}\, \delta_{j \alpha_k}\, \delta_{j \beta_k} h_j^{(i_1 \cdots i_{k-1})}(\alpha)\, \overline{h^{(i_1 \cdots i_{k-1})}(\beta)} \\
&\quad \times |h_j^{\alpha_1 \cdots \alpha_{k-1}}\rangle \langle h_j^{\beta_1 \cdots \beta_{k-1}}| \\
&= \sum_{\alpha_1 \cdots \alpha_{k-1}} \sum_j |h_j^{\alpha_1 \cdots \alpha_{k-1}}\rangle \langle h_j^{\alpha_1 \cdots \alpha_{k-1}}| = R.
\end{aligned}$$

It follows from Theorem 5.2 that the system (L_j, R) determines a state ω on $\mathcal{O}_n$ whose restriction to UHF_n satisfies (5.2).

To finish the proof of Proposition 7.1, we need only check that the representation (7.2)–(7.3) is irreducible on $\mathscr{H} = L^2(X)$ when restricted to UHF_n.

Let $T \in \mathscr{B}(\mathscr{H})$ and assume $T\pi(a) = \pi(a)\, T$, $\forall a \in \mathrm{UHF}_n$. Recall UHF_n contains the canonical m.a.s.a. generated by

$$e_{i_1 i_1}^{(1)} \otimes \cdots e_{i_m i_m}^{(m)} \sim s_{i_1} \cdots s_{i_m} s_{i_m}^* \cdots s_{i_1}^*$$

and the representation yields

$$\begin{aligned}
&\pi(s_{i_1} \cdots s_{i_m} s_{i_m}^* \cdots s_{i_1}^*)\, \xi(x_1, x_2, ...) \\
&\qquad = \delta_{i_1 x_1}\, \delta_{i_2 x_2} \cdots \delta_{i_m x_m} \xi(x_1, x_2, ...), \qquad \forall \xi \in L^2(X).
\end{aligned}$$

It follows that there exists $f \in L^\infty(X)$ such that $T = m_f$, i.e., that T is a multiplication operator on $L^2(X)$, $\xi \mapsto f\xi$. But T also commutes with the other operators in UHF_n, and these act as:

$$\begin{aligned}
&\pi(s_{i_1} \cdots s_{i_m} s^*_{j_m} \cdots s^*_{j_1})\, \xi(x_1, x_2, ...)\\
&\quad = \pi(e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(m)}_{i_m j_m})\, \xi(x_1, x_2, ...)\\
&\quad = u(x_1, ...)\, u(x_2, ...) \cdots u(x_m, ...)\, \bar{u}(j_m, x_{m+1}, ...) \cdots\\
&\qquad \times \bar{u}(j_1, ..., j_m, x_{m+1}, ...)\, \delta_{i_1 x_1} \cdots \delta_{i_m x_m} \xi(j_1, ..., j_m, x_{m+1}, ...)\\
&\quad = \mathscr{F}_{k,m}(i, j, x)\, \delta_{i_1 x_1} \cdots \delta_{i_m x_m} \xi(j_1, ..., j_m, x_{m+1}, ...)
\end{aligned}$$

(see (7.4) above).

Since T is a multiplication operator, it also commutes with

$$\xi \mapsto \delta_{i_1 x_1} \cdots \delta_{i_m x_m} \xi(j_1, ..., j_m, x_{m+1}, ...).$$

This is because $\pi(s_{i_1} \cdots s_{i_m} s^*_{j_m} \cdots s^*_{j_1})$ is the product of a unitary multiplication operator and the latter operator, and T commutes with the former, and thus with the latter. A little computation then shows that the function f in $T = m_f$ must satisfy

$$f(i_1, ..., i_m, x_{m+1}, ...) = f(j_1, ..., j_m, x_{m+1}, ...)$$

for all i, j multi-indices $x \in X$, and therefore be constant on X. It follows that the commutant of $\pi(\mathrm{UHF}_n)$ on $L^2(X)$ is one dimensional, which is the asserted irreducibility. This ends the proof of Proposition 7.1.

We showed that when u is given as in (7.8) and ω is the corresponding state, then $\pi_\omega|_{\mathrm{UHF}_n}$ is irreducible. Thus ω defines a *shift* on $\mathscr{B}(\mathscr{H})$, and by [BJP, Lemma 5.4] two shifts defined from ω and ω' coincide iff there exists $g \in U(n)$ such that $\omega' = \omega \circ \tau_g$.

In conclusion, there is associated with every $k, n \in \mathbb{N}$ and function

$$u \colon \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{(k+1) \text{ times}} \to \mathbb{T} \tag{7.8}$$

the following complementary data:

(i) $\omega_u \in P_k$.

(ii) $\pi_{\omega_u} \in \mathrm{Rep}_S(\mathcal{O}_n, \mathscr{H}) = \{\pi \in \mathrm{Rep}(\mathcal{O}_n, \mathscr{H}),\ |\pi|_{\mathrm{UHF}_n} \text{ is irreducible}\}$.

(iii) $\alpha_u(A) = \sum \pi_{\omega_u}(s_i)\, A \pi_{\omega_u}(s_i)^*$ for all $A \in \mathscr{B}(\mathscr{H})$, a shift on $\mathscr{B}(\mathscr{H})$.

(iv) $L_u \in \mathscr{L}(\mathbb{C}^n, M_{n^k})$, $R_u \in M_{n^k}$.

For (iv), note that a system $\{L_j\} \in M_{n^k}$ determines an $L \in \mathscr{L}(\mathbb{C}^n, M_{n^k})$ by setting for $y \in \mathbb{C}^n$

$$L(y) := \sum_{j=1}^{n} y_j L_j.$$

DEFINITION 7.1. We say that an element ω in P_k is *diagonal* if it can be represented by a function u as in (7.8).

Specifically, there is a function

$$u: \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k+1 \text{ times}} \to \mathbb{T}$$

and a basis for $\mathbb{C}^n$ such that, in the basis, L is represented as follows $L(|j\rangle) = L_j (j \in \mathbb{Z}_n)$ and

$$L_j \, |i_1 \cdots i_k\rangle = n^{-1/2} u(i_1, ..., i_k, j) \times |i_2 i_3 \cdots i_k j\rangle$$

and

$$R \, |i_1 \cdots i_k\rangle = n^{-1} \sum_{\alpha} u(\alpha, i_2, ..., i_k) \, \bar{u}(i_1, i_2, ..., i_k) \, |\alpha i_1 \cdots i_{k-1}\rangle.$$

We showed in Proposition 3.1 that two diagonal (or arbitrary) states $\omega, \omega' \in \bigcup_{k=0}^{\infty} P_k$ determine conjugate shifts iff there is a $g \in U(n)$ such that $\omega'_\infty = \omega_\infty \circ \tau_g$. This means that conjugacy classes of shifts correspond to $U(n)$-orbits with the group $U(n)$ acting on the data in any one of the forms (i) or (iv).

We now describe the diagonal elements in $\bigcup_0^\infty P_k$ as a "cross section" for the associated orbit space.

THEOREM 7.3. *Consider two diagonal elements in $\bigcup_0^\infty P_k$ (relative to the same basis in $\mathbb{C}^n$) corresponding to functions u and u'. Then the corresponding shifts are conjugate iff there exists a k such that u and u' are both functions of $k+1$ variables*:

$$\underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k+1 \text{ times}} \to \mathbb{T}$$

and there exists a $g \in U(n)$ *such that*

$$u'(i_0, i_1, i_2, ..., i_k)\, \delta_{i_1 j_1}\, \delta_{i_2 j_2} \cdots \delta_{i_k j_k}$$

$$= \sum_{j_0} \sum_{p_1} \cdots \sum_{p_k} g(j_0, i_0)\, \overline{g(p_1, i_1)}\, g(p_1, j_1)\, \overline{g(p_2, i_2)}\, g(p_2, j_2) \cdots$$

$$\times \overline{g(p_k, i_k)}\, g(p_k, j_k) \times u(j_0, p_1, p_2, ..., p_k)$$

for all $(i_0, ..., i_k) \in \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k+1}$.

For the proof we need the following result which relates the state ω and the corresponding tensor L, and the transformation rule for the $U(n)$-coaction.

LEMMA 7.4. *If a state* $\omega \in S_k$ *is given by the tensor* $L \in \mathscr{L}(\mathbb{C}^n, M_{n^k})$ *and* $g \in U(n)$, *then the elements*

$$L^g(x) := \underbrace{(g^{-1} \otimes \cdots \otimes g^{-1})}_{k}\, L(gx)(g \otimes \cdots \otimes g) \qquad \forall x \in \mathbb{C}^n,$$

$$R^g := (g^{-1} \otimes \cdots \otimes g^{-1})\, R(g \otimes \cdots \otimes g) \tag{7.10}$$

determines the state $\omega \circ \tau_g$.

Proof. Let $\rho := \omega|_{M_{n^k}}$ where M_{n^k} is viewed as the subalgebra

$$\mathfrak{A}_k \simeq M_{n^k} \subset \mathrm{UHF}_n \subset \mathcal{O}_n$$

and let $\mathrm{Ad}_k(g) = \underbrace{g \otimes \cdots \otimes g}_{k} \cdot \underbrace{g^{-1} \otimes \cdots \otimes g^{-1}}_{k}$. Then it follows that

$$(\omega \circ \tau_g)|_{M_{n^k}} = \rho \circ \mathrm{Ad}_k(g).$$

We have for $\forall x, y \in \mathbb{C}^n$,

$$\rho(L(x)\, L(y)^*) = \sum \sum x_i \bar{y}_j \rho(L_i L_j^*)$$

$$= \sum \sum x_i \bar{y}_j (\omega \circ \sigma^k)(e_{ij})$$

$$= (\omega \circ \sigma^k)(s_x s_y^*) = \omega \circ \sigma^k(e_{xy}) = \omega_\infty(e_{xy}).$$

If $g \in U(n)$, and L^g is as in (7.10), then

$$\begin{aligned}(\rho \circ \mathrm{Ad}_k(g))(L^g(x)\, L^g(y)^*) &= \rho(\mathrm{Ad}_k(g)(\mathrm{Ad}_k(g^{-1})\, L(gx)\mathrm{Ad}_k(g^{-1})\, L(gy)^*)) \\ &= \rho(L(gx)\, L(gy)^*) \\ &= (\omega \circ \sigma^k)(s_{gx} s_{gy}^*) \\ &= (\omega \circ \sigma^k)(\tau_g(s_x s_y^*)) \\ &= (\omega \circ \tau_g) \circ \sigma^k(e_{xy}),\end{aligned}$$

and this formula shows that $\omega \circ \tau_g$ is determined by the tensor L^g as specified.

The formula for R^g is computed in a similar fashion.

Remark. We say that some L as in the lemma is in reduced form if $L(x) \in PM_{n^k}P$, $\forall x \in \mathbb{C}^n$, where P is the support projection for $\rho := \omega|_{M_{n^k}}$. If L and L' are in reduced form and ω and ω' are the respective states, then (for $g \in U(n)$) we have $L^g = L'$ iff $\omega_\infty \circ \tau_g = \omega'_\infty$. When $u \equiv 1$ the elements $\{L(y)\}_{y \in \mathbb{C}^n} \subset M_{n^k}$ are represented on $\mathbb{C}^{n^k} = \underbrace{\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k}$ as follows, see (2.10)–(2.11) above: Let $w := n^{-1/2}(\underbrace{1, ..., 1}_{n}) \in \mathbb{C}^n$. Then

$$L(y)(x^1 \otimes \cdots \otimes x^k) = \langle w \mid x^1 \rangle\, x^2 \otimes \cdots \otimes x^k \otimes y$$

and

$$L(y)^*\,(x^1 \otimes \cdots \otimes x^k) = \langle y \mid x^k \rangle\, w \otimes x^1 \otimes \cdots \otimes x^{k-1}.$$

When $u \colon \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k+1} \to \mathbb{T}$ is introduced, the formulas hold with the following modification: The vector $w = (w_i)_{i=1}^n$ becomes

$$w_i := n^{-1/2} u(i, \underbrace{...}_{k}) =: u_0(i, ...)$$

and

$$u(i, i_1, i_2, ..., i_k)$$

is viewed as a diagonal matrix for each $(i_1, ..., i_k)$

$$L^u(y)(x^1 \otimes \cdots \otimes x^k) = \langle \bar{u}_0(\cdots) \mid x^1 \rangle\, x^2 \otimes \cdots \otimes x^k \otimes y$$

with the variables $\underbrace{\cdots}_{1 \text{ to } k}$ acting on the tensor $x^2 \otimes \cdots \otimes x^k \otimes y$. Similarly $u(i_0, i_1, ..., i_k)$ can be interpreted as the corresponding dual operator for $L^u(y)^*$.

Proof of Theorem 7.3. Elements in $\mathbb{C}^n$ will be denoted $y, x^1, ..., x^k$. A basis $\{|i\rangle\}_{i=1}^n$ for $\mathbb{C}^n$ will be fixed such that

$$x^\nu = \sum_i x_i^\nu \, |i\rangle, \; \nu = 1, ..., k$$

with summation indices i ranging over $\mathbb{Z}_n$. If

$$u \colon \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k+1} \to \mathbb{T},$$

the contracted function: $\underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k \text{ times}} \to \mathbb{C}$ is defined by

$$\langle \bar{x}^1 \mid u(\cdot, i_2, ..., i_{k+1})\rangle := \sum_{i_1} x_{i_1}^1 u(i_1, i_2, ..., i_{k+1}).$$

Functions f on $\mathbb{Z}_n$ will be identified with diagonal matrices $\begin{pmatrix} f(1) & & 0 \\ & \ddots & \\ 0 & & f(n) \end{pmatrix}$, and similarly functions on $\underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k}$ will be identified with diagonal elements in $M_{n^k} = \underbrace{M_n \otimes \cdots \otimes M_n}_{k}$. We then have

$$\begin{aligned} L(y)(x^1 \otimes \cdots \otimes x^k) &= \sum_j \sum_{i_1} \cdots \sum_{i_k} y_j x_{i_1}^1 \cdots x_{i_k}^k \times L_j \, |i_1 \cdots i_k\rangle \\ &= n^{-1/2} \sum_j \sum_{i_1} \cdots \sum_{i_k} y_j x_{i_1}^1 \cdots x_{i_k}^k \\ &\quad \times u(i_1, ..., i_k, j) \, |i_2, ..., i_k, j\rangle \\ &= n^{-1/2} \langle \bar{x}^1 | \, u(\cdot, \underbrace{\cdots}_{k})\rangle \, |x^2 \otimes \cdots \otimes x^k \otimes y\rangle. \end{aligned}$$

Hence, for $g \in U(n)$, we have

$$L^g(y)\,|x^1 \otimes \cdots \otimes x^k\rangle = \underbrace{(g^{-1} \otimes \cdots \otimes g^{-1})}_{k}\, L(gy)\,|gx^1 \otimes \cdots \otimes gx^k\rangle$$

$$= n^{-1/2} \underbrace{(g^{-1} \otimes \cdots \otimes g^{-1})}_{k} \langle \overline{gx^1}|\, u(\cdot, \underbrace{\cdots}_{k})\rangle$$

$$\times (|gx^2 \otimes \cdots \otimes gx^k \otimes gy\rangle)$$

$$= n^{-1/2} \langle \overline{gx^1}|\, \mathrm{Ad}_k(g^{-1})\, u\,(\cdot, \underbrace{\cdots}_{k})\rangle$$

$$\times (|x^2 \otimes \cdots \otimes x^k \otimes y\rangle)$$

$$= n^{-1/2} \langle \bar{x}^1|\, \mathrm{Ad}_k(g^{-1})\, u_{\bar{g}}\,(\cdot, \underbrace{\cdots}_{k})$$

$$\times (|x^2 \otimes \cdots \otimes x^k \otimes y\rangle)$$

where

$$u_{\bar{g}}(\cdot, \underbrace{\cdots}_{k}) = \sum_j g(j, i)\, u(j, \underbrace{\cdots}_{k}).$$

Recalling the formula

$$\mathrm{Ad}(g^{-1}) \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix} = \left(\sum_{p=1}^{n} \overline{g(p, i)}\, g(p, j)\, f_p \right), \tag{7.11}$$

the desired formula (7.9) in the theorem follows.

Remark. When u is given as in (7.8), then it is only for a very special subset in $U(n)$ that $\omega \circ \tau_g$ is diagonal in the same basis.

Let $k, n \in \mathbb{N}$ be fixed. The transformation rule from the expression on the right hand side in (7.9) holds for general diagonal elements in P_k. The $U(n)$ coaction refers to the manifold $\mathscr{L}$ of all tensors subject to the conditions in Theorem 6.1 above. We may define an inner product for elements L and L' in $\mathscr{L}$ as

$$\langle L \mid L' \rangle = \mathrm{trace}_{M_{n^k}} \left(\sum_{j=1}^{n} L_j^* L_j' \right)$$

and Lemma 7.4 then implies that the $U(n)$-coaction $L \mapsto L^g$ extends to a unitary coaction on the linearization, i.e., we have

$$\langle L^g \mid L'^g \rangle = \langle L \mid L' \rangle \qquad \text{for} \quad \forall L, L' \in \mathscr{L}.$$

By a slight abuse of notation, we will use $\mathbb{T}^n \times S_n$ to denote the subgroup of $U(n)$ with the property that $g \in \mathbb{T}^n \times S_n$ iff each row and each column of g has only one nonzero element, and this element is then necessarily a phase factor. Thus $\mathbb{T}^n \times S_n$ identifies with the semidirect product of the n-torus $\mathbb{T}^n$ by the permutation group S_n of n elements, acting on $\mathbb{T}^n$ by permuting coordinates.

THEOREM 7.5. *For any $u \in C(\mathbb{Z}_n^{k+1}, \mathbb{T})$, the $U(n)$-orbit $\{L_u^g \mid g \in U(n)\}$ in $\mathscr{L}_{n,k}$ intersects the diagonal elements for $g \in \mathbb{T}^n \times S^n$, and if $g = (\rho_1, ..., \rho_n) \times \sigma \in \mathbb{T}^n \times S_n$, we have*

$$u^g(i_0, i_1, ..., i_k) = \rho_{\sigma(i_0)} u(\sigma(i_0), \sigma(i_1), ..., \sigma(i_k)).$$

Conversely, for a dense open subset of $C(\mathbb{Z}_n^{k+1}, \mathbb{T})$, the $U(n)$-orbit in $\mathscr{L}_{n,k}$ intersects the diagonal elements only for $g \in \mathbb{T}^n \times S^n$.

Remark 1. For a general $u \in C(\mathbb{Z}_n^{k+1}, \mathbb{T})$ the intersection could be larger. For example, if $u(x_0, x_1, ..., x_k) = 1$ for all $x_0, x_1, ..., x_k$, then the set of g such that L_u^g is diagonal is the set of all $g \in U(n)$ transforming

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \text{ into a vector of the form } \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_1 \end{pmatrix}$$

where $|\rho_i| = 1$ for $i = 1, ..., n$.

Remark 2. For the dense open subset of $C(\mathbb{Z}_n^{k+1}, \mathbb{T})$ we shall take the set of u with the property that for any $(i_1, ..., i_k) \in \mathbb{Z}_n^k$ there exists a pair $i, j \in \mathbb{Z}_n$ such that

$$u(i, i_1, ..., i_k) \neq u(j, i_1, ..., i_k),$$

but if $k \geqslant 2$ this is not the optimal choice.

Proof. Fix the function $u = u(x_0, ..., x_k)$ and assume that $g \in U(n)$ is an element such that $\omega_u \circ \tau_g$ is diagonal. This means that the u' defined by formula (7.9) is a function of $k+1$ variables such that

$$|u'(x_0, ..., x_k)| = 1$$

for all $x_0, ..., x_k \in \mathbb{Z}_n$. Now, identify u with the finite sequence

$$\bar{U}(i_0) = [u(i_0, i_1, ..., i_k)\, \delta_{i_1 j_1} \cdots \delta_{i_k j_k}]$$

of $n^k \times n^k$ unitary diagonal matrices, i.e., i_0 labels the n matrices, and $(i_1, ..., i_k, ; j_1 \cdots j_k)$ labels the matrix entries. Formula (7.9) in conjunction with formula (7.11) then says that $g \in U(n)$ is such that $\omega_u \circ \tau_g$ is diagonal if and only if

$$\bar{U}'(i_0) = \sum_{j_0} g(j_0, i_0) (\underbrace{g^{-1} \otimes \cdots \otimes g^{-1}}_{k \text{ factors}}) \, \bar{U}(j_0) (\underbrace{g \otimes \cdots \otimes g}_{k \text{ factors}})$$

is a new family of $n^k \times n^k$ unitary diagonal matrices.

From this formula we first see that if $g \in \mathbb{T}^n \times S_n$ then $\bar{U}'(i_0)$ is diagonal since $\bar{U}(j_0)$ is so, and the first part of the theorem follows. Next note that $g = (\rho_0, ..., \rho_n) \times \sigma$ corresponds to the matrix

$$g(i, j) = \rho_i \, \delta_{i, \sigma(j)}$$

in $U(n)$, and, inserting this into the formula (7.9), the formula for u^g follows.

Now, multiply (7.12) by $\overline{g(k_0, i_0)}$ and sum over i_0 to obtain

$$\sum_{i_0} \overline{g(k_0, i_0)} \, \bar{U}'(i_0) = (\underbrace{g^{-1} \otimes \cdots \otimes g^{-1}}_{k \text{ factors}}) \, \bar{U}(k_0) (\underbrace{g \otimes \cdots \otimes g}_{k \text{ factors}})$$

for all $k_0 \in \mathbb{Z}_n$. But by Stone-Weierstrass's theorem, if u has the property in Remark 2, the $*$-algebra generated by $\bar{U}(1), ..., \bar{U}(n)$ is the $*$-algebra $\mathscr{D}$ of all diagonal operators in M_{n^k}. Since $\bar{U}'(1), ..., \bar{U}'(n)$ are assumed to be diagonal, it thus follows from the relation above that

$$(g^{-1})^{\otimes k} \mathscr{D} g^{\otimes k} \subseteq \mathscr{D}.$$

From a standard result of Weyl, [Hel], it follows that $g^{\otimes k} \in \mathbb{T}^{n^k} \times S_{n^k}$, and hence $g \in \mathbb{T}^n \times S_n$. This ends the proof of Theorem 7.5.

THEOREM 7.6. *Let $k, n \in \mathbb{N}$ be given, and let*

$$u \colon \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k+1} \to \mathbb{T}$$

be a function and let the system $L_j = L_j^u$ depending on u be given as in (7.6). *Then the elements*

$$1, L_i L_j^*, ..., L_{i_p} \cdots L_{i_1} L_{j_1}^* \cdots L_{j_p}^*, ...$$

span all of M_{n^k}, i.e., the system is minimal *in the sense of* [*FNW*2].

Proof. The result follows from a brute force calculation, or from the clustering for ω_∞, which in turn follows from (7.13), below, and [FNW2, Theorem 1.5]. When applied to the present example, [FNW2] yields the asserted minimality property for $\{L_i\}_{i=1}^n$ if we check that, for $\forall p \in \mathbb{N}$, $\forall A \in \mathfrak{A}_p \simeq M_{n^p}$ and $\forall B \in \mathrm{UHF}_n \subset \mathcal{O}_n$, $\lim_{j\to\infty} \omega_\infty(A\sigma^{p+j}(B)) = \omega_\infty(A)\,\omega_\infty(B)$. Recall, since $\omega = \omega^u$ satisfies $\omega_\infty = \omega \circ \sigma^k$, the desired clustering property is implied by the following:

Lemma 7.7. *Let $u\colon X \to \mathbb{T}$ be given and suppose it is a function of $k+1$ variables, and let $\omega = \omega^u$ be the corresponding state. Then for all $p \in \mathbb{N}$, all $A \in \mathfrak{A}_p$, and all $B \in UHF_n \subset \mathcal{O}_n$, we have*

$$\omega(A\sigma^{p+2k}(B)) = \omega(A)(\omega \circ \sigma^k)(B). \tag{7.13}$$

Proof. Let $m > 2k$. Then

$$\omega(e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(m)}_{i_m j_m}) = n^{-m} \int_X \prod_{h=0}^{\infty} u(\sigma^h(i, x))\, \overline{u(\sigma^h(j, x))}\, d\mu(x) \tag{7.14}$$

where $d\mu(x)$ is the Haar measure on X which here involves only a finite number of summations, and where u is viewed as a function on $X = \prod_1^\infty \mathbb{Z}_n$ but depending only on the first $k+1$ variables;

$$(i, x) := (i_1, i_2, \ldots, i_m, x_1, x_2, \ldots) \in X;$$

and

$$\sigma(y_1, y_2, \ldots) := (y_2, y_3, \ldots) \qquad \text{for} \quad \forall y \in X.$$

The “infinite” product is really finite, i.e., the last factors $u(\sigma^h(i, x)) \neq 1$ are

$$u(i_{m-k+1}, \ldots, i_m, x_1) \cdots u(i_m, x_1, x_2, \ldots, x_k).$$

For the evaluation of the left hand side in (7.13) we may restrict to terms $e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(m)}_{i_m j_m}$ with $m > 2k$, and the subindices of the form

$$(i_1 \cdots i_p r_{p+1} \cdots r_{2p+2k} i_{2p+2k+1} \cdots i_m)$$

and

$$(j_1 \cdots j_p r_{p+1} \cdots r_{2p+2k} j_{2p+2k+1} \cdots j_m).$$

We take $A = e^{(1)}_{i_1 j_1} \otimes \cdots \otimes e^{(p)}_{i_p j_p}$ and similarly for B. Then the result follows where the factors are written out in $\omega(A\sigma^{p+2k}(B))$ and terms of the form $u(r_q \cdots r_{q+k})\, \bar{u}(r_q \cdots r_{q+k})$ are cancelled. (Recall u maps into $\mathbb{T}$ so $u(x)\, \bar{u}(x) = |u(x)|^2 = 1$ for $\forall x \in X$.)

8. DENSITY OF STRONGLY ASYMPTOTICALLY SHIFT INVARIANT STATES IN THE ASYMPTOTICALLY SHIFT INVARIANT STATES

Let us use the terminology that a pure state ω of UHF_n is *asymptotically shift invariant* if it is in P, i.e., if

$$\lim_{m\to\infty} \|(\omega\circ\sigma-\omega)|_{A_m^c}\|=0$$

or

$$\lim_{m\to\infty} \|\omega\circ\sigma^{m+1}-\omega\circ\sigma^m\|=0.$$

We say that ω is *strongly asymptotically shift invariant* if there is a $k\in\mathbb{N}$ such that $\omega\in P_k$, i.e.,

$$\omega\circ\sigma^{k+1}=\omega\circ\sigma^k.$$

We will now address the question how large $\bigcup_k P_k$ is in P. The answer is that it is less than norm dense:

PROPOSITION 8.1. *There is a state $\omega\in P$ such that if $\varphi\in\bigcup_k P_k$, then*

$$\|(\omega-\varphi)|_{A_m^c}\|=1$$

for all $m\in\mathbb{N}$.

Proof. Let ω_m be a sequence of pure states on M_n such that

$$\sum_{m=1}^{\infty} \|\omega_m-\omega_{m+1}\|^2<+\infty$$

but $\{\omega_m \mid m\geqslant M\}$ is dense in the pure state space of M_n for all $M\in\mathbb{N}$ (so in particular $\sum_{m=1}^{\infty}\|\omega_m-\omega_{m+1}\|=+\infty$). (Such a sequence may be constructed as follows: Let φ_m be any dense sequence in the pure state space of M_n. The φ_m are vector states given by unit vectors in $\mathbb{C}^n$, and we may assume $\langle\xi_m,\xi_{m+1}\rangle\geqslant 0$ where ξ_m is a unit vector corresponding to φ_m. By rotating ξ_m into ξ_{m+1} through a sequence of m^2 equal angles, we obtain m^2+1 pure states $\varphi_{m,0}=\varphi_m, \varphi_{m,2}, ..., \varphi_{m,m^2}=\varphi_{m+1}$ such that $\|\varphi_{m,k}-\varphi_{m,k+1}\|\leqslant\pi/m^2$ for $k=0, ..., m^2-1$, and thus $\sum_{k=0}^{m^2-1}\|\varphi_{m,k}-\varphi_{m,k+1}\|^2\leqslant m^2(\pi/m^2)^2=\pi^2/m^2$. Now let ω_m be the sequence $\varphi_1, \varphi_{2,0}, ..., \varphi_{2,4}=\varphi_{3,0}, ..., \varphi_{3,9}=\varphi_{4,0}, ...$ Then $\{\omega_m\}$ is dense, and $\sum_m\|\omega_m-\omega_{m+1}\|^2\leqslant\sum_m \pi^2/m^2<+\infty$.)

Let ω be the corresponding infinite product state on $\mathrm{UHF}_n=\bigotimes_{m=1}^{\infty} M_n$,

$$\omega=\bigotimes_{m=1}^{\infty}\omega_m.$$

By [BJP, Example 5.5], $\omega\in P$. Let $\epsilon>0$ and choose $\varphi\in P_k$ such that $\|(\varphi-\omega)|_{A_l^c}\|\leqslant\epsilon$ for some $l\in\mathbb{N}$. But this would imply $\|\omega_{m_1}-\omega_{m_2}\|\leqslant 2\epsilon$ for all $m_1, m_2\geqslant l$, and as $\{\omega_m \mid m\geqslant M\}$ is dense, it follows that $2\epsilon\geqslant 2$. The proposition follows.

Remark. By a simple argument, one may replace 1 by 2 in Proposition 8.1.

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[BJ(a)] O. Bratteli and P. E. T. Jorgensen, Iterated function systems and permutation representations of the Cuntz algebra, Oslo preprint, UiO Pure Mathematics, No. 12, June 1996.

[BJ(b)] O. Bratteli and P. E. T. Jorgensen, Isometries, shifts, Cuntz algebras and multiresolution wavelet analysis of scale N, Oslo preprint, UiO Pure Mathematics, No. 25, November 1996.

[BJ(c)] O. Bratteli and P. E. T. Jorgensen, A connection between multiresolution wavelet theory of scale N and representations of the Cuntz algebra $\mathcal{O}_N$, preprint, November 1996; to appear *in* "Proceedings of the Rome Conference on Operator Algebras and Quantum Field Theory" (J. Roberts, Ed.).

[J(a)] P. E. T. Jorgensen, A duality for endomorphisms of von Neumann algebras, *J. Math. Phys.* **37** (1996), 1521–1538.

[J(b)] P. E. T. Jorgensen, Harmonic analysis of fractal processes via C^*-algebras, *Math. Nachr.*, to appear.

[DP] K. R. Davidson and D. R. Pitts, Free semigroup algebras, preprint, 1996.

These papers continue the study of classes of representations described by (7.2)–(7.3) in the present paper. In [DP], the representations go under the name "atomic representations," and they are studied (independently) and classified up to equivalence of irreducibles, but the framework is different. The papers [BJ(a)–(c), J(a), J(b)] are concerned with decomposition series of representations of $\mathcal{O}_N$, a geometric model for estimating multiplicities, and applications to the theory of tilings (with fractal boundaries) and wavelet multiresolutions.

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