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Suborthogonal double covers of the complete graph by stars

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Abstract

In this paper, we consider suborthogonal double covers as a generalization of the well-known concept of orthogonal double covers. We look for a collection of subgraphs of the complete graph K_n all being identical copies of a given graph G (called pages) such that every edge of K_n is contained in exactly two pages and two pages have at most one edge in common. Necessary and sufficient conditions are proved for suborthogonal double covers of K_n by stars.

Keywords: Star; Edge partition; Suborthogonal; ODC

1. Introduction

An *orthogonal double cover* (*ODC*) of a complete graph $\mathscr{K}$ by a graph $\mathscr{G}$ is a set of subgraphs of $\mathscr{K}$ each isomorphic to $\mathscr{G}$, such that every edge of $\mathscr{K}$ is contained in exactly two subgraphs and each two subgraphs have exactly one edge in common. ODCs have been studied in several directions. Various constructions and statements concerning the existence of ODCs can be found in [1] and its references. However, for several graphs, e.g. the path of three edges, the non-existence of an ODC has been proved, but it is possible to construct suborthogonal double covers for several of these graphs (cf. [4]).

The study of suborthogonality is motivated by the equivalence between suborthogonal double covers of the complete graph by a smaller complete graph and super-simple designs. The existence of super-simple designs has been studied by Gronau and Mullin [2]. Wilson [5] proved the existence of a decomposition of a complete graph whenever n is large enough. Wilson-type results on suborthogonal double covers can be found in [3,4].

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For any positive integer n let $\mathscr{K}_n$ denote the complete graph on n vertices labeled by $0, 1, 2, \ldots, n-1$, if nothing else is presupposed. By $\mathscr{G} = (V, E)$ we denote any nonempty simple graph (i.e. a graph without loops and multiple edges) with vertex set $V(\mathscr{G})$ of v elements and edge set $E(\mathscr{G})$ of e elements.

Definition. A *suborthogonal double cover* (*SODC*) of $\mathscr{K}_n$ by a simple graph $\mathscr{G}$ is a set $S = \{\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_s\}$ of subgraphs of $\mathscr{K}_n$, called pages, isomorphic to $\mathscr{G}$ such that

I every edge of $\mathscr{K}_n$ is contained in exactly two pages,
II $|E(\mathscr{G}_i) \cap E(\mathscr{G}_j)| \leqslant 1 \ \forall i \neq j$, i.e. two different pages have at most one edge in common.

An SODC differs from an ODC in the second condition, where for ODCs the edge sets of different pages are required to have exactly one edge in common. This leads to rather restrictive parameter conditions for ODCs, where $s = n$, i.e. the number of pages of an ODC is fixed by the number of vertices of $\mathscr{K}_n$, and $|E(\mathscr{G})| = e = n - 1$, the number of edges, too.

We want to give necessary conditions generated by the above definition. Since these conditions are quite self-explanatory, we will give only a short remark on every item.

Lemma 1 (Necessary conditions). *For an SODC of the complete graph $\mathscr{K}_n$ on n vertices by a graph $\mathscr{G}$ on v vertices with e edges, where d denotes the gcd of the vertex degrees of $\mathscr{G}$ and s denotes the number of pages, the following conditions hold*:

(i) $v \leqslant n$ by counting the vertices of $\mathscr{G}$ and $\mathscr{K}_n$,
(ii) $n(n-1) \equiv 0 \pmod{e}$ caused by the edge numbers,
(iii) $2(n-1) \equiv 0 \pmod{d}$ by vertex degrees,
(iv)
$$\frac{n(n-1)}{2} \leqslant \binom{s}{2}$$
$$\Leftrightarrow n \leqslant s = \frac{n(n-1)}{e}$$
$$\Leftrightarrow e \leqslant n-1,$$

since the number of edges to be covered twice has to be less or equal the number of pairs of pages.

2. SODCs by stars

Let $\mathscr{S}_e$ be the star on $e + 1 = v$ vertices and with e edges. Since SODCs of $\mathscr{K}_n$ by a single edge clearly exist for all natural numbers n, we presuppose e to be at least 2. We call the single vertex with degree e the centre of the star. See Figs. 1 and 2.

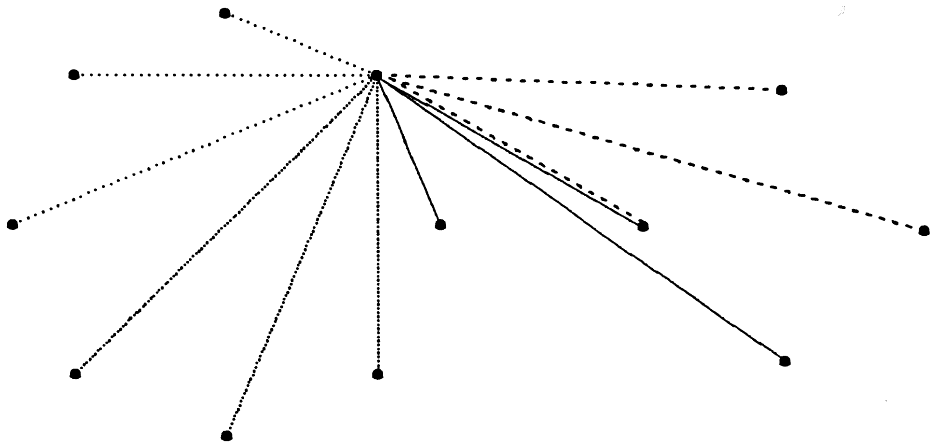

Fig. 1. A set of $t+1$ stars with the same centre in A.

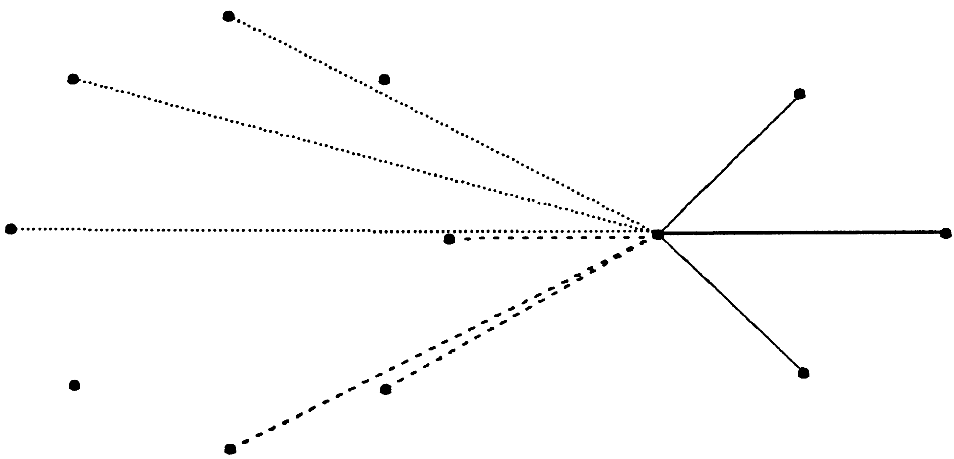

Fig. 2. A set of t stars with the same centre in B.

Theorem 2. *There exists an SODC of $\mathscr{K}_n$ by a star $\mathscr{S}_e$ iff*

$$v \leqslant n \ (\textit{resp. } e < n) \tag{1}$$

$$n(n-1) \equiv 0 \,(\operatorname{mod} e) \tag{2}$$

$$\left\lceil \frac{n-1}{e} \right\rceil e - (n-1) \leqslant \binom{\lceil \frac{n-1}{e} \rceil}{2}. \tag{3}$$

Before proving the result we will present some notation. Let

$$s := \frac{n(n-1)}{e} \tag{4}$$

again denote the number of pages, and

$$t{:=}\left\lfloor \frac{s}{n} \right\rfloor = \left\lfloor \frac{n-1}{e} \right\rfloor \tag{5}$$

denote the maximum number of stars which could have the same vertex of $\mathscr{K}_n$ as their centre without using an edge twice, since every star has e edges and each vertex of $\mathscr{K}_n$ is incident to $n-1$ edges.

Let furthermore,

$$r{:=}s - tn, \quad \text{where } 0 \leqslant r < n,\ r \in \mathbb{N}. \tag{6}$$

Proof (*Theorem* 2)**.** If $s \equiv 0 \pmod n$, i.e. $n-1 \equiv 0 \pmod e$ the assumption is clear by using each vertex of $\mathscr{K}_n$ as the centre of exactly $t = \frac{s}{n} = \frac{n-1}{e}$ stars. So the only interesting case is $r > 0$.

First we prove the necessity of (1)–(3). Obviously, (1) and (2) are equal to conditions (i) and (ii) in Lemma 1, respectively. It remains to check (3). By (4) and (6) we get

$$\frac{n(n-1)}{e} = tn + r, \quad 0 < r < n.$$

Since $r > 0$ in every possible SODC of $\mathscr{K}_n$ by $\mathscr{S}_e$, there has to be at least one vertex X of $V(\mathscr{K}_n)$ as the centre of $\hat{t} > t$ stars. For this vertex we know

$$\hat{t}e - (n-1) \leqslant \binom{\hat{t}}{2},$$

i.e. the least number of edges to be covered twice has to be at most the number of pairs of stars which could share an edge. By elementary observations on the concave function $f(x){=}xe{-}\binom{x}{2}$ we can show this equation implying $\hat{t}{=}t{+}1$, i.e. $(t{+}1)e{-}(n{-}1) \leqslant \binom{t+1}{2}$. This proves the necessity of (3).

Indeed, $t+1$ will be the maximum number of stars having the same centre. To prepare the construction we want to provide some arithmetic relations. We divide the vertex set of $\mathscr{K}_n$ into two distinct subsets $A = \{1_a, 2_a, \dots, r_a\}$ of r vertices and $B = \{1_b, 2_b, \dots, (n-r)_b\}$ of the remaining $n-r$ vertices. We will construct an SODC where each vertex of A is the centre of $t+1$ stars and each vertex of B is the centre of t stars. Therefore, we define

$$l_0{:=}(n-1) - te \quad \text{and} \quad l_2{:=}(t+1)e - (n-1), \tag{7}$$

such that l_2 is the number of edges covered twice by $t+1$ stars having the same centre and using all possible edges, and l_0 is the number of edges not being covered by t stars with the same centre using only different edges. By (4) and (6), both l_0 and l_2 are nonnegative.

Because of (6) we get the correct number of stars, and hence the correct number of edges, too. This implies

$$rl_2 = (n-r)l_0. \tag{8}$$

Furthermore, we have

$$n > e = l_0 + l_2,$$

and by (8)

$$n > l_0 + \frac{l_0(n-r)}{r}.$$

Hence

$$nr > l_0 r + l_0(n-r),$$

that is $r > l_0$ and analogously by (8)

$$n - r > l_2. \tag{9}$$

By (5) and (7), condition (3) is equivalent to

$$\binom{t+1}{2} \geqslant l_2. \tag{10}$$

Inequalities (9) and (10) are what we need for the construction. Now let us build the SODC. We take any vertex of set A, say 1_a. Since 1_a is going to be the centre of $t+1$ stars, we have to cover $(t+1)e$ edges.

Step 1: We choose l_2 of the $\binom{t+1}{2}$ pairs one can form of the $t+1$ stars and label them with nos. $1, 2, \ldots, l_2$ (possible because of (10)). Then every star of pair i gets edge $(1_a, i_b)$ (possible because of (9)), that are $2l_2$ edges, altogether. Single edges to the remaining $(n-r)-l_2$ vertices Y_b and $r-1$ vertices X_a can be distributed arbitrarily on the remaining $(t+1)e - 2l_2$ star edges. Now we have $(t+1)e$ edges forming the first set of $(t+1)$ stars with the same centre.

Step 2: This set of stars is going to be rotated in the two sets A and B by adding $1 \pmod a$ and $l_2 \pmod b$ to every vertex, respectively. After rotation, exactly the edges we need are left to form the desired t stars with centre in Y_b for each element Y_b of B. The exact situation after rotation is illustrated in Table 1.

Table 1

Vertices	Connected with every	
X_a	$X'_a \neq X_a$	Twice
	Y_b: $Y \in \{l_2(X-1)+1, l_2(X-1)+2, \ldots,$ $l_2(X-1)+l_2 \equiv l_2 X \pmod{n-r}\}$	Twice
	Y_b: $Y \in \{1, 2, \ldots, (l_2(X-1),$ $l_2X+1, l_2X+2, \ldots, n-r \pmod{n-r}\}$	Once
Y_b	$Y'_b \neq Y_b$	Never
and hence with		
Y_b	l_0 of the X_a	Twice
	$r - l_0$ of the X_a	Once

Step 3: For every Y_b we form the desired t stars in an arbitrary way using the remaining edges.

$$\begin{array}{l} r - l_0 \text{ edges } (Y_b, X_a) \\ \underline{\text{and } n - r - 1 \text{ edges } (Y_b, Y'_b) \quad (Y \neq Y') \text{ (one in every direction)}} \\ = \quad te \text{ edges.} \end{array}$$

Since for condition (II) only stars with the same centre are of importance, this construction gives us the SODC.

Example. Let us consider the SODC of $\mathcal{K}_{12}$ by a star of 3 edges, then we have $n=12$ and $e=3$, and hence we get $s=44$, $t=3$ and $r=8$.

$$\begin{array}{ll} A = \{0_a, 1_a, 2_a, 3_a, 4_a, 5_a, 6_a, 7_a\} & B = \{0_b, 1_b, 2_b, 3_b\} \\ \text{rotation} + 1 \pmod 8 & +(t+1)e - (n-1) = 1 \pmod 4. \end{array}$$

Corollary 3. *In the directed case we consider stars where the centre is a source. An SODC of the directed $\mathcal{K}_n$ by directed stars exists iff*

$$v \leqslant n \ (\textit{resp. } e < n), \tag{11}$$

$$(n-1) \equiv 0 \pmod e. \tag{12}$$

Proof. The assumption is clear, since two stars sharing the centre never share an edge and vice versa, that is for every edge each endpoint has to be the centre once.

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