JOURNAL OF DIFFERENTIAL EQUATIONS 53, 139–145 (1984)

When All Solutions of $x' = -\sum q_i(t)\, x(t - T_i(t))$ Oscillate*

BRIAN R. HUNT† AND JAMES A. YORKE‡

University of Maryland, College Park, Maryland 20742

Received June 21, 1982; revised October 12, 1982

In this paper the long-term behavior of solutions to the equation in the title are examined, where $q_i(t)$ and $T_i(t)$ are positive. In particular, it is shown that if $\liminf_{t\to\infty} \sum_{i=1}^{n} T_i(t)\, q_i(t) > 1/e$, all solutions oscillate about 0 infinitely often.

1. INTRODUCTION

Ladas and Stavroulakis [4] observe that all nonzero solutions (for $t \geqslant 0$) to the scalar differential delay equation

$$x'(t) = -qx(t - T), \tag{1.1}$$

(where $'$ means d/dt, and T and q are positive constants) must oscillate about 0 (and must, in fact, have infinitely many zeroes) if and only if

$$Tq > 1/e. \tag{1.2}$$

We call Tq the "torque" for Eq. (1.1), for in a sense q represents the magnitude of a force and T represents from how far away (in time rather than space) this force is applied.

Ladas, Sficas, and Stavroulakis [3, 5] study an extension of Eq. (1.1) to

$$x'(t) = -\sum_{i=1}^{n} q_i x(t - T_i), \tag{1.3}$$

where q_i and T_i are positive constants. In this paper we examine a further extension to

$$x'(t) = -\sum_{i=1}^{n} q_i(t)\, x(t - T_i(t)), \tag{1.4}$$

* This research was partially supported under NSF Grant MCS81-17967.

† Department of Mathematics. Current address: Mathematics Department, Stanford University, Stanford, California.

‡ Institute for Physical Science and Technology and Department of Mathematics.

0022-0396/84 $3.00

505/53/2-2

where T_i and q_i are continuous and positive-valued on $[0, \infty)$. In mechanics, the total torque is obtained by summing the individual torques. Letting $\tau_i(t) = T_i(t)\, q_i(t)$ be the torque due to the ith term in Eq. (1.4), we define the total torque for Eq. (1.4) to be $\tau(t) = \sum_{i=1}^{n} \tau_i(t)$. Our main result is Theorem 1.1.

THEOREM 1.1. *If there is a uniform upper bound T_0 on the T_i's* and

$$\liminf_{t\to\infty} \tau(t) > \frac{1}{e}, \tag{1.5}$$

then all solutions to Eq. (1.4) *must oscillate.*

The theorem is sharp in that the lower bound $1/e$ cannot be improved. However, even when the T_i's and q_i's are constant there are cases in which Eq. (1.5) does not hold but all solutions do oscillate.

In Section 2 we study the special case in which the q_i's and T_i's are constants. Theorem 1.1 is proved in Section 3. Before we proceed, though, we should clarify what we mean by "solutions" to Eq. (1.4). "Solutions" are assumed to be defined on $[-T_0, \infty)$ and Eq. (1.4) must be satisfied for $t > 0$.

2. The Constant Case

In this section we assume all T_i and q_i are constant and, as before, $\tau_i = T_i q_i$ and $\tau = \sum_{i=1}^{n} \tau_i$. Then Eq. (1.5) becomes

$$\tau > 1/e \tag{2.1}$$

and $T_0 = \max T_i$.

One might try to find a counterexample to Theorem 1.1 for which there is a "real exponential" solution, i.e., $x(t) = \exp(-\lambda t)$ for some real λ. Substituting $x(t)$ in Eq. (1.3) we get

$$-\lambda e^{-\lambda t} = -\sum_{i=1}^{n} q_i e^{-\lambda(t - T_i)} = -\sum_{i=1}^{n} q_i e^{\lambda T_i} e^{-\lambda t},$$

or, equivalently,

$$\lambda = \sum_{i=1}^{n} q_i e^{\lambda T_i}. \tag{2.2}$$

Thus, if Eq. (2.2) has a real solution λ, Eq. (1.3) has a nonoscillatory solution. Tramov [10] has proven that the converse is also true. Another proof appears in [3].

To prove Theorem 1.1 in the constant case we shown that Eq. (2.1) implies the nonexistence of a real solution λ to Eq. (2.2). Equation (2.2) is equivalent to

$$1 = \frac{1}{\lambda} \sum_{i=1}^{n} q_i e^{\lambda T_i} = \sum_{i=1}^{n} \frac{q_i}{\lambda} e^{\lambda T_i}. \tag{2.3}$$

Each $(q_i/\lambda)\exp(\lambda T_i)$ is minimized by setting $\lambda = 1/T_i$ and, thus, has its minimum equal to $T_i q_i e = \tau_i e$. So if Eq. (2.1) holds,

$$\sum_{i=1}^{n} \frac{q_i}{\lambda} e^{\lambda T_i} \geqslant \sum_{i=1}^{n} \tau_i e = \tau e > 1,$$

and Eq. (2.3) has no solution.

Ladas and Stavroulakis [5] obtain four conditions which imply that all solutions to Eq. (1.3) oscillate. Three of these are spcial cases of Eq. (2.1) in fact, but the fourth is independent of Eq. (2.1). This condition,

$$\left(\sum_{i=1}^{n} T_i\right)\left(\prod_{i=1}^{n} q_i\right)^{1/n} > \frac{1}{e}, \tag{2.4}$$

does, however, imply that no real solution to Eq. (2.2) exists. We see this by using the arithmetic mean–geometric mean inequality:

$$\sum_{i=1}^{n} \frac{q_i}{\lambda} e^{\lambda T_i} \geqslant n \left(\prod_{i=1}^{n} \frac{q_i}{\lambda} e^{\lambda T_i}\right)^{1/n} = \frac{n}{\lambda}\left(\prod_{i=1}^{n} q_i\right)^{1/n} \exp\left(\frac{\lambda}{n} \sum_{i=1}^{n} T_i\right).$$

This expression as a function of λ has minimum $(\sum_{i=1}^{n} T_i)(\prod_{i=1}^{n} q_i)^{1/n} e$, so if Eq. (2.4) holds, then Eq. (2.3), and, hence, Eq. (2.2) has no real solution.

As we have already remarked, Eq. (2.1) and (2.4) are independent; we give two numerical examples to demonstrate this. If $T_1 = T_2 = \frac{1}{3}$, $q_1 = \frac{1}{4}$, and $q_2 = 1$, Eq. (2.1) holds but Eq. (2.4) does not. If $T_1 = 1$, $T_2 = \frac{1}{12}$, $q_1 = \frac{1}{4}$, and $q_2 = 1$, Eq. (2.4) holds but Eq. (2.1) does not. The advantage of working with these two equations rather than Eq. (2.2) directly is that Eq. (2.1) and Eq. (2.4) are explicit, while determining whether or not a real solution to Eq. (2.2) exists may be quite a problem in itself. But the existence of a solution to Eq. (2.2) is a sharper condition than the others.

3. Proof of Main Result

Now we prove Theorem 1.1. Assume the contrary, that there is a nonoscillatory solution $x(t)$ to Eq. (1.4) although Eq. (1.5) holds. Since the negative of a solution to Eq. (1.4) is also a solution, we may assume that $x(t)$ is positive for t sufficiently large. We may further assume, without loss of generality, that $x(t)$ is positive and decreasing on all of $[-T_0, \infty)$, translating the original solution to the left if necessary. Since Eq. (1.5) holds, we may also assume that there exists a $\tau_0 > 1/e$ such that $\tau(t) \geqslant \tau_0$ for $t \geqslant 0$, again translating to the left if necessary. Then $y(t) = -\ln x(t)$ exists and is increasing on $[-T_0, \infty)$, and

$$y'(t) = \sum_{i=1}^{n} q_i(t)\, e^{y(t) - y(t - T_i(t))}. \tag{3.1}$$

We wish to construct a "delay" $T(t) \leqslant T_0$ for which a solution $y_0(t)$ exists for

$$y_0'(t) = q(t)\, e^{y_0(t) - y_0(t - T(t))},$$

where $q(t) \equiv T_0/T(t)$ and y_0 grows as slowly as possible. In particular, we will have $y_0(t) \leqslant y(t)$ when both are defined, and we will reach a contradiction when we show that $y_0(t) \to \infty$ for $t \to t_* < \infty$. Specifically, let $y_0(t)$ satisfy

$$y_0'(t) = \inf_{0 < T \leqslant T_0} \frac{\tau_0}{T}\, e^{y_0(t) - y_0(t - T)} \qquad (t > 0), \tag{3.2}$$

where on $[-T_0, 0]$, $y_0(t)$ is constant and less than $y(t)$. Then letting

$$f(t, T) = \frac{\tau_0}{T}\, e^{y_0(t) - y_0(t - T)}$$

and observing that for all t for which $y_0(t)$ is defined $f(t, T) \to \infty$ as $T \to 0$, we see that there is, in fact, a function $T(t)$ on $(0, \infty)$ for which

$$y_0'(t) = f(t, T(t)) = \frac{\tau_0}{T(t)}\, e^{y_0(t) - y_0(t - T(t))} \qquad (t > 0) \tag{3.3}$$

and $0 < T(t) \leqslant T_0$.

Since y is increasing and $y_0'(t) = 0$ on $[-T_0, 0]$, $y'(t) > y_0'(t)$ on this interval. Assume that for some $t \geqslant 0$, $y(t)$ and $y_0(t)$ are defined and $y'(t) \leqslant y_0'(t)$. Let t_0 be the infimum of all such t. Then $y'(t_0) \leqslant y_0'(t_0)$ while $y'(t) > y_0'(t)$ on $[-T_0, t_0)$. But then

$$\begin{aligned} y'(t_0) &= \sum_{i=1}^{n} q_i(t) e^{y(t_0) - y(t_0 - T_i(t))} > \sum_{i=1}^{n} q_i(t) e^{y_0(t_0) - y_0(t_0 - T_i(t))} \\ &\geqslant \sum_{i=1}^{n} q_i(t)\, T_i(t) \inf_{0 < T \leqslant T_0} \frac{1}{T} e^{y_0(t_0) - y_0(t_0 - T)} \\ &= \frac{\sum_{i=1}^{n} T_i(t)\, q_i(t)}{\tau_0}\, y_0'(t_0) \geqslant y_0'(t_0), \end{aligned}$$

a contradiction. So $y'(t) > y_0'(t)$ whenever the two are defined.

The continuous function $y_0(t)$ satisfies $y_0'(t)=0$ on $[-T_0, 0)$ while $y_0'(t)>0$ on $(0,\infty)$. Suppose that $y_0'(t)$ is not an increasing function on $(0,\infty)$. Let t_0 be the supremum of all t for which y_0' is increasing on $(0, t]$. Since $y_0(t)$ is continuous, f is continuous, so $y_0'(t)$ is continuous on $(0,\infty)$. Therefore, there exists $t_1 > t_0$ such that $y_0'(t) < 2y_0'(t_0)$ for $t \in [t_0, t_1]$. Now if $T < T_1$, where $T_1 = \tau_0/(2y_0'(t_0))$, then

$$f(t, T) > \frac{\tau_0}{T} > 2y_0'(t_0),$$

so, in fact, for all $t \in [t_0, t_1]$, $y_0'(t) = \inf f(t, T)$, where the infimum is taken over all $T \in [T_1, T_0]$. We may choose $t_2 \in (t_0, t_1)$ close enough to t_0 so that $t_2 - T_1 < t_0$ and $y_0'(t) > y_0'(t_2 - T_1)$ for $t \in [t_0, t_2]$, because y_0' is increasing on $[0, t_0]$ and continuous on $[0,\infty)$. Since y_0' is continuous, $\partial f/\partial t$ exists and is continuous. If $T \in [T_1, T_0]$ and $t \in [t_0, t_2]$, then

$$\begin{aligned}\left.\frac{\partial f}{\partial t}\right|_{(t,T)} &= \frac{\tau_0}{T} e^{y_0(t)-y_0(t-T)}(y_0'(t) - y_0'(t-T))\\ &\geqslant \frac{\tau_0}{T} e^{y_0(t)-y_0(t-T)}(y_0'(t) - y_0'(t_2 - T_1)) > 0.\end{aligned}$$

Thus $y_0'(t)$ is increasing on $[t_0, t_2]$, and, therefore, on $[0, t_2]$, contradicting the definition of t_0. We conclude that $y_0'(t)$ is increasing on $[0,\infty)$.

Recall that we defined $T(t) \leqslant T_0$ on $(0,\infty)$ so that

$$f(t, T(t)) = \inf_{0<T\leqslant T_0} f(t, T).$$

Since $y_0(t)$ is continuous, $\partial f/\partial T$ exists and is continuous. So if $T(t) \neq T_0$,

$$0 = \left.\frac{\partial f}{\partial T}\right|_{(t,T(t))} = \frac{\tau_0}{(T(t))^2} e^{y_0(t)-y_0(t-T(t))}(T(t)\, y_0'(t-T(t)) - 1),$$

and

$$T(t) = 1/y_0'(t - T(t)).$$

If $T(t) = T_0$, $(\partial f/\partial T)|_{(t,T_0)} \leqslant 0$, so $T_0 \leqslant 1/(y_0'(t - T_0))$. By Eq. (3.3),

$$\begin{aligned}y_0'(T_0) &= \frac{\tau_0}{T(T_0)} e^{y_0(T_0)-y_0(T_0-T(T_0))} \geqslant \frac{\tau_0}{T(T_0)} e^{T(T_0)\, y_0'(T_0 - T(T_0))}\\ &\geqslant \tau_0 e y_0'(T_0 - T(T_0)) \geqslant \tau_0 e y_0'(0),\end{aligned}$$

where $y_0'(0)$ is the right-hand derivative of y at 0. Similarly,

$$y_0'(2T_0) \geqslant \tau_0 e y_0'(T_0) \geqslant (\tau_0 e)^2 y_0'(0)$$

and, in general, if k is an integer,

$$y_0'(kT_0) \geqslant (\tau_0 e)^k y_0'(0).$$

Since $\tau_0 e > 1$, there exists t_0 for which $y_0'(t_0) > 1/T_0$. Let $a = y_0'(t_0)$, $b = \tau_0 e$, $t_1 = t_0 + T_0$, and

$$t_{n+1} = t_n + \frac{1}{ab^{n-1}}$$

for $n \geqslant 1$. Then if $t \geqslant t_1$,

$$\begin{aligned} y_0'(t) &= \frac{\tau_0}{T(t)} e^{y_0(t) - y_0(t - T(t))} \geqslant \frac{\tau_0}{T(t)} e^{T(t)\, y_0'(t - T(t))} \\ &\geqslant \tau_0 e y_0'(t - T(t)) \geqslant b y_0'(t_0) = ab. \end{aligned}$$

Since $y_0'(t - T(t)) \geqslant a$, $T(t) \leqslant 1/a$. Similarly, if $t \geqslant t_2$ then $t - T(t) \geqslant t_2 - 1/a = t_1$, so $y_0'(t) \geqslant b y_0'(t_1) \geqslant ab^2$ and $T(t) \leqslant 1/y_0'(t_1) \leqslant 1/(ab)$. Reasoning inductively, we conclude that for all n, $y_0'(t_n) \geqslant ab^n$. But as $n \to \infty$,

$$t_n \to t_1 + \frac{1/a}{1 - (1/b)}$$

while $y_0'(t_n) \geqslant ab^n \to \infty$.

Equation (3.2) implies that $y_0(t) \to \infty$ when $y_0'(t)$ does. Thus $y_0(t) \to \infty$ as t approaches some $t_* < \infty$; and we are finished. ∎

A stronger condition than Eq. (1.5), analagous to Eq. (2.2) for the constant case, is

$$\inf_{\lambda, t \in (0,\infty)} \frac{1}{\lambda} \sum_{i=1}^{n} q_i(t) e^{\lambda T_i(t)} > 1. \tag{3.4}$$

However, there may be a positive solution to Eq. (1.4) with Eq. (3.4) being satisfied. An example follows. Choose $b > 0$, and let $a = b/(1 - \exp(-b)) > 1$. Let $T_1(t) = 0$, $T_2(t) = 1$, and

$$g(t) = e^{bt}(1 - e^{b}),$$

$$q_1(t) = a(g(t) - 1),$$

and

$$q_2(t) = ae^{-g(t)}.$$

Differentiating $(1/\lambda)(q_1(t) + q_2(t)e^{\lambda})$ with respect to λ we find that for given t it is minimized at $\lambda = g(t)$ and the minimum is a. Thus Eq. (3.4) is satisfied. But substituting $x(t) = \exp(-\exp(bt))$ into Eq. (1.4) we find that it is a solution. Notice, though, that $q_1(t) \to \infty$ as $t \to \infty$. This example is one motivation for the

Conjecture. If there are constants q_0 and T_0 for which $0 \leqslant q_i(t) \leqslant q_0$ and $0 \leqslant T_i(t) \leqslant T_0$ for $1 \leqslant i \leqslant n$ and $t \geqslant 0$, then Eq. (3.4) implies that all solutions to (1.4) oscillate.

References

1. T. Kusano, On even order functional differential equations with advanced and retarded arguments, *J. Differential Equations* **45** (1982), 75–84.
2. G. Ladas, Sharp conditions for oscillations caused by delays, *Applicable Anal.* **9** (1979), 93–98.
3. G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, Necessary and sufficient conditions for oscillations, *Amer. Math. Monthly* **90** (1983), 637–640; see also Necessary and sufficient conditions for oscillations of higher order delay differential equations, *Trans. Amer. Math. Soc.*, in press.
4. G. Ladas and I. P. Stavroulakis, On delay differential inequalities of first order, *Abstract Amer. Math. Soc.* **1**(6) (1980), 577.
5. G. Ladas and I. P. Stavroulakis, Oscillations caused by several retarded and advanced arguments, *J. Differential Equations* **44** (1982), 134–152.
6. G. S. Ladde, Oscillations caused by retarded perturbations of first order linear ordinary differential equations, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **63** (1977), 351–359.
7. A. D. Myshkis, "Linear Differential Equations with Delayed Argument," GITTL, Moscow/Leningrad, 1951. [Russian]
8. V. N. Sevelo and N. V. Varech, "Asymptotic Methods in the Theory of Nonlinear Oscillations," Naukova Fumka, Kiev, 1979. [Russian]
9. Y. G. Sficas and V. A. Staikos, Oscillations of differential equations with deviating arguments, *Funkcial. Ekvac.* **19** (1976), 35–43.
10. M. I. Tramov, Conditions for oscillatory solutions of first order differential equations with a delayed argument, *Izv. Vyssh. Uchebn. Zaved. Mat.* **19**(3) (1975), 92–96.
11. M. I. Tramov, On the oscillation of the solutions of differential equations with a delayed argument, *in* "Aspects of Computational and Applied Mathematics," *Tashkent* **5** (1971), 31–36.