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On the convolution equation related to the diamond kernel of Marcel Riesz

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Abstract

In this paper, we study the distribution $e^{\alpha t}\diamondsuit^k\delta$ where $\diamondsuit^k$ is introduced and named as the Diamond operator iterated k-times ($k=0,1,2,\ldots$) and is definded by

$$\diamondsuit^k=\left(\left(\frac{\partial^2}{\partial t_1^2}+\frac{\partial^2}{\partial t_2^2}+\cdots+\frac{\partial^2}{\partial t_p^2}\right)^2-\left(\frac{\partial^2}{\partial t_{p+1}^2}+\frac{\partial^2}{\partial t_{p+2}^2}+\cdots+\frac{\partial^2}{\partial t_{p+q}^2}\right)^2\right)^k,$$

where $t=(t_1,t_2,\ldots,t_n)$ is a variable and $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is a constant and both are the points in the n-dimensional Euclidean space R^n, δ is the Dirac-delta distribution with $\diamondsuit^0\delta=\delta$ and $p+q=n$ (the dimension of R^n)

At first, the properties of $e^{\alpha t}\diamondsuit^k\delta$ are studied and later we study the application of $e^{\alpha t}\diamondsuit^k\delta$ for solving the solutions of the convolution equation

$$(e^{\alpha t}\diamondsuit^k\delta)*u(t)=e^{\alpha t}\sum_{r=0}^{m}c_r\diamondsuit^r\delta.$$

We found that its solutions related to the Diamond Kernel of Marcel Riesz and moreover, the type of solutions such as, the classical solution (the ordinary function) or the tempered distributions depending on m,k and α. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

From [2, Theorem 3.1], the equation $\diamondsuit^k u(t)=\delta$ has $(-1)^k S_{2k}(t)*R_{2k}(t)$ as an elementary solution and is called the Diamond Kernel of Marcel Riesz where $S_{2k}(t)$ and $R_{2k}(t)$ are defined by (2.1) and (2.2), respectively, with $\gamma=2k$ where γ is nonnegative.

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Consider the convolution equation

$$(\mathrm{e}^{\alpha t}\Diamond^k\delta)*u(t)=\mathrm{e}^{\alpha t}\sum_{r=0}^{m}c_r\Diamond^r\delta. \tag{1.1}$$

In finding the type of solutions $u(t)$ of Eq. (1.1), we use the method of convolution of the tempered distribution. Before going to that point, some definitions and basic concepts are needed.

2. Preliminaries

Definition 2.1. Let the function $S_\gamma(t)$ be defined by

$$S_\gamma=2^{-\gamma}\pi^{-n/2}\Gamma\left(\frac{n-\gamma}{2}\right)\frac{|t|^{\gamma-n}}{\Gamma(\gamma/2)}, \tag{2.1}$$

where γ is a complex parameter, n is the dimension of R^n, $t=(t_1,t_2,\ldots,t_n)\in R^n$ and $|t|=\sqrt{(t_1^2+\cdots+t_n^2)}$. Now S_γ is an ordinary function if $\mathrm{Re}(\gamma)\geqslant n$ and is a distribution of γ if $\mathrm{Re}(\gamma)<n$.

Definition 2.2. Let $t=(t_1,t_2,\ldots,t_n)$ be the point of R^n and write $v=t_1^2+t_2^2+\cdots+t_p^2-t_{p+1}^2-t_{p+2}^2-\cdots-t_{p+q}^2$, $p+q=n$. Denote by $\Gamma_+=\{t\in R^n\colon\ t_1>0 \text{ and } v>0\}$ the set of an interior of the forward cone and $\bar{\Gamma}$ is the closure of Γ.

For any complex number γ, define

$$R_\gamma(t)=\begin{cases}\frac{v^{(\gamma-n)/2}}{K_n(\gamma)} & \text{if } t\in\Gamma_+,\\ 0 & \text{if } t\notin\Gamma_+,\end{cases}$$

where $K_n(\gamma)$ is given by the formula

$$K_n(\gamma)=\frac{\pi^{(n-1)/2}\Gamma(\frac{2+\gamma-n}{2})\Gamma(\frac{1-\gamma}{2})\Gamma(\gamma)}{\Gamma(\frac{2+\gamma-p}{2})\Gamma(\frac{p-\gamma}{2})}.$$

The function $R_\gamma(t)$ was introduced by Nozaki [3, p.72]. It is well known that $R_\gamma(t)$ is an ordinary function if $Re(\gamma)\geqslant n$ and is a distribution of α if $Re(\gamma)<n$. Let $\operatorname{supp}R_\gamma(t)$ denote the support of $R_\gamma(t)$ and suppose that $\operatorname{supp}R_\gamma(t)\subset\bar{\Gamma}_+$.

Lemma 2.1. *$S_\gamma(t)$ and $R_\gamma(t)$ are homogeneous distributions of order $\alpha-n$. Moreover they are tempered distribution.*

Proof. Since $S_\gamma(t)$ and $R_\gamma(t)$ satisfy the Euler equation

$$\sum_{i=1}^{n} t_i \frac{\partial R_\gamma(t)}{\partial t_i} = (\alpha - n)R_\gamma(t),$$

$$\sum_{i=1}^{n} t_i \frac{\partial S_\gamma(t)}{\partial t_i} = (\alpha - n)S_\gamma(t),$$

then they are homogeneous distribution of order $\alpha - n$ by Donoghue [1, pp. 154,155] that proved that every homogeneous distribution is a tempered distribution. □

Lemma 2.2 (The convolution of tempered distribution). *The convolution $S_\gamma(t) * R_\gamma(t)$ exists and is a tempered distribution.*

Proof. Choose $\operatorname{supp} R_\gamma(t) = K \subset \overline{\Gamma}_+$ where K is a compact set. Then $R_\gamma(t)$ is a tempered distribution with compact support and by Donoghue [1, pp. 156–159], $S_\gamma(t) * R_\gamma(t)$ exists and is a tempered distribution. □

3. The properties of $e^{\alpha t} \Diamond^k \delta$

Lemma 3.1.

$$e^{\alpha t} \Diamond^k \delta = L^k \delta, \tag{3.1}$$

where L is the partial differential operator of Diamond type and is defined by

$$\begin{aligned}
L \equiv{}& \Diamond + \sum_{r=1}^{n} \alpha_r^2 \Box - 2\sum_{r=1}^{n}\sum_{i=1}^{r}\left(\alpha_r \frac{\partial^3}{\partial t_i^2 \partial t_r} + \alpha_i \frac{\partial^3}{\partial t_i \partial t_r^2}\right) \\
&+ 2\sum_{r=1}^{n}\sum_{j=p+1}^{p+q}\left(\alpha_r \frac{\partial^3}{\partial t_j^2 \partial t_r} + \alpha_j \frac{\partial^3}{\partial t_j \partial t_r^2}\right) \\
&+ 4\left(\sum_{r=1}^{n}\sum_{i=1}^{p} \alpha_r \alpha_j \frac{\partial^2}{\partial t_i \partial t_r} - \sum_{r=1}^{n}\sum_{j=p+1}^{p+q} \alpha_r \alpha_j \frac{\partial^2}{\partial t_j \partial t_r}\right) \\
&- 2\sum_{r=1}^{n} \alpha_r^2 \left(\sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial t_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial t_j}\right) \\
&+ \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2\right) \Delta - 2\left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2\right) \sum_{r=1}^{n} \alpha_r \frac{\partial}{\partial t_r} \\
&+ \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2\right) \sum_{r=1}^{n} \alpha_r^2,
\end{aligned} \tag{3.2}$$

where $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial t_i^2} - \sum_{j=p}^{p+q} \frac{\partial^2}{\partial t_j^2}$, $\Delta = \sum_{i=1}^{p} \frac{\partial^2}{\partial t_i^2}$, $p+q=n$. *Actually* $\Diamond = \Box\Delta$ *and* $e^{\alpha t}\Diamond^k\delta$ *is a tempered distribution of order* $4k$.

Proof. For $k=1$, we have

$$\langle e^{\alpha t}\Diamond\delta, \varphi(t)\rangle = \langle \delta, \Diamond e^{\alpha t}\varphi(t)\rangle,$$

where $\varphi(t) \in \mathscr{S}$ the Schwartz space. By computing directly we obtain

$$\Diamond e^{\alpha t}\varphi(t) = e^{\alpha t}M\varphi(t), \tag{3.3}$$

where M is the partial differential operator of the form (3.2) whose the coefficients of the third term, the fourth term, the sixth term and the eighth term of the right-hand side of Eq. (3.2) have opposite signs.

Thus $\langle \delta, \Diamond e^{\alpha t}\varphi(t)\rangle = \langle \delta, e^{\alpha t}M\varphi(t)\rangle = M\varphi(0)$.

By the properties of δ and its partial derivatives with the linear differential operator M, we obtain $M\varphi(0) = \langle L\delta, \varphi\rangle$ where L defined by Eq. (3.2). It follows that $e^{\alpha t}\Diamond\delta = L\delta$. Now

$$\underbrace{(e^{\alpha t}\Diamond\delta) * (e^{\alpha t}\Diamond\delta) * \cdots * (e^{\alpha t}\Diamond\delta)}_{k\text{-times}} = \underbrace{(L\delta) * (L\delta) * \cdots * (L\delta)}_{k\text{-times}},$$

we have $e^{\alpha t}(\delta * \Diamond^k\delta) = \delta * (L^k\delta)$. Thus $e^{\alpha t}\Diamond^k\delta = L^k\delta$. It follows that, for any k, we obtain Eq. (3.1). Since δ has a compact support, hence by Schwartz [4], δ and $L^k\delta$ are tempered distributions and $L^k\delta$ has order $4k$. It follows that $e^{\alpha t}\Diamond^k\delta$ is a tempered distribution of order $4k$. □

Lemma 3.2 (Boundedness property). $|\langle e^{\alpha t}\Diamond^k\delta, \varphi(t)\rangle| \leqslant K$ *where* K *is a constant and* $\varphi \in \mathscr{S}$.

Proof.

$$\begin{aligned}\langle e^{\alpha t}\Diamond^k\delta, \varphi(t)\rangle &= \langle \Diamond^k\delta, e^{\alpha t}\varphi(t)\rangle \\ &= \langle \Diamond^{k-1}\delta, \Diamond e^{\alpha t}\varphi(t)\rangle \\ &= \langle \Diamond^{k-1}\delta, e^{\alpha t}M\varphi(t)\rangle,\end{aligned}$$

where M is defined by Eq. (3.3). By keeping on operate $\Diamond k-1$ times we obtain

$$\begin{aligned}\langle e^{\alpha t}\Diamond^k\delta, \varphi t\rangle &= \langle \delta, e^{\alpha t}M^k\varphi t\rangle \\ &= M^k\varphi(0).\end{aligned}$$

Since $\varphi(0)$ is bounded and also $M^k\varphi(0)$ is bounded. It follows that $|\langle e^{\alpha t}\Diamond^k\delta, \varphi(t)| = |M^k \varphi(0)| \leqslant K$. □

4. The application of $e^{\alpha t}\Diamond^k\delta$

Given $u(t)$ is an distribution and by Lemma 3.1, we have

$$(e^{\alpha t}\Diamond^k\delta) * u(t) = (L^k\delta) * u(t) = L^k u(t),$$

where L is defined by (3.2).

Theorem 4.1. *Given the linear partial differential equation of the form*

$$(e^{\alpha t}\diamondsuit^k\delta)*u(t)=L^k u(t)=\delta. \tag{4.1}$$

Then $u(t)=e^{\alpha t}(-1)^k S_{2k}(t)*R_{2k}(t)$ *is an elementary solution of* (4.1) *or The Diamond Kernel of Marcel Riesz of* (4.1) *where* $S_{2k}(t)$ *and* $R_{2k}(t)$ *are defined by* (2.1) *and* (2.2) *respectively with* $\gamma=2k$.

Proof. By Kananthai [2, Lemma 2.4], $(-1)^k S_{2k}(t)$ is an elementary solution of the Laplace operator Δ^k iterated k-times and also by Trione [5], $R_{2k}(t)$ is an elementary solution of the ultra-hyperbolic operator $\Box^k$ iterated k-times (that is $\Delta^k(-1)^k S_{2k}(t)=\delta$ and $\Box^k R_{2k}(t)=\delta$).

Now $\diamondsuit^k=\Box^k\Delta^k$, consider $e^{\alpha t}(\Box^k\Delta^k\delta)*R_{2k}(t)=\delta$ By Lemma 2.2 with $\gamma=2k$, $(-1)^k S_{2k}(t)*R_{2k}(t)$ exists and is a tempered distribution.

Convolving both sides of the above equation by $e^{\alpha t}[(-1)^k S_{2k}(t)*R_{2k}(t)]$ we obtain

$$e^{\alpha t}[\Delta^k(-1)^k S_{2k}(t)*\Box^k R_{2k}(t)*u(t)]=[e^{\alpha t}(-1)^k S_{2k}(t)*R_{2k}(t)]*\delta$$

$$(e^{\alpha t}\delta)*u(t)=e^{\alpha t}(-1)^k S_{2k}(t)*R_{2k}(t).$$

It follows that $u(t)=e^{\alpha t}(-1)^k S_{2k}(t)*R_{2k}(t)$. $\Box$

Theorem 4.2. *Given the convolution equation*

$$(e^{\alpha t}\diamondsuit^k\delta)*u(t)=e^{\alpha t}\sum_{r=0}^{m}c_r\diamondsuit^r\delta, \tag{4.2}$$

then the type of solutions of (4.2) *depend on the relationship between* k,m *and* α *are as the following cases.*

(1) *If* $m<k$ *and* $m=0$ *then* (4.2) *has the solution* $u(t)=e^{\alpha t}[c_0(-1)^k S_{2k}(t)*R_{2k}(t)]$ *and* $u(t)$ *is an ordinary function for* $2k\geqslant n$ *with any* α *and is a tempered distribution for* $2k<n$ *and for some* $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ *with* $\alpha_i<0$ $(i=1,2,\ldots,n)$.

(2) *If* $0<m<k$, *then the solution of* (4.2) *is*

$$u(t)=e^{\alpha t}\left[\sum_{r=1}^{m}c_r(-1)^{k-r}S_{2k-2r}(t)*R_{2k-2r}(t)\right]$$

which is an ordinary function for $2k-2r\geqslant n$ *with any* α *and is a tempered distribution if* $2k-2r<n$ *for some* α *with* $\alpha_i<0$ $(i=1,2,\ldots,n)$.

(3) *If* $m\geqslant k$ *and for any* α *and suppose that* $k\leqslant m\leqslant M$, *then* (4.2) *has* $u(t)=e^{\alpha t}\sum_{r=k}^{M}c_r\diamondsuit^{r-k}\delta$ *as a solution which is the singular distribution.*

Proof. (1) For $m<k$ and $m=0$, then (4.2) becomes $(e^{\alpha t}\diamondsuit^k\delta)*u(t)=e^{\alpha t}C_0\delta=C_0 e^{\alpha t}\delta=C_0\delta$. By Theorem 4.1 we obtain

$$u(t)=C_0 e^{\alpha t}((-1)^k S_{2k}(t)*R_{2k}(t)).$$

Now, by (2.1) and (2.2), $S_{2k}(t)$ and $R_{2k}(t)$ are ordinary functions respectively for $2k\geqslant n$. It follows that $u(t)$ is an ordinary function for any constant α. If $2k<n$ then $S_{2k}(t)$ and $R_{2k}(t)$ are the analytic

functions except at the origin and by Lemma 2.1 $S_{2k}(t)$ and $R_{2k}(t)$ are tempered distributions and by Lemma 2.2, $(-1)^k S_{2k}(t) * R_{2k}(t)$ exists and is a tempered distribution.

Now, for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i < 0$ $(i = 1, 2, \ldots, n)$ we have $e^{\alpha t}$ is a slow growth function and also its partial derivative is a slow growth. It follows that $e^{\alpha t}[(-1)^k S_{2k}(t) * R_{2k}(t)]$ is also a tempered distribution.

(2) For $0 < m < k$, we have

$$(e^{\alpha t} \diamondsuit^k \delta) * u(t) = e^{\alpha t}[c_1 \diamondsuit \delta + c_2 \diamondsuit^2 \delta + \cdots + c_m \diamondsuit^m \delta].$$

Convolving both sides by $e^{\alpha t}[(-1)^k S_{2k}(t) * R_{2k}(t)]$, we obtain

$$\begin{aligned} u(t) &= e^{\alpha t}[c_1 \diamondsuit((-1)^k S_{2k}(t) * R_{2k}(t)) + c_2 \diamondsuit^2((-1)^k S_{2k}(t) * R_{2k}(t)) \\ &\quad + \cdots + c_m \diamondsuit^k((-1)^k S_{2k}(t) * R_{2k}(t))] \\ &= e^{\alpha t}[c_1(-1)^{k-1} S_{2k-2}(t) * R_{2k-2}(t) + c_2(-1)^{k-2} S_{2k-4}(t) * R_{2k-4}(t) \\ &\quad + \cdots + c_m(-1)^{k-m} S_{2k-2m}(t) * R_{2k-2m}(t)] \\ &= e^{\alpha t}\left[\sum_{r=1}^{m}(-1)^{k-r} S_{2k-2r}(t) * R_{2k-2r}(t)\right] \end{aligned}$$

by Theorem 4.1 and by Kananthai [2, Theorem 3.2] for $r < k$. Similarly, as in the case (1) $e^{\alpha t}$ $[\sum_{r=1}^{m}(-1)^{k-r} S_{2k-2r}(t) * R_{2k-2r}(t)]$ is the ordinary function if $2k - 2r \geqslant n$ and for any α, and is a tempered distribution if $2k - 2r < n$ and for some α with $\alpha_i < 0$ $(i = 1, 2, \ldots, n)$.

(3) For $m \geqslant k$ and for any α, suppose that $k \leqslant m \leqslant M$ we have

$$(e^{\alpha t} \diamondsuit^k \delta) * u(t) = c_k e^{\alpha t} \diamondsuit^k \delta + c_{k+1} e^{\alpha t} \diamondsuit^{k+1} \delta + \cdots + c_M e^{\alpha t} \diamondsuit^k \delta.$$

Convoluing both sides by $e^{\alpha t}((-1)^k S_{2k}(t) * R_{2k}(t))$ and by Kananthai [2, Theorem 3.2] for $k \leqslant m \leqslant M$, we obtain

$$u(t) = c_k e^{\alpha t} \delta + c_{k+1} e^{\alpha t} \diamondsuit \delta + c_{k+2} e^{\alpha t} \diamondsuit^2 \delta + \cdots + c_M e^{\alpha t} \diamondsuit^{M-k} \delta$$

or $u(t) = e^{\alpha t} \sum_{r=k}^{M} c_r \diamondsuit^{r-k} \delta$. Since $e^{\alpha t} \diamondsuit^{r-k} \delta = L^{r-k}$ for $k \leqslant r \leqslant M$ and L is defined by (3.2). Thus L^{r-k} is a singular distribution. It follows that $u(t)$ is a singular distribution. That completes the proof. □

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