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Letter to the Editor

A new proof of some polynomial inequalities related to pseudo-splines ☆

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Abstract

Pseudo-splines of type I were introduced in [I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (2003) 1–46] and [Selenick, Smooth wavelet tight frames with zero moments, Appl. Comput. Harmon. Anal. 10 (2000) 163–181] and type II were introduced in [B. Dong, Z. Shen, Pseudo-splines, wavelets and framelets, Appl. Comput. Harmon. Anal. 22 (2007) 78–104]. Both types of pseudo-splines provide a rich family of refinable functions with B-splines, interpolatory refinable functions and refinable functions with orthonormal shifts as special examples. In [B. Dong, Z. Shen, Pseudo-splines, wavelets and framelets, Appl. Comput. Harmon. Anal. 22 (2007) 78–104], Dong and Shen gave a regularity analysis of pseudo-splines of both types. The key to regularity analysis is Proposition 3.2 in [B. Dong, Z. Shen, Pseudo-splines, wavelets and framelets, Appl. Comput. Harmon. Anal. 22 (2007) 78–104], which also appeared in [A. Cohen, J.P. Conze, Régularité des bases d'ondelettes et mesures ergodiques, Rev. Mat. Iberoamericana 8 (1992) 351–365] and [I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia, 1992] for the case $l = N - 1$. In this note, we will give a new insight into this proposition.

Keywords: Refinement equations; Orthogonal wavelets; Pseudo-splines; Regularity

1. Introduction

Pseudo-splines of type I were first introduced in [3] and [8] in order to construct tight framelets with required approximation order of the truncated frame series. Pseudo-splines of type II were introduced in [4] to construct symmetric or antisymmetric tight framelets with required approximation order. Both types of pseudo-splines provide a rich family of compactly supported refinable functions which includes B-splines, orthogonal refinable functions and interpolatory refinable functions as its special cases (see [4]). A comprehensive study, especially the regularity analysis, was given in [4]. They were then extended and extensively studied in [5] and [6]. The refinement mask of a pseudo-spline of type I with order (N, l) is given by

$$\left|\widehat{{}_1a_{N,l}}(\omega)\right|^2 := \cos^{2N}\frac{\omega}{2}\sum_{j=0}^{l}\binom{N+l}{j}\sin^{2j}\frac{\omega}{2}\cos^{2(l-j)}\frac{\omega}{2}, \quad \omega \in [-\pi, \pi]. \tag{1.1}$$

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The refinement mask of a pseudo-spline of type II with order (N, l) is given by

$$\widehat{{}_2a_{N,l}}(\omega) := \cos^{2N}\frac{\omega}{2}\sum_{j=0}^{l}\binom{N+l}{j}\sin^{2j}\frac{\omega}{2}\cos^{2(l-j)}\frac{\omega}{2}, \quad \omega\in[-\pi,\pi],$$

where $N \geqslant 1$ and $0 \leqslant l \leqslant N-1$. We note that $|\widehat{{}_1a_{N,l}}(\omega)|^2 = \widehat{{}_2a_{N,l}}(\omega)$. Hence, $\widehat{{}_1a_{N,l}}(\omega)$ is a square root of $\widehat{{}_2a_{N,l}}(\omega)$ and it is a 2π-periodic trigonometric polynomial with real coefficients by Fejér–Riesz lemma. The mask $\widehat{{}_ka_{N,l}}$ ($k = 1, 2$) completely determines the corresponding refinable function ${}_k\phi$ ($k = 1$ or 2) which we call *pseudo-splines*.

Let $P_{N,l}(x) := \sum_{j=0}^{l}\binom{N-1+j}{j}x^j$, by (1) of Lemma 2.2 in [4], $\sum_{j=0}^{l}\binom{N+l}{j}\sin^{2j}\frac{\omega}{2}\cos^{2(l-j)}\frac{\omega}{2} = P_{n,l}(\sin^2\frac{\omega}{2})$ for all $\omega\in[-\pi,\pi]$. Therefore, when $l = N-1$, pseudo-splines of type I are refinable functions with orthonormal shifts given in [2] and pseudo-splines of type II are the interpolatory refinable functions which were first studied by Dubuc in [7]. The pseudo-splines with order $(N, 0)$ are B-splines. The other pseudo-splines fill in the gap between the B-spline and orthogonal or interpolator refinable functions. In [4], Dong and Shen gave a regularity analysis of pseudo-splines of both types. The key to regularity analysis is Proposition 3.2 in [4]. This proposition is important and difficult to prove. In [1] and [2], it takes several technics including numeric computation to prove it for the case $l = N-1$. In [4], by using many steps to prove that a polynomial is decreasing, Dong and Shen gave a complete proof of this proposition. In this note, by using an auxiliary polynomial and some concave properties, we will give a new insight into Proposition 3.2 in [4].

2. Four lemmas

In this section, we establish four technical lemmas which will be used to prove our main result.

Lemma 2.1. *For given nonnegative integers N and l, let $P_{N,l}(x) := \sum_{j=0}^{l}\binom{N-1+j}{j}x^j$. Then*

$$P'_{N,l}(x) = \frac{N}{1-x}\left[P_{N,l}(x) - \binom{N+l}{l}x^l\right], \tag{2.1}$$

$$P''_{N,l}(x) = \frac{N}{(1-x)^2}\left[(N+1)P_{N,l}(x) - \binom{N+l}{l}\left(N+1+\frac{l(1-x)}{x}\right)x^l\right]. \tag{2.2}$$

Proof. Borrowing an idea from [1] and [4], to prove (2.1), by the definition of $P_{N,l}(x)$,

$$\begin{aligned}\frac{1-x}{N}P'_{N,l}(x) &= \frac{1-x}{N}\sum_{j=1}^{l}\binom{N-1+j}{j}\,j\,x^{j-1} = (1-x)\sum_{j=0}^{l-1}\binom{N+j}{j}x^j\\ &= 1+\sum_{j=1}^{l}\left[\binom{N+j}{j}-\binom{N+j-1}{j-1}\right]x^j - \binom{N+l}{l}x^l = P_{N,l}(x) - \binom{N+l}{l}x^l.\end{aligned}$$

Therefore (2.1) holds. To prove (2.2), taking the derivative of (2.1),

$$\begin{aligned}P''_{N,L}(x) &= \frac{N}{(1-x)^2}\left[P_{N,l}(x)-\binom{N+l}{l}x^l\right] + \frac{N}{1-x}\left[P'_{N,l}(x) - \binom{N+l}{l}lx^{l-1}\right]\\ &= \frac{N}{(1-x)^2}\left[P_{N,l}(x)-\binom{N+l}{l}x^l\right] + \frac{N^2}{(1-x)^2}\left[P_{N,l}(x)-\binom{N+l}{l}x^l\right] - \frac{N}{(1-x)}\binom{N+l}{l}lx^{l-1}\\ &= \frac{N}{(1-x)^2}\left[(N+1)P_{N,l}(x) - \binom{N+l}{l}\left(N+1+\frac{l(1-x)}{x}\right)x^l\right]. \qquad \square\end{aligned}$$

Lemma 2.2. *For given integers N and l, if $0 < l \leqslant N-1$,*

$$2\cdot\frac{N+2\sqrt{l}-l}{(N+\sqrt{l}-l)^2+N} + \frac{1}{l} \leqslant \frac{N+2\sqrt{l+1}-l-1}{(N+\sqrt{l+1}-l-1)^2+N}\,\frac{N+l}{l}. \tag{2.3}$$

Proof. By multiplying the common denominator of (2.3), it suffices to show that

$$\big[2l(N+2\sqrt{l}-l)+(N+\sqrt{l}-l)^2+N\big]\big[(N+\sqrt{l+1}-l-1)^2+N\big] \\ \leqslant [N+2\sqrt{l+1}-l-1](N+l)\big[(N+\sqrt{l}-l)^2+N\big]. \quad (2.4)$$

Note that the first factor can be divided by $N+l$. That is,

$$2l(N+2\sqrt{l}-l)+(N+\sqrt{l}-l)^2+N=(N+l)(N+2\sqrt{l}-l+1).$$

After this factorization, by direct computation and simplification, inequality (2.4) is equivalent to

$$(N+l)(1+\sqrt{l}-\sqrt{l+1})\big[\big(\sqrt{l(l+1)}-N\big)(\sqrt{l+1}-1+\sqrt{l})+(\sqrt{l+1}-1-\sqrt{l})\sqrt{l+1}\big]\leqslant 0.$$

By the fact $1\leqslant l+1\leqslant N$ and direct computation, the last inequality holds. □

Lemma 2.3. *For given nonnegative integers N and l, let $P_{N,l}(x):=\sum_{j=0}^{l}\binom{N-1+j}{j}x^j$ and $g(x):=P_{N,l}(x)(l+1-\sqrt{l+1})^2+x^2P''_{N,l}(x)-2xP'_{N,l}(x)(l+1-\sqrt{l+1})$. Then $g(1/2)\leqslant 0$ for $l\leqslant N-1$.*

Proof. By the definition of g and Eqs. (2.1) and (2.2),

$$\begin{aligned} g\left(\frac{1}{2}\right) &= P_{N,l}\left(\frac{1}{2}\right)(l+1-\sqrt{l+1})^2+\left(N^2+N\right)P_{N,l}\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^l\binom{N+l}{l}\left(N+N^2+Nl\right) \\ &\quad -2N(l+1-\sqrt{l+1})\left[P_{N,l}\left(\frac{1}{2}\right)-\binom{N+l}{l}\left(\frac{1}{2}\right)^l\right] \\ &= P_{N,l}\left(\frac{1}{2}\right)\left[(l+1-\sqrt{l+1}-N)^2+N\right]-\left(\frac{1}{2}\right)^l\binom{N+l}{l}N(N+2\sqrt{l+1}-l-1). \end{aligned}$$

Then $g(\frac{1}{2})\leqslant 0$ follows from

$$2^lP_{N,l}\left(\frac{1}{2}\right)\leqslant N\cdot\frac{N+2\sqrt{l+1}-l-1}{(N+\sqrt{l+1}-l-1)^2+N}\binom{N+l}{l}. \quad (2.5)$$

We now prove (2.5) by induction. Equation (2.5) is obviously true for $l=0$. Suppose (2.5) holds for l_0. Now consider $l=l_0+1\leqslant N-1$,

$$\begin{aligned} 2^{l_0+1}P_{N,l_0+1}\left(\frac{1}{2}\right) &= 2\cdot 2^{l_0}P_{N,l_0}\left(\frac{1}{2}\right)+\binom{N+l_0}{l_0+1} \\ &\leqslant \left[2N\cdot\frac{N+2\sqrt{l_0+1}-l_0-1}{(N+\sqrt{l_0+1}-l_0-1)^2+N}+\frac{N}{l_0+1}\right]\binom{N+l_0}{l_0} \quad \text{(by inductive hypothesis)} \\ &\leqslant N\cdot\left[\frac{N+2\sqrt{l_0+2}-l_0-2}{(N+\sqrt{l_0+2}-l_0-2)^2+N}\frac{N+l_0+1}{l_0+1}\right]\binom{N+l_0}{l_0} \quad \text{(by (2.3))} \\ &= N\cdot\frac{N+2\sqrt{l_0+2}-l_0-2}{(N+\sqrt{l_0+2}-l_0-2)^2+N}\binom{N+l_0+1}{l_0+1}. \end{aligned}$$

That is, (2.5) holds for $l=l_0+1$. □

Lemma 2.4. *Suppose $p(x)=\sum_{j=0}^{l}c_jx^j$ is a polynomial with nonnegative coefficients c_j, $j=0,1,\ldots,l$. Let $g(x):=p(x)(l+1-\sqrt{l+1})^2+x^2p''(x)-2xp'(x)(l+1-\sqrt{l+1})$. If $g(x_0)\leqslant 0$ for some positive number x_0, then*

$$p(x)p(y)\leqslant p^2\left(\frac{x+y}{2}\right)\quad \forall x,y\in[x_0,+\infty). \quad (2.6)$$

Proof. If $p(x)\equiv 0$ or $l=0$, then obviously (2.6) holds. Hence we assume that $p(x)>0$ in $(0,+\infty)$ and $l>0$. By the definition of $p(x)$ and $g(x)$, $g(x)=\sum_{j=0}^{l}c_jd(j)x^j$, where $d(t):=(l+1-\sqrt{l+1})^2+t(t-1)-2t(l+1-\sqrt{l+1})$.

It is easy to check that $d(t) = (l + 1 - \sqrt{l+1} - t)^2 - t$, so $d(l + 1) = 0 = d(l + 1 - 2\sqrt{l+1} + 1)$. Hence there exists an integer $L \in [0, l]$, such that $d(j) \geqslant 0$ for $j = 0, \dots, L$ and $d(j) \leqslant 0$ for $j = L + 1, \dots, l$. Thus, each term $c_j d(j)x^{j-L}$, $j = 0, \dots, l$ either has a nonpositive power $j - L$ or a nonpositive coefficient $c_j d(j)$. Therefore, $g(x)x^{-L}$ is decreasing in $(0, +\infty)$. If $g(x_0) \leqslant 0$ for some $x_0 > 0$, then $g(x_0)x_0^{-L} \leqslant 0$ and therefore $g(x)x^{-L} \leqslant 0$ for all $x \geqslant x_0$. Hence $g(x) \leqslant 0$ for all $x \geqslant x_0$. That is,

$$p(x)(l + 1 - \sqrt{l+1})^2 + x^2 p''(x) \leqslant 2xp'(x)(l + 1 - \sqrt{l+1}) \quad \forall x \in [x_0, +\infty). \tag{2.7}$$

Hence

$$\left[p(x)(l + 1 - \sqrt{l+1})^2\right]\left[x^2 p''(x)\right] \leqslant \left[xp'(x)(l + 1 - \sqrt{l+1})\right]^2 \quad \forall x \in [x_0, +\infty). \tag{2.8}$$

The above inequality is same as $p(x)p''(x) \leqslant [p'(x)]^2$ for all $x \in [x_0, +\infty)$. Therefore, function $\ln p(x)$ is concave downward in $[x_0, +\infty)$. Hence $\ln p(x) + \ln p(y) \leqslant 2 \ln p(\frac{x+y}{2})$ for all $x, y \in [x_0, \infty)$, which implies (2.6). □

Remark. In the above proof, we have proved (2.7) instead of (2.8). The reason for (2.8) to follow from (2.7) is that $a + b \leqslant 2c$ implies $ab \leqslant c^2$. Here, the quantities for a and b are very close to each other by choosing the factors $(l + 1 - \sqrt{l+1})^2$ and x^2.

3. Main result

This section establishes the main result of this note.

Theorem 3.1. *For given integers N and l with $0 \leqslant l \leqslant N - 1$, let $P_{N,l}(x) := \sum_{j=0}^{l} \binom{N-1+j}{j} x^j$. Then*

$$P_{N,l}(x) \leqslant P_{N,l}\left(\frac{3}{4}\right) \quad \forall x \in \left[0, \frac{3}{4}\right], \tag{3.1}$$

$$P_{N,l}(x)P_{N,l}\left[4x(1-x)\right] \leqslant \left[P_{N,l}\left(\frac{3}{4}\right)\right]^2 \quad \forall x \in \left[\frac{3}{4}, 1\right]. \tag{3.2}$$

Proof. Since $P_{N,l}(x)$ is increasing in $[0, \infty)$, (3.1) holds. We now prove (3.2). It is easy to check that $4x(1 - x) \leqslant \frac{3}{2} - x$ for all $x \in [\frac{3}{4}, 1]$. Hence it suffices to prove that

$$P_{N,l}(x)P_{N,l}\left(\frac{3}{2} - x\right) \leqslant \left[P_{N,l}\left(\frac{3}{4}\right)\right]^2 \quad \forall x \in \left[\frac{3}{4}, 1\right]. \tag{3.3}$$

On the other hand, by Lemmas 2.3 and 2.4,

$$P_{N,l}(x)P_{N,l}(y) \leqslant \left[P_{N,l}\left(\frac{x+y}{2}\right)\right]^2 \quad \forall x, y \in \left[\frac{1}{2}, 1\right],$$

which implies (3.3). Hence (3.2) holds. □

Remark. By Theorem 3.1 in [4], our main result leads to an optimal decay estimate on the Fourier transform of pseudo-splines.

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