



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Extending structures II: The quantum version

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ARTICLE INFO

Article history:

Received 9 November 2010

Available online 29 April 2011

Communicated by Nicolás Andruskiewitsch

MSC:

16T10

16T05

16S40

Keywords:

Crossed product

Bicrossed product

The factorization problem

ABSTRACT

Let A be a Hopf algebra and H a coalgebra. We shall describe and classify up to an isomorphism all Hopf algebras E that factorize through A and H : that is E is a Hopf algebra such that A is a Hopf subalgebra of E , H is a subcoalgebra in E with $1_E \in H$ and the multiplication map $A \otimes H \rightarrow E$ is bijective. The tool we use is a new product, we call it the unified product, in the construction of which A and H are connected by three coalgebra maps: two actions and a generalized cocycle. Both the crossed product of an Hopf algebra acting on an algebra and the bicrossed product of two Hopf algebras are special cases of the unified product. A Hopf algebra E factorizes through A and H if and only if E is isomorphic to a unified product of A and H . All such Hopf algebras E are classified up to an isomorphism that stabilizes A and H by a Schreier type classification theorem. A coalgebra version of lazy 1-cocycles as defined by Bichon and Kassel plays the key role in the classification theorem.

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Introduction

Let \mathcal{C} be a category whose objects are sets endowed with various algebraic structures (S) and \mathcal{D} be a category such that there exists a forgetful functor $F : \mathcal{C} \rightarrow \mathcal{D}$, i.e. a functor that forgets some of the structures (S). To illustrate, the following are forgetful functors:

$$F : Gr \rightarrow Set, \quad F : Lie \rightarrow Vec, \quad F : Hopf \rightarrow CoAlg, \quad F : Hopf \rightarrow Alg$$

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where $\mathcal{G}r$, $\mathcal{S}et$, $\mathcal{L}ie$, $\mathcal{V}ec$, $\mathcal{H}opf$, $\mathcal{C}oAlg$, $\mathcal{A}lg$ are the categories of all groups, sets, Lie algebras, vector spaces, Hopf algebras, coalgebras and respectively algebras. In this context we formulate a general problem which may be of interest for many areas of mathematics:

Extending structures problem (ES). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a forgetful functor and consider two objects $C \in \mathcal{C}$, $D \in \mathcal{D}$ such that $F(C)$ is a subobject of D in \mathcal{D} . Describe and classify all mathematical structures (S) that can be defined on D such that D becomes an object of \mathcal{C} and C is a subobject of D in the category \mathcal{C} (the classification is up to an isomorphism that stabilizes C and a certain type of fixed quotient D/C).

The classification part of the ES-problem is a challenge for the introduction of new types of cohomology. The ES-problem generalizes and unifies two famous and still open problems in the theory of groups: the *extension problem* of Hölder [9] and the *factorization problem* of Ore [14]. Let us explain this. In [1] we formulated the ES-problem at the level of groups, corresponding to the forgetful functor $F : \mathcal{G}r \rightarrow \mathcal{S}et$: if A is a group and E a set such that $A \subseteq E$ [1, Corollary 2.10], describe all group structures (E, \cdot) that can be defined on the set E such that A is a subgroup of (E, \cdot) . In order to do that we have introduced a new product for groups, called the unified product [1, Theorem 2.6], such that both the crossed product (the tool for the extension problem) and the bicrossed product (the tool for the factorization problem) of two groups are special cases of it. The unified product for groups is associated to a group A and a new hidden algebraic structure $(H, *)$, connected by two actions and a generalized cocycle satisfying some compatibility conditions.

We now take a step forward and formulate the ES-problem at the level of Hopf algebras corresponding to the forgetful functor $F : \mathcal{H}opf \rightarrow \mathcal{C}oAlg$:

(H-C) Extending structures problem. Let A be a Hopf algebra and E a coalgebra such that A is a subcoalgebra of E . Describe and classify all Hopf algebra structures that can be defined on E such that A is a Hopf subalgebra of E .

There is of course a dual version of the ES-problem corresponding to the forgetful functor $F : \mathcal{H}opf \rightarrow \mathcal{A}lg$ to be addressed somewhere else. If at the level of groups the ES-problem is elementary, for Hopf algebras the problem is more difficult. Indeed, let A be a group and E a set such that $A \subseteq E$. For a field k we look at the extension $k[A] \subseteq k[E]$, where $k[A]$ is the group algebra that is a Hopf algebra and a subcoalgebra in the group-like coalgebra $k[E]$. Assume now that (E, \cdot) is a group structure on the set E such that A is a subgroup of (E, \cdot) . Thus, we obtain an extension of Hopf algebras $k[A] \subseteq k[E]$. This extension of Hopf algebras has a remarkable property: let $H \subseteq E$ be a system of representatives for the right cosets of the subgroup A in the group (E, \cdot) such that $1_E \in H$. Since the map $u : A \times H \rightarrow E$, $u(a, h) = a \cdot h$ is bijective, we obtain that the multiplication map

$$k[A] \otimes k[H] \rightarrow k[E], \quad a \otimes h \mapsto a \cdot h$$

is bijective, i.e. the Hopf algebra $k[E]$ factorizes through the Hopf subalgebra $k[A]$ and the subcoalgebra $k[H]$. This is not valid for arbitrary extensions of Hopf algebras. Therefore, we have to restrict the (H-C) extending structures problem to those Hopf algebras E that factorize through a given Hopf subalgebra A and a given subcoalgebra H : we called this the **restricted (H-C) ES-problem** and we shall give a complete answer to it in the present paper. It turns out that H is not only a subcoalgebra of E but will be endowed additionally with a hidden algebraic structure that will play the role of the system of representatives for congruence in the theory of groups.

The paper is organized as follows: In the first section we recall the classical constructions of the crossed product of a Hopf algebra H acting on an algebra A and of the bicrossed product (double cross product in Majid's terminology) of two Hopf algebras H and A , as the product that we define will generalize both of them. In Section 2 we define the concept of an extending structure of a bialgebra A consisting of a system $\Omega(A) = (H, \triangleleft, \triangleright, f)$, where H is a coalgebra and an unitary not necessarily associative algebra such that A and H are connected by three coalgebra maps $\triangleleft : H \otimes A \rightarrow H$, $\triangleright : H \otimes A \rightarrow A$, $f : H \otimes H \rightarrow A$ satisfying some natural normalization conditions (Definition 2.1). For a bialgebra extending structure $\Omega(A) = (H, \triangleleft, \triangleright, f)$ of A we define a product $A \rtimes_{\Omega(A)} H = A \rtimes H$ and call it the unified product: both the crossed product of an Hopf algebra acting on an algebra and the bicrossed product (double cross product in Majid's terminology) of two Hopf algebras are special

cases of the unified product. Theorem 2.4 gives necessary and sufficient conditions for $A \ltimes H$ to be a bialgebra, which is precisely the Hopf algebra version of [1, Theorem 2.6] proven for the group case that served as a model for us. The seven compatibility conditions in Theorem 2.4 are very natural and, mutatis-mutandis, are the ones (with two reasonable deformations via the right action \triangleleft) that appear in the construction of the crossed product and the bicrossed product of two Hopf algebras. Theorem 2.7 proves that a Hopf algebra E factorizes through a Hopf subalgebra A and a subcoalgebra H if and only if E is isomorphic to a unified product of A and H and gives the answer for the first part of the restricted (H-C) ES-problem.

Section 3 is devoted to the classification part of the restricted (H-C) ES-problem. Our view point descends from the classical classification theorem of Schreier at the level of groups: all extensions of an abelian group K by a group Q are classified by the second cohomology group $H^2(Q, K)$ [15, Theorem 7.34]. Let A be a Hopf algebra. Two Hopf algebra extending structures $\Omega(A) = (H, \triangleleft, \triangleright, f)$ and $\Omega'(A) = (H, \triangleleft', \triangleright', f')$ are called equivalent if there exists $\varphi : A \ltimes' H \rightarrow A \ltimes H$ a left A -module, a right H -comodule and a Hopf algebra map. As in group extension theory we shall prove that any such morphism $\varphi : A \ltimes' H \rightarrow A \ltimes H$ is an isomorphism and the following diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{i_A} & A \ltimes H & \xrightarrow{\pi_H} & H \\
 \text{Id}_A \downarrow & & \downarrow \varphi & & \downarrow \text{Id}_H \\
 A & \xrightarrow{i_A} & A \ltimes' H & \xrightarrow{\pi_H} & H
 \end{array}$$

is commutative. Theorem 3.4 shows that any such morphism $\varphi : A \ltimes' H \rightarrow A \ltimes H$ is uniquely determined by a coalgebra lazy 1-cocycle: i.e. a unitary coalgebra map $u : H \rightarrow A$ such that:

$$h_{(1)} \otimes u(h_{(2)}) = h_{(2)} \otimes u(h_{(1)})$$

for all $h \in H$. Corollary 3.6 is the Schreier type classification theorem for unified products: the part of the second cohomology group from the theory of groups will be played now by a special quotient set $H^2_{l,c}(H, A, \triangleleft)$. Also, a classification result for bicrossed product of two Hopf algebras is derived from Theorem 3.4.

1. Preliminaries

Throughout this paper, k will be a field. Unless specified otherwise, all algebras, coalgebras, bialgebras, tensor products and homomorphisms are over k . For a coalgebra C , we use Sweedler’s Σ -notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc. (summation understood). Let A be a bialgebra and H a coalgebra. H is called a *right A -module coalgebra* if there exists $\triangleleft : H \otimes A \rightarrow H$ a morphism of coalgebras such that (H, \triangleleft) is a right A -module. For a k -linear map $f : H \otimes H \rightarrow A$ we denote $f(g, h) = f(g \otimes h)$; f is the *trivial map* if $f(g, h) = \varepsilon_H(g)\varepsilon_H(h)1_A$, for all $g, h \in H$. Similarly, the k -linear maps $\triangleleft : H \otimes A \rightarrow H$, $\triangleright : H \otimes A \rightarrow A$ are the *trivial actions* if $h \triangleleft a = \varepsilon_A(a)h$ and respectively $h \triangleright a = \varepsilon_H(h)a$, for all $a \in A$ and $h \in H$. For further computations, the fact that $\triangleleft : H \otimes A \rightarrow H$, $\triangleright : H \otimes A \rightarrow A$ and $f : H \otimes H \rightarrow A$ are coalgebra maps can be written explicitly as follows:

$$\Delta_H(h \triangleleft a) = h_{(1)} \triangleleft a_{(1)} \otimes h_{(2)} \triangleleft a_{(2)}, \quad \varepsilon_A(h \triangleleft a) = \varepsilon_H(h)\varepsilon_A(a) \tag{1}$$

$$\Delta_A(h \triangleright a) = h_{(1)} \triangleright a_{(1)} \otimes h_{(2)} \triangleright a_{(2)}, \quad \varepsilon_A(h \triangleright a) = \varepsilon_H(h)\varepsilon_A(a) \tag{2}$$

$$\Delta_A(f(g, h)) = f(g_{(1)}, h_{(1)}) \otimes f(g_{(2)}, h_{(2)}), \quad \varepsilon_A(f(g, h)) = \varepsilon_H(g)\varepsilon_H(h) \tag{3}$$

for all $g, h \in H, a \in A$.

Crossed product of Hopf algebras. The crossed product of a Hopf algebra H acting on a k -algebra A was introduced independently in [5] and [8] as a generalization of the crossed product of groups acting on k -algebras. Let H be a Hopf algebra, A a k -algebra and two k -linear maps $\triangleright : H \otimes A \rightarrow A$, $f : H \otimes H \rightarrow A$ such that

$$\begin{aligned}
 h \triangleright 1_A &= \varepsilon_H(h)1_A, & 1_H \triangleright a &= a \\
 h \triangleright (ab) &= (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), & f(h, 1_H) &= f(1_H, h) = \varepsilon_H(h)1_A
 \end{aligned}$$

for all $h \in H$, $a, b \in A$. The crossed product $A \#_f H$ of A with H is the k -module $A \otimes H$ with the multiplication given by

$$(a \# h)(c \# g) := a(h_{(1)} \triangleright c) f(h_{(2)}, g_{(1)}) \# h_{(3)} g_{(2)} \tag{4}$$

for all $a, c \in A$, $h, g \in H$, where we denoted $a \otimes h$ by $a \# h$. It can be proved [13, Lemma 7.1.2] that $A \#_f H$ is an associative algebra with identity element $1_A \# 1_H$ if and only if the following two compatibility conditions hold:

$$[g_{(1)} \triangleright (h_{(1)} \triangleright a)] f(g_{(2)}, h_{(2)}) = f(g_{(1)}, h_{(1)}) ((g_{(2)} h_{(2)}) \triangleright a) \tag{5}$$

$$(g_{(1)} \triangleright f(h_{(1)}, l_{(1)})) f(g_{(2)}, h_{(2)} l_{(2)}) = f(g_{(1)}, h_{(1)}) f(g_{(2)} h_{(2)}, l) \tag{6}$$

for all $a \in A$, $g, h, l \in H$. The first compatibility is called the twisted module condition while (6) is called the cocycle condition. The crossed product $A \#_f H$ was studied only as an algebra extension of A , being an essential tool in Hopf-Galois extensions theory. If, in addition, we suppose that A is also a Hopf algebra, a natural question arises: *when does the crossed product $A \#_f H$ have a Hopf algebra structure with the coalgebra structure given by the tensor product of coalgebras?* In case that \triangleright and f are coalgebra maps, as a consequence of Theorem 2.4, we will show in Example 2.5 that $A \#_f H$ is a Hopf algebra if and only if the following two compatibility conditions hold:

$$\begin{aligned}
 g_{(1)} \otimes g_{(2)} \triangleright a &= g_{(2)} \otimes g_{(1)} \triangleright a \\
 g_{(1)} h_{(1)} \otimes f(g_{(2)}, h_{(2)}) &= g_{(2)} h_{(2)} \otimes f(g_{(1)}, h_{(1)})
 \end{aligned}$$

for all $g, h \in H$ and $a, b \in A$.

Bicrossed product of Hopf algebras. The bicrossed product of Hopf algebras was introduced by Majid in [11, Proposition 3.12] under the name of double cross product. We shall adopt the name of bicrossed product from [10, Theorem 2.3]. A *matched pair* of bialgebras is a system $(A, H, \triangleleft, \triangleright)$, where A and H are bialgebras, $\triangleleft : H \otimes A \rightarrow H$, $\triangleright : H \otimes A \rightarrow A$ are coalgebra maps such that (A, \triangleright) is a left H -module coalgebra, (H, \triangleleft) is a right A -module coalgebra and the following compatibility conditions hold:

$$1_H \triangleleft a = \varepsilon_A(a)1_H, h \triangleright 1_A = \varepsilon_H(h)1_A \tag{7}$$

$$g \triangleright (ab) = (g_{(1)} \triangleright a_{(1)}) ((g_{(2)} \triangleleft a_{(2)}) \triangleright b) \tag{8}$$

$$(gh) \triangleleft a = (g \triangleleft (h_{(1)} \triangleright a_{(1)})) (h_{(2)} \triangleleft a_{(2)}) \tag{9}$$

$$g_{(1)} \triangleleft a_{(1)} \otimes g_{(2)} \triangleright a_{(2)} = g_{(2)} \triangleleft a_{(2)} \otimes g_{(1)} \triangleright a_{(1)} \tag{10}$$

for all $a, b \in A, g, h \in H$. Let $(A, H, \triangleleft, \triangleright)$ be a matched pair of bialgebras; the *bicrossed product* $A \bowtie H$ of A with H is the k -module $A \otimes H$ with the multiplication given by

$$(a \bowtie h) \cdot (c \bowtie g) := a(h_{(1)} \triangleright c_{(1)}) \bowtie (h_{(2)} \triangleleft c_{(2)})g \tag{11}$$

for all $a, c \in A, h, g \in H$, where we denoted $a \otimes h$ by $a \bowtie h$. $A \bowtie H$ is a bialgebra with the coalgebra structure given by the tensor product of coalgebras and moreover, if A and H are Hopf algebras, then $A \bowtie H$ has an antipode given by the formula:

$$S(a \bowtie h) := (1_A \bowtie S_H(h)) \cdot (S_A(a) \bowtie 1_H) \tag{12}$$

for all $a \in A$ and $h \in H$ [12, Theorem 7.2.2].

2. Bialgebra extending structures and unified products

In this section we shall introduce the unified product for bialgebras; this will be the tool for answering the restricted (H-C) ES-problem. First we need the following:

Definition 2.1. Let A be a bialgebra. An *extending datum* of A is a system $\Omega(A) = (H, \triangleleft, \triangleright, f)$ where:

- (i) $H = (H, \Delta_H, \varepsilon_H, 1_H, \cdot)$ is a k -module such that $(H, \Delta_H, \varepsilon_H)$ is a coalgebra, $(H, 1_H, \cdot)$ is an unitary, not necessarily associative k -algebra, such that

$$\Delta_H(1_H) = 1_H \otimes 1_H \tag{13}$$

- (ii) The k -linear maps $\triangleleft : H \otimes A \rightarrow H, \triangleright : H \otimes A \rightarrow A, f : H \otimes H \rightarrow A$ are morphisms of coalgebras such that the following normalization conditions hold:

$$h \triangleright 1_A = \varepsilon_H(h)1_A, \quad 1_H \triangleright a = a, \quad 1_H \triangleleft a = \varepsilon_A(a)1_H, \quad h \triangleleft 1_A = h \tag{14}$$

$$f(h, 1_H) = f(1_H, h) = \varepsilon_H(h)1_A \tag{15}$$

for all $h \in H, a \in A$.

Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ an extending datum of A . We denote by $A \times_{\Omega(A)} H = A \times H$ the k -module $A \otimes H$ together with the multiplication:

$$(a \times h) \bullet (c \times g) := a(h_{(1)} \triangleright c_{(1)})f(h_{(2)} \triangleleft c_{(2)}, g_{(1)}) \times (h_{(3)} \triangleleft c_{(3)}) \cdot g_{(2)} \tag{16}$$

for all $a, c \in A$ and $h, g \in H$, where we denoted $a \otimes h \in A \otimes H$ by $a \times h$.

Definition 2.2. Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ be an extending datum of A . The object $A \times H$ introduced above is called the *unified product of A and $\Omega(A)$* if $A \times H$ is a bialgebra with the multiplication given by (16), the unit $1_A \times 1_H$ and the coalgebra structure given by the tensor product of coalgebras, i.e.:

$$\Delta_{A \times H}(a \times h) = a_{(1)} \times h_{(1)} \otimes a_{(2)} \times h_{(2)} \tag{17}$$

$$\varepsilon_{A \times H}(a \times h) = \varepsilon_A(a)\varepsilon_H(h) \tag{18}$$

for all $h \in H, a \in A$. In this case the extending datum $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is called a *bialgebra extending structure* of A . The maps \triangleright and \triangleleft are called the *actions* of $\Omega(A)$ and f is called the $(\triangleright, \triangleleft)$ -cocycle of $\Omega(A)$. A bialgebra extending structure $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is called a *Hopf algebra extending structure* of A if $A \times H$ has an antipode.

The multiplication given by (16) has a rather complicated formula; however, for some specific elements we obtain easier forms which will be useful for future computations.

Lemma 2.3. *Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ an extending datum of A . The following cross-relations hold:*

$$(a \times 1_H) \bullet (c \times g) = ac \times g \tag{19}$$

$$(a \times g) \bullet (1_A \times h) = af(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)} \tag{20}$$

$$(a \times g) \bullet (b \times 1_H) = a(g_{(1)} \triangleright b_{(1)}) \times g_{(2)} \triangleleft b_{(2)} \tag{21}$$

Proof. Straightforward using the normalization conditions (13)–(15). \square

It follows from (19) that the map $i_A : A \rightarrow A \times H$, $i_A(a) := a \times 1_H$, for all $a \in A$, is a k -algebra map and

$$(a \times 1_H) \bullet (1_A \times g) = a \times g \tag{22}$$

for all $a \in A$ and $g \in H$. Hence the set $T := \{a \times 1_H \mid a \in A\} \cup \{1_A \times g \mid g \in H\}$ is a system of generators as an algebra for $A \times H$ and this observation will turn out to be essential in proving the next theorem which provides necessary and sufficient conditions for $A \times H$ to be a bialgebra: it is the Hopf algebra version of [1, Theorem 2.6] where the unified product for groups is constructed.

Theorem 2.4. *Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ an extending datum of A . The following statements are equivalent:*

- (1) $A \times H$ is an unified product;
 - (2) The following compatibilities hold:
 - (2a) $\Delta_H : H \rightarrow H \otimes H$ and $\varepsilon_H : H \rightarrow k$ are k -algebra maps;
 - (2b) (H, \triangleleft) is a right A -module structure;
 - (2c) $(g \cdot h) \cdot l = (g \triangleleft f(h_{(1)}, l_{(1)})) \cdot (h_{(2)} \cdot l_{(2)})$;
 - (2d) $g \triangleright (ab) = (g_{(1)} \triangleright a_{(1)})[(g_{(2)} \triangleleft a_{(2)}) \triangleright b]$;
 - (2e) $(g \cdot h) \triangleleft a = [g \triangleleft (h_{(1)} \triangleright a_{(1)})] \cdot (h_{(2)} \triangleleft a_{(2)})$;
 - (2f) $[g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)})]f(g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)}), h_{(3)} \triangleleft a_{(3)}) = f(g_{(1)}, h_{(1)})[(g_{(2)} \cdot h_{(2)}) \triangleright a]$;
 - (2g) $(g_{(1)} \triangleright f(h_{(1)}, l_{(1)}))f(g_{(2)} \triangleleft f(h_{(2)}, l_{(2)}), h_{(3)} \cdot l_{(3)}) = f(g_{(1)}, h_{(1)})f(g_{(2)} \cdot h_{(2)}, l)$;
 - (2h) $g_{(1)} \triangleleft a_{(1)} \otimes g_{(2)} \triangleright a_{(2)} = g_{(2)} \triangleleft a_{(2)} \otimes g_{(1)} \triangleright a_{(1)}$;
 - (2i) $g_{(1)} \cdot h_{(1)} \otimes f(g_{(2)}, h_{(2)}) = g_{(2)} \cdot h_{(2)} \otimes f(g_{(1)}, h_{(1)})$
- for all $g, h, l \in H$ and $a, b \in A$.

Before going into the proof of the theorem, we have a few observations on the relations (2a)–(2i) in Theorem 2.4. Although they look rather complicated at first sight, they are in fact quite natural and can be interpreted as follows: (2a) and (2b) show that $(H, \Delta_H, \varepsilon_H, 1_H, \cdot)$ is a non-associative bialgebra and a right A -module coalgebra via \triangleleft . (2c) measures how far $(H, 1_H, \cdot)$ is from being an associative algebra. (2d), (2e) and (2h) are exactly, mutatis–mutandis, the compatibility conditions (7)–(10) appearing in the definition of a matched pair of bialgebras. (2f) and (2g) are deformations via the action \triangleleft of the twisted module condition (5) and respectively of the cocycle condition (6) which appears in the definition of the crossed product for Hopf algebras. (2i) is a symmetry condition for the cocycle f similar to (2h). Both relations are trivially fulfilled if, for example, H is cocommutative or f is the trivial cocycle.

Proof. We prove Theorem 2.4 in several steps. From (19) and (21) it is straightforward that $1_A \times 1_H$ is a unit for the algebra $(A \times H, \bullet)$. Next, we prove that $\varepsilon_{A \times H}$ given by (18) is an algebra map if and only if $\varepsilon_H : H \rightarrow k$ is an algebra map. For $h, g \in H$ we have:

$$\begin{aligned} \varepsilon_{A \times H}((1_A \times h) \bullet (1_A \times g)) &\stackrel{(20)}{=} \varepsilon_{A \times H}(f(h_{(1)}, g_{(1)}) \times g_{(2)} \cdot h_{(2)}) \\ &\stackrel{(3)}{=} \varepsilon_H(h_{(1)})\varepsilon_H(g_{(1)})\varepsilon_H(g_{(2)} \cdot h_{(2)}) \\ &= \varepsilon_H(g \cdot h) \end{aligned}$$

and $\varepsilon(1_A \times h)\varepsilon(1_A \times g) = \varepsilon_H(h)\varepsilon_H(g)$. Thus, if $\varepsilon_{A \times H}$ is a k -algebra map then ε_H is a k -algebra map. Conversely, suppose that ε_H is a k -algebra map. Then, we have:

$$\begin{aligned} \varepsilon_{A \times H}((a \times h) \bullet (c \times g)) &= \varepsilon_A(a)\varepsilon_H(h_{(1)})\varepsilon_A(c_{(1)})\varepsilon_H(h_{(2)})\varepsilon_A(c_{(2)})\varepsilon_H(g_{(1)})\varepsilon_H(h_{(3)}) \\ &\quad \times \varepsilon_A(c_{(3)})\varepsilon_H(g_{(2)}) \\ &= \varepsilon_A(a)\varepsilon_H(h)\varepsilon_A(c)\varepsilon_H(g) \\ &= \varepsilon_{A \times H}(a \times h)\varepsilon_{A \times H}(c \times g) \end{aligned}$$

for all $a, c \in A$ and $h \in H$ i.e. $\varepsilon_{A \times H}$ is an algebra map.

The next step is to prove that $\Delta_{A \times H}$ is a k -algebra map if and only if $\Delta_H : H \rightarrow H \otimes H$ is a k -algebra map and the relations (2h), (2i) hold. Observe that $\Delta_{A \times H}(1_A \times 1_H) \stackrel{(13)}{=} 1_A \times 1_H \otimes 1_A \times 1_H$. Since $T = \{a \times 1_H \mid a \in A\} \cup \{1_A \times g \mid g \in H\}$ generates $A \times H$ as an algebra, $\Delta_{A \times H}$ is a k -algebra map if and only if $\Delta_{A \times H}(xy) = \Delta_{A \times H}(x)\Delta_{A \times H}(y)$ for all $x, y \in T$. First, observe that:

$$\begin{aligned} \Delta_{A \times H}((a \times 1_H) \bullet (b \times 1_H)) &= \Delta_{A \times H}(ab \times 1_H) \\ &= a_{(1)}b_{(1)} \times 1_H \otimes a_{(2)}b_{(2)} \times 1_H \\ &= (a_{(1)} \times 1_H \otimes a_{(2)} \times 1_H)(b_{(1)} \times 1_H \otimes b_{(2)} \times 1_H) \\ &= \Delta_{A \times H}(a \times 1_H)\Delta_{A \times H}(b \times 1_H) \end{aligned}$$

and

$$\begin{aligned} \Delta_{A \times H}((a \times 1_H) \bullet (1_A \times g)) &= \Delta_{A \times H}(a \times g) \\ &= a_{(1)} \times g_{(1)} \otimes a_{(2)} \times g_{(2)} \\ &= (a_{(1)} \times 1_H \otimes a_{(2)} \times 1_H)(1_A \times g_{(1)} \otimes 1_A \times g_{(2)}) \\ &= \Delta_{A \times H}(a \times 1_H)\Delta_{A \times H}(1_A \times g) \end{aligned}$$

for all $a, b \in A, g \in H$. There are two more relations to consider; for $g, h \in H$ we have:

$$\begin{aligned} \Delta_{A \times H}((1_A \times g) \bullet (1_A \times h)) &\stackrel{(20)}{=} \Delta_{A \times H}(f(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)}) \\ &\stackrel{(3)}{=} f(g_{(1)(1)}, h_{(1)(1)}) \times (g_{(2)} \cdot h_{(2)})_{(1)} \\ &\quad \otimes f(g_{(1)(2)}, h_{(1)(2)}) \times (g_{(2)} \cdot h_{(2)})_{(2)} \\ &= f(g_{(1)}, h_{(1)}) \times (g_{(3)} \cdot h_{(3)})_{(1)} \\ &\quad \otimes f(g_{(2)}, h_{(2)}) \times (g_{(3)} \cdot h_{(3)})_{(2)} \end{aligned}$$

and

$$\begin{aligned} \Delta_{A \times H}(1_A \times g)\Delta_{A \times H}(1_A \times h) &= (1_A \times g_{(1)} \otimes 1_A \times g_{(2)})(1_A \times h_{(1)} \otimes 1_A \times h_{(2)}) \\ &\stackrel{(20)}{=} f(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)} \otimes f(g_{(3)}, h_{(3)}) \times g_{(4)} \cdot h_{(4)} \end{aligned}$$

Thus $\Delta_{A \times H}((1_A \times g) \bullet (1_A \times h)) = \Delta_{A \times H}(1_A \times g)\Delta_{A \times H}(1_A \times h)$ if and only if

$$\begin{aligned} f(g_{(1)}, h_{(1)}) \times (g_{(3)} \cdot h_{(3)})_{(1)} \otimes f(g_{(2)}, h_{(2)}) \times (g_{(3)} \cdot h_{(3)})_{(2)} \\ = f(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)} \otimes f(g_{(3)}, h_{(3)}) \times g_{(4)} \cdot h_{(4)} \end{aligned}$$

We show now that this relation holds if and only if $\Delta_H : H \rightarrow H \otimes H$ is a k -algebra map and (2i) holds. Indeed, suppose first that the above relation holds. By applying $\varepsilon_A \otimes Id \otimes \varepsilon_A \otimes Id$ to it we obtain $\Delta_H(g \cdot h) = g_{(1)} \cdot h_{(1)} \otimes g_{(2)} \cdot h_{(2)}$, i.e. Δ_H is a k -algebra map. Furthermore, if we apply $\varepsilon_A \otimes Id \otimes Id \otimes \varepsilon_H$ to it we obtain $g_{(1)} \cdot h_{(1)} \otimes f(g_{(2)}, h_{(2)}) = g_{(2)} \cdot h_{(2)} \otimes f(g_{(1)}, h_{(1)})$, i.e. (2i). Conversely, suppose that Δ_H is a k -algebra map and (2i) holds. We then have:

$$\begin{aligned} f(g_{(1)}, h_{(1)}) \times (g_{(3)} \cdot h_{(3)})_{(1)} \otimes f(g_{(2)}, h_{(2)}) \times (g_{(3)} \cdot h_{(3)})_{(2)} \\ = f(g_{(1)}, h_{(1)}) \times g_{(3)} \cdot h_{(3)} \otimes f(g_{(2)}, h_{(2)}) \times g_{(4)} \cdot h_{(4)} \\ = f(g_{(1)}, h_{(1)}) \times \underline{g_{(2)(2)} \cdot h_{(2)(2)}} \otimes \underline{f(g_{(2)(1)}, h_{(2)(1)})} \times g_{(3)} \cdot h_{(3)} \\ \stackrel{(2i)}{=} f(g_{(1)}, h_{(1)}) \times g_{(2)(1)} \cdot h_{(2)(1)} \otimes f(g_{(2)(2)}, h_{(2)(2)}) \times g_{(3)} \cdot h_{(3)} \\ = f(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)} \otimes f(g_{(3)}, h_{(3)}) \times g_{(4)} \cdot h_{(4)} \end{aligned}$$

as needed. To end with, for the last family of generators we have:

$$\begin{aligned} \Delta_{A \times H}((1_A \times g) \bullet (a \times 1_H)) &\stackrel{(21)}{=} \Delta_{A \times H}(g_{(1)} \triangleright a_{(1)} \times g_{(2)} \triangleleft a_{(2)}) \\ &\stackrel{(1), (2)}{=} g_{(1)} \triangleright a_{(1)} \times g_{(3)} \triangleleft a_{(3)} \otimes g_{(2)} \triangleright a_{(2)} \times g_{(4)} \triangleleft a_{(4)} \end{aligned}$$

and

$$\begin{aligned} \Delta_{A \times H}(1_A \times g)\Delta_{A \times H}(a \times 1_H) &= (1_A \times g_{(1)} \otimes 1_A \times g_{(2)})(a_{(1)} \times 1_H \otimes a_{(2)} \times 1_H) \\ &\stackrel{(21)}{=} g_{(1)} \triangleright a_{(1)} \times g_{(2)} \triangleleft a_{(2)} \otimes g_{(3)} \triangleright a_{(3)} \times g_{(4)} \triangleleft a_{(4)} \end{aligned}$$

Thus $\Delta_{A \times H}((1_A \times g) \bullet (a \times 1_H)) = \Delta_{A \times H}(1_A \times g)\Delta_{A \times H}(a \times 1_H)$ if and only if

$$\begin{aligned} g_{(1)} \triangleright a_{(1)} \times g_{(3)} \triangleleft a_{(3)} \otimes g_{(2)} \triangleright a_{(2)} \times g_{(4)} \triangleleft a_{(4)} \\ = g_{(1)} \triangleright a_{(1)} \times g_{(2)} \triangleleft a_{(2)} \otimes g_{(3)} \triangleright a_{(3)} \times g_{(4)} \triangleleft a_{(4)} \end{aligned}$$

This relation is equivalent to the compatibility condition (2h): indeed, by applying $\varepsilon_A \otimes Id \otimes Id \otimes \varepsilon_H$ to it we obtain (2h). Conversely suppose that (2h) holds. Then:

$$\begin{aligned} g_{(1)} \triangleright a_{(1)} \times \underline{g_{(3)} \triangleleft a_{(3)} \otimes g_{(2)} \triangleright a_{(2)}} \times g_{(4)} \triangleleft a_{(4)} \\ \stackrel{(2h)}{=} g_{(1)} \triangleright a_{(1)} \times g_{(2)} \triangleleft a_{(2)} \otimes g_{(3)} \triangleright a_{(3)} \times g_{(4)} \triangleleft a_{(4)} \end{aligned}$$

as needed.

To resume, we proved until now that $\Delta_{A \times H}$ and $\varepsilon_{A \times H}$ are k -algebra maps if and only if the relations (2a), (2h), (2i) hold. In what follows we shall prove, in the hypothesis that $\Delta_{A \times H}$ and $\varepsilon_{A \times H}$ are k -algebra maps, that the multiplication given by (16) is associative if and only if the compatibility conditions (2b)–(2g) hold. This will end the proof. We make use again of the fact that T generates $A \times H$ as an algebra. Thus \bullet is associative if and only if $x \bullet (y \bullet z) = (x \bullet y) \bullet z$, for all $x, y, z \in T$. To start with, we will prove that:

$$(a \times 1_H) \bullet (y \bullet z) = [(a \times 1_H) \bullet y] \bullet z$$

for all $a \in A$ and $y, z \in T$. Indeed, we have:

$$\begin{aligned} (a \times 1_H) \bullet ((1_A \times g) \bullet (b \times 1_H)) &\stackrel{(21)}{=} (a \times 1_H) \bullet (g_{(1)} \triangleright b_{(1)} \times g_{(2)} \triangleleft b_{(2)}) \\ &\stackrel{(19)}{=} a(g_{(1)} \triangleright b_{(1)}) \times (g_{(2)} \triangleleft b_{(2)}) \\ &\stackrel{(21)}{=} (a \times g) \bullet (b \times 1_H) \\ &= ((a \times 1_H) \bullet (1_A \times g)) \bullet (b \times 1_H) \end{aligned}$$

and

$$\begin{aligned} (a \times 1_H) \bullet ((1_A \times g) \bullet (1_A \times h)) &\stackrel{(20)}{=} (a \times 1_H) \bullet (f(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)}) \\ &\stackrel{(19)}{=} af(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)} \\ &\stackrel{(20)}{=} (a \times g) \bullet (1_A \times h) \\ &\stackrel{(19)}{=} ((a \times 1_H) \bullet (1_A \times g)) \bullet (1_A \times h) \end{aligned}$$

The other two possibilities for choosing the elements of T can also be proven by a straightforward computation. Thus \bullet is associative if and only if $(1_A \times g) \bullet (y \bullet z) = [(1_A \times g) \bullet y] \bullet z$, for all $g \in H, y, z \in T$. First we note that:

$$\begin{aligned} (1_A \times g) \bullet ((a \times 1_H) \bullet (1_A \times h)) &= (1_A \times g) \bullet (a \times h) \\ &= (g_{(1)} \triangleright a_{(1)})f(g_{(2)} \triangleleft a_{(2)}, h_{(1)}) \times (g_{(3)} \triangleleft a_{(3)}) \cdot h_{(2)} \\ &\stackrel{(20)}{=} (g_{(1)} \triangleright a_{(1)} \times g_{(2)} \triangleleft a_{(2)}) \bullet (1_A \times h) \\ &\stackrel{(21)}{=} ((1_A \times g) \bullet (a \times 1_H)) \bullet (1_A \times h) \end{aligned}$$

On the other hand:

$$\begin{aligned} (1_A \times g) \bullet ((b \times 1_H) \bullet (c \times 1_H)) &= (1_A \times g) \bullet (bc \times 1_H) \\ &\stackrel{(21)}{=} g_{(1)} \triangleright (b_{(1)}c_{(1)}) \times g_{(2)} \triangleleft (b_{(2)}c_{(2)}) \end{aligned}$$

and

$$\begin{aligned} ((1_A \times g) \bullet (b \times 1_H)) \bullet (c \times 1_H) &\stackrel{(21)}{=} (g_{(1)} \triangleright b_{(1)} \times g_{(2)} \triangleleft b_{(2)}) \bullet (c \times 1_H) \\ &\stackrel{(21)}{=} (g_{(1)} \triangleright b_{(1)})(g_{(2)} \triangleleft b_{(2)}) \triangleright c_{(1)} \times (g_{(3)} \triangleleft b_{(3)}) \triangleleft c_{(2)} \end{aligned}$$

Hence $(1_A \times g) \bullet ((b \times 1_H) \bullet (c \times 1_H)) = ((1_A \times g) \bullet (b \times 1_H)) \bullet (c \times 1_H)$ if and only if

$$g_{(1)} \triangleright (b_{(1)}c_{(1)}) \times g_{(2)} \triangleleft (b_{(2)}c_{(2)}) = (g_{(1)} \triangleright b_{(1)})((g_{(2)} \triangleleft b_{(2)}) \triangleright c_{(1)}) \times (g_{(3)} \triangleleft b_{(3)}) \triangleleft c_{(2)} \quad (23)$$

for all $b, c \in A$ and $g \in H$. We show now that the relation (23) is equivalent to the compatibility conditions (2b) and (2d). Indeed, by applying $\varepsilon_A \otimes Id$ and respectively $Id \otimes \varepsilon_H$ in (23) we obtain relations (2b) respectively (2d). Conversely, suppose that relations (2b) and (2d) hold. We then have:

$$\begin{aligned} &(g_{(1)} \triangleright b_{(1)})((g_{(2)} \triangleleft b_{(2)}) \triangleright c_{(1)}) \times (g_{(3)} \triangleleft b_{(3)}) \triangleleft c_{(2)} \\ &= (g_{(1)(1)} \triangleright b_{(1)(1)})((g_{(1)(2)} \triangleleft b_{(1)(2)}) \triangleright c_{(1)}) \times (g_{(2)} \triangleleft b_{(2)}) \triangleleft c_{(2)} \\ &\stackrel{(2d)}{=} g_{(1)} \triangleright (b_{(1)}c_{(1)}) \otimes (g_{(2)} \triangleleft b_{(2)}) \triangleleft c_{(2)} \\ &\stackrel{(2b)}{=} g_{(1)} \triangleright (b_{(1)}c_{(1)}) \otimes g_{(2)} \triangleleft (b_{(2)}c_{(2)}) \end{aligned}$$

i.e. (23) holds. Now we deal with the last two cases. Since \triangleright is a coalgebra map we obtain:

$$\begin{aligned} (1_A \times g) \bullet ((1_A \times h) \bullet (a \times 1_H)) &\stackrel{(21)}{=} (1_A \times g) \bullet (h_{(1)} \triangleright a_{(1)} \times h_{(2)} \triangleleft a_{(2)}) \\ &\stackrel{(2)}{=} (g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)}))f(g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)}), h_{(4)} \triangleleft a_{(4)}) \\ &\quad \times [g_{(3)} \triangleleft (h_{(3)} \triangleright a_{(3)})] \cdot (h_{(5)} \triangleleft a_{(5)}) \end{aligned}$$

and

$$\begin{aligned} ((1_A \times g) \bullet (1_A \times h)) \bullet (a \times 1_H) &\stackrel{(20)}{=} (f(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)}) \bullet (a \times 1_H) \\ &\stackrel{(21)}{=} f(g_{(1)}, h_{(1)})[(g_{(2)} \cdot h_{(2)}) \triangleright a_{(1)}] \times (g_{(3)} \cdot h_{(3)}) \triangleleft a_{(2)} \end{aligned}$$

Thus $(1_A \times g) \bullet ((1_A \times h) \bullet (a \times 1_H)) = ((1_A \times g) \bullet (1_A \times h)) \bullet (a \times 1_H)$ if and only if

$$\begin{aligned} &(g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)}))f(g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)}), h_{(4)} \triangleleft a_{(4)}) \times [g_{(3)} \triangleleft (h_{(3)} \triangleright a_{(3)})] \cdot (h_{(5)} \triangleleft a_{(5)}) \\ &= f(g_{(1)}, h_{(1)})[(g_{(2)} \cdot h_{(2)}) \triangleright a_{(1)}] \times (g_{(3)} \cdot h_{(3)}) \triangleleft a_{(2)} \end{aligned}$$

We shall prove, using (2h), that this relation is equivalent to the compatibility conditions (2e) and (2f). Indeed, by applying $Id \otimes \varepsilon_H$ and respectively $\varepsilon_A \otimes Id$ to it we obtain (2e) and respectively (2f). Conversely, suppose that relations (2e) respectively (2f) hold. We denote LHS the left-hand side of the above

$$\begin{aligned} \text{LHS} &= (g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)}))f(g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)}), h_{(3)(2)} \triangleleft a_{(3)(2)}) \\ &\quad \times [g_{(3)} \triangleleft (h_{(3)(1)} \triangleright a_{(3)(1)})] \cdot (h_{(4)} \triangleleft a_{(4)}) \\ &\stackrel{(2h)}{=} (g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)}))f(g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)}), h_{(3)} \triangleleft a_{(3)}) \\ &\quad \times [g_{(3)} \triangleleft (h_{(4)} \triangleright a_{(4)})] \cdot (h_{(5)} \triangleleft a_{(5)}) \\ &\stackrel{(2f)}{=} f(g_{(1)}, h_{(1)})[(g_{(2)} \cdot h_{(2)}) \triangleright a_{(1)}] \otimes [g_{(3)} \triangleleft (h_{(3)} \triangleright a_{(2)})] \cdot (h_{(4)} \triangleleft a_{(3)}) \\ &\stackrel{(2e)}{=} f(g_{(1)}, h_{(1)})[(g_{(2)} \cdot h_{(2)}) \triangleright a_{(1)}] \otimes (g_{(3)} \cdot h_{(3)}) \triangleleft a_{(2)} \end{aligned}$$

as needed. Only one associativity relation remains to be verified:

$$\begin{aligned} (1_A \times g) \bullet ((1_A \times h) \bullet (1_A \times l)) &\stackrel{(20)}{=} (1_A \times g) \bullet (f(h_{(1)}, l_{(1)}) \times h_{(2)} \cdot l_{(2)}) \\ &\stackrel{(3)}{=} (g_{(1)} \triangleright f(h_{(1)}, l_{(1)})) f(g_{(2)} \triangleleft f(h_{(2)}, l_{(2)}), h_{(4)} \cdot l_{(4)}) \\ &\quad \times (g_{(3)} \triangleleft f(h_{(3)}, l_{(3)})) \cdot (h_{(5)} \cdot l_{(5)}) \end{aligned}$$

and

$$\begin{aligned} ((1_A \times g) \bullet (1_A \times h)) \bullet (1_A \times l) &\stackrel{(20)}{=} (f(g_{(1)}, h_{(1)}) \times g_{(2)} \cdot h_{(2)}) \bullet (1_A \times l) \\ &\stackrel{(20)}{=} f(g_{(1)}, h_{(1)}) f(g_{(2)} \cdot h_{(2)}, l_{(1)}) \times (g_{(3)} \cdot h_{(3)}) \cdot l_{(2)} \end{aligned}$$

Hence $(1_A \times g) \bullet ((1_A \times h) \bullet (1_A \times l)) = ((1_A \times g) \bullet (1_A \times h)) \bullet (1_A \times l)$ if and only if

$$\begin{aligned} (g_{(1)} \triangleright f(h_{(1)}, l_{(1)})) f(g_{(2)} \triangleleft f(h_{(2)}, l_{(2)}), h_{(4)} \cdot l_{(4)}) \otimes (g_{(3)} \triangleleft f(h_{(3)}, l_{(3)})) \cdot (h_{(5)} \cdot l_{(5)}) \\ = f(g_{(1)}, h_{(1)}) f(g_{(2)} \cdot h_{(2)}, l_{(1)}) \otimes (g_{(3)} \cdot h_{(3)}) \cdot l_{(2)} \end{aligned}$$

for all $g, h, l \in H$. We shall prove, using (2i), that this relation is equivalent to the compatibility conditions (2c) and (2g). Indeed, by applying $Id \otimes \varepsilon_H$ and respectively $\varepsilon_A \otimes Id$ to it we obtain (2c) and respectively (2g). Conversely, suppose that relations (2c) and (2g) hold and denote LHS' the left-hand side of the above relation. Then:

$$\begin{aligned} \text{LHS}' &= (g_{(1)} \triangleright f(h_{(1)}, l_{(1)})) f(g_{(2)} \triangleleft f(h_{(2)}, l_{(2)}), \underline{h_{(3)(2)} \cdot l_{(3)(2)}}) \\ &\quad \otimes (g_{(3)} \triangleleft \underline{f(h_{(3)(1)}, l_{(3)(1)})}) \cdot (h_{(4)} \cdot l_{(4)}) \\ &\stackrel{(2i)}{=} \underline{(g_{(1)} \triangleright f(h_{(1)}, l_{(1)})) f(g_{(2)} \triangleleft f(h_{(2)}, l_{(2)}), h_{(3)} \cdot l_{(3)})} \\ &\quad \otimes (g_{(3)} \triangleleft f(h_{(4)}, l_{(4)})) \cdot (h_{(5)} \cdot l_{(5)}) \\ &\stackrel{(2g)}{=} f(g_{(1)}, h_{(1)}) f(g_{(2)} \cdot h_{(2)}, l_{(1)}) \otimes (g_{(3)} \triangleleft f(h_{(3)}, l_{(2)})) \cdot (h_{(4)} \cdot l_{(3)}) \\ &= f(g_{(1)}, h_{(1)}) f(g_{(2)} \cdot h_{(2)}, l_{(1)}) \otimes \underline{(g_{(3)} \triangleleft f(h_{(3)(1)}, l_{(2)(1)})) \cdot (h_{(3)(2)} \cdot l_{(2)(2)})} \\ &\stackrel{(2c)}{=} f(g_{(1)}, h_{(1)}) f(g_{(2)} \cdot h_{(2)}, l_{(1)}) \otimes (g_{(3)} \cdot h_{(3)}) \cdot l_{(2)} \end{aligned}$$

as needed and the proof is now finished. \square

Examples 2.5. 1. Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ an extending datum of A such that the cocycle f is trivial, that is $f(g, h) = \varepsilon_H(g)\varepsilon_H(h)1_A$, for all $g, h \in H$.

Then $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is a bialgebra extending structure of A if and only if H is a bialgebra and $(A, H, \triangleleft, \triangleright)$ is a matched pair of bialgebras. In this case, the associated unified product $A \times H = A \bowtie H$ is the bicrossed product of bialgebras constructed in (11).

Conversely, a matched pair of bialgebras can be deformed using a coalgebra lazy cocycle in order to obtain a bialgebra extending structure as follows. Let $(A, H, \triangleleft, \triangleright)$ be a matched pair of bialgebras such that A has antipode S_A and $u : H \rightarrow A$ a coalgebra lazy 1-cocycle in the sense of Definition 3.3 such that $h \triangleleft u(g) = h\varepsilon_H(g)$, for all $h \in H$ and $g \in G$. Then $\Omega(A) = (H, \triangleleft, \triangleright', f')$ is a bialgebra extending structure of A , where \triangleright' and f' are given by

$$h \triangleright' c = u(h_{(1)})(h_{(2)} \triangleright c_{(1)})S_A(u(h_{(3)} \triangleleft c_{(2)}))$$

$$f'(h, g) = u(h_{(1)})(h_{(2)} \triangleright u(g_{(1)}))S_A(u(h_{(3)}g_{(2)}))$$

for all $h, g \in H$ and $c \in A$.

2. Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ an extending datum of A such that the action \triangleleft is trivial, that is $h \triangleleft a = \varepsilon_A(a)h$, for all $h \in H$ and $a \in A$.

Then $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is a bialgebra extending structure of A if and only if H is an usual bialgebra and the following compatibility conditions are fulfilled:

- (a) The twisted module condition (5) and the cocycle condition (6) hold;
- (b) $g \triangleright (ab) = (g_{(1)} \triangleright a)(g_{(2)} \triangleright b)$;
- (c) $g_{(1)} \otimes g_{(2)} \triangleright a = g_{(2)} \otimes g_{(1)} \triangleright a$;
- (d) $g_{(1)}h_{(1)} \otimes f(g_{(2)}, h_{(2)}) = g_{(2)}h_{(2)} \otimes f(g_{(1)}, h_{(1)})$

for all $g, h \in H$ and $a, b \in A$.

In this case, the associated unified product $A \ltimes H = A \#_f H$ is the crossed product constructed in (4). In particular, if A is a bialgebra, the crossed product $A \#_f H$ is a bialgebra with the coalgebra structure given by the tensor product of coalgebras if and only if the compatibility conditions (c) and (d) above hold.

Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ a bialgebra extending structure of A . Then $i_A : A \rightarrow A \ltimes H$, $i_A(a) = a \times 1_H$, for all $a \in A$ is an injective bialgebra map, $i_H : H \rightarrow A \ltimes H$, $i_H(h) = 1_A \otimes h$, for all $h \in H$ is an injective coalgebra map and

$$u : A \otimes H \rightarrow A \ltimes H, \quad u(a \otimes h) = i_A(a) \bullet i_H(h) = (a \times 1_H) \bullet (1_A \otimes h) = a \ltimes h$$

for all $a \in A$ and $h \in H$ is bijective, i.e. the unified product $A \ltimes H$ factorizes through A and H . The next theorem shows the converse of this remark: any bialgebra E that factorizes through a subbialgebra of A and a subcoalgebra H is isomorphic to a unified product. In order to avoid complicated computations we use the following elementary remark:

Lemma 2.6. *Let E be a bialgebra, L a coalgebra and $u : L \rightarrow E$ an isomorphism of coalgebras. Then there exists a unique algebra structure on L such that $u : L \rightarrow E$ is an isomorphism of bialgebras given by:*

$$l \cdot l' := u^{-1}(u(l)u(l')), \quad 1_L := u^{-1}(1_E)$$

for all $l, l' \in L$. Furthermore, if E has an antipode S_E , then L is a Hopf algebra with the antipode $S_L := u^{-1} \circ S_E \circ u$.

Proof. Straightforward: the algebra structure on L is obtained by transferring the algebra structure from E via the isomorphism of coalgebras u . The multiplication on L is a coalgebra map since it is a composition of coalgebra maps. \square

Theorem 2.7. *Let E be a bialgebra, $A \subseteq E$ a subbialgebra, $H \subseteq E$ a subcoalgebra such that $1_E \in H$ and the multiplication map $u : A \otimes H \rightarrow E$, $u(a \otimes h) = ah$, for all $a \in A, h \in H$ is bijective.*

Then, there exists $\Omega(A) = (H, \triangleleft, \triangleright, f)$ a bialgebra extending structure of A such that $u : A \ltimes H \rightarrow E$, $u(a \ltimes h) = ah$ is an isomorphism of bialgebras. Furthermore, if E is a Hopf algebra then $A \ltimes H$ is a Hopf algebra.

Proof. Since E is a bialgebra, the multiplication $m_E : E \otimes E \rightarrow E$ is a coalgebra map. Thus $u : A \otimes H \rightarrow E$ is in fact an isomorphism of coalgebras, with its inverse $u^{-1} : E \rightarrow A \otimes H$ which is also a coalgebra map. The k -linear map

$$\mu : H \otimes A \rightarrow A \otimes H, \quad \mu(h \otimes a) := u^{-1}(ha)$$

for all $h \in H$ and $a \in A$ is a coalgebra map as a composition of coalgebra maps. We define the actions $\triangleright, \triangleleft$ by the formulas:

$$\triangleright : H \otimes A \rightarrow A, \quad \triangleright := (Id \otimes \varepsilon_H) \circ \mu \tag{24}$$

$$\triangleleft : H \otimes A \rightarrow H, \quad \triangleleft := (\varepsilon_A \otimes Id) \circ \mu \tag{25}$$

They are coalgebra maps as compositions of coalgebra maps. Moreover, the normalization conditions (14) and (15) are trivially fulfilled. More explicitly, \triangleright and \triangleleft are given as follows: let $h \in H$ and $c \in A$. Since u is a bijective map, there exists a unique element $\sum_j \alpha_j \otimes l_j \in A \otimes H$ such that $hc = \sum_j \alpha_j l_j$. Then:

$$h \triangleright c = \sum_j \alpha_j \varepsilon_H(l_j), \quad h \triangleleft c = \sum_j \varepsilon_A(\alpha_j) l_j$$

Next we construct the coalgebra maps $f : H \otimes H \rightarrow A$ and $\cdot : H \otimes H \rightarrow H$. The k -linear map

$$\nu : H \otimes H \rightarrow A \otimes H, \quad \nu(h \otimes g) := u^{-1}(hg)$$

for all $h, g \in H$ is a coalgebra map as a composition of coalgebra maps. We define:

$$f : H \otimes H \rightarrow A, \quad f := (Id \otimes \varepsilon_H) \circ \nu \tag{26}$$

$$\cdot : H \otimes H \rightarrow H, \quad \cdot := (\varepsilon_A \otimes Id) \circ \nu \tag{27}$$

They are coalgebra maps as compositions of coalgebra maps. The normalization conditions $1_E \cdot h = h \cdot 1_E = h$ and $f(h, 1_E) = f(1_E, h) = \varepsilon_H(h)1_A$, for all $h \in H$ are trivially fulfilled.

In order to prove that $\Omega(A) = (H, \triangleleft, \triangleright, f, \cdot)$ is a bialgebra extending structure of A we use Lemma 2.6 and then Theorem 2.4: the unique algebra structure that can be defined on $A \otimes H$ such that u becomes an isomorphism of bialgebras is given by:

$$\begin{aligned} (a \otimes h) \bullet (c \otimes g) &= u^{-1}(u(a \otimes h)u(c \otimes g)) \\ &= u^{-1}(ahcg) \end{aligned}$$

This algebra structure on $A \otimes H$ coincides with the one given by (16) on a unified product if and only if

$$u^{-1}(ahcg) = a(h_{(1)} \triangleright c_{(1)})f(h_{(2)} \triangleleft c_{(2)}, g_{(1)}) \otimes (h_{(3)} \triangleleft c_{(3)}) \cdot g_{(2)}$$

Since u is a bijective map the above formula holds if and only if:

$$hcg = (h_{(1)} \triangleright c_{(1)})f(h_{(2)} \triangleleft c_{(2)}, g_{(1)})(h_{(3)} \triangleleft c_{(3)}) \cdot g_{(2)} \tag{28}$$

holds for all $c \in A$ and $h, g \in H$. Therefore, the proof is finished if we prove that the relation (28) holds in the bialgebra E . Let $c \in A$ and $h, g \in H$. Then there exists a unique element $\sum_{j=1}^n \alpha_j \otimes l_j \in A \otimes H$ such that:

$$hc = \sum_{j=1}^n \alpha_j l_j \tag{29}$$

Hence $h \triangleright c = \sum_{j=1}^n \varepsilon_H(l_j) \alpha_j$ and $h \triangleleft c = \sum_{j=1}^n \varepsilon_A(\alpha_j) l_j$. Moreover, for any $j = 1, \dots, n$ there exists a unique element $\sum_{i=1}^m A_{ji} \otimes Z_i \in A \otimes H$ such that:

$$l_j g = \sum_{i=1}^m A_{ji} Z_i \tag{30}$$

Using relations (26) and (27) we obtain:

$$f(l_j, g) = \sum_{i=1}^m \varepsilon_H(Z_i) A_{ji}, \quad l_j \cdot g = \sum_{i=1}^m \varepsilon_A(A_{ji}) Z_i \tag{31}$$

and

$$hcg = \sum_{i,j=1}^{m,n} \alpha_j A_{ji} Z_i \tag{32}$$

In what follows we use the fact that m_E, \triangleright and \triangleleft are coalgebra maps. For example, by applying Δ to the relation (29) we obtain:

$$\begin{aligned} h_{(1)} \triangleright c_{(1)} \otimes h_{(2)} \triangleleft c_{(2)} \otimes h_{(3)} \triangleleft c_{(3)} &= \sum_{j=1}^n \varepsilon_H(l_{j(1)}) \alpha_{j(1)} \otimes \varepsilon_A(\alpha_{j(2)}) l_{j(2)} \otimes \varepsilon_A(\alpha_{j(3)}) l_{j(3)} \\ &= \sum_{j=1}^n \varepsilon_H(l_{j(1)}) \alpha_j \otimes l_{j(2)} \otimes l_{j(3)} \\ &= \sum_{j=1}^n \alpha_j \otimes l_{j(1)} \otimes l_{j(2)} \end{aligned}$$

Thus, we have:

$$h_{(1)} \triangleright c_{(1)} \otimes h_{(2)} \triangleleft c_{(2)} \otimes h_{(3)} \triangleleft c_{(3)} = \sum_{j=1}^n \alpha_j \otimes l_{j(1)} \otimes l_{j(2)} \tag{33}$$

Moreover, by applying Δ to the relation (30) and using the relation (31) we obtain:

$$f(l_{j(1)}, g_{(1)}) \otimes l_{j(1)} \cdot g_{(2)} = \sum_{i=1}^m \varepsilon_H(Z_{i(1)}) A_{ji(1)} \otimes \varepsilon_A(A_{ji(2)}) Z_{i(2)} = \sum_{i=1}^m A_{ji} \otimes Z_i \tag{34}$$

We denote by RHS the right-hand side of (28). Then:

$$\begin{aligned} \text{RHS} &\stackrel{(33)}{=} \sum_{j=1}^n \alpha_j f(l_{j(1)}, g_{(1)}) l_{j(2)} \cdot g_{(2)} \\ &\stackrel{(34)}{=} \sum_{i,j=1}^{m,n} \alpha_j A_{ji} Z_i \\ &\stackrel{(32)}{=} hcg \end{aligned}$$

Thus the relation (28) holds true and the proof is now finished since $u^{-1}(1_E) = 1_A \otimes 1_E$. We use Theorem 2.4 in order to obtain that $\Omega(A) = (H, \triangleleft, \triangleright, f, \cdot)$ is a bialgebra extending structure of A . Moreover, if E is a Hopf algebra then $A \times H$ is also a Hopf algebra with the antipode given by $S_{A \times H} = u^{-1} \circ S_E \circ u$ according to Lemma 2.6. \square

Remark. A special case of Theorem 2.7 above in which A is a normal Hopf subalgebra of E is given in [3]. Moreover, in [6] is proved that if both A and H are Hopf subalgebras of E and A is normal then E is a semi-direct product.

Next we construct an antipode for the unified product $A \times H$.

Proposition 2.8. *Let A be a Hopf algebra with an antipode S_A and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ a bialgebra extending structure of A such that there exists an antimorphism of coalgebras $S_H : H \rightarrow H$ such that*

$$h_{(1)} \cdot S_H(h_{(2)}) = S_H(h_{(1)}) \cdot h_{(2)} = \varepsilon_H(h) 1_H \tag{35}$$

for all $h \in H$. Then the unified product $A \times H$ is a Hopf algebra with the antipode $S : A \times H \rightarrow A \times H$ given by:

$$S(a \times g) := (S_A[f(S_H(g_{(2)}), g_{(3)})] \times S_H(g_{(1)})) \bullet (S_A(a) \times 1_H) \tag{36}$$

for all $a \in A$ and $g \in H$.

Proof. Let $a \times g \in A \times H$. Since the multiplication \bullet on $A \times H$ is associative we have:

$$\begin{aligned} &S(a_{(1)} \times g_{(1)}) \bullet (a_{(2)} \times g_{(2)}) \\ &= (S_A[f(S_H(g_{(2)}), g_{(3)})] \times S_H(g_{(1)})) \bullet (S_A(a_{(1)}) \times 1_H) \bullet (a_{(2)} \times g_{(4)}) \\ &\stackrel{(19)}{=} (S_A[f(S_H(g_{(2)}), g_{(3)})] \times S_H(g_{(1)})) \bullet (S_A(a_{(1)}) a_{(2)} \times g_{(4)}) \\ &= \varepsilon_A(a) (S_A[f(S_H(g_{(2)}), g_{(3)})] \times S_H(g_{(1)})) \bullet (1_A \times g_{(4)}) \\ &\stackrel{(20)}{=} \varepsilon_A(a) S_A(f(S_H(g_{(2)}), g_{(3)})) f(S_H(g_{(1)}_{(1)}), g_{(4)}_{(1)}) \times S_H(g_{(1)}_{(2)}) \cdot g_{(4)}_{(2)} \\ &= \varepsilon_A(a) S_A(f(S_H(g_{(1)}_{(2)}), g_{(2)})) f(S_H(g_{(1)}_{(1)}_{(1)}), g_{(3)}_{(1)}) \times S_H(g_{(1)}_{(1)}_{(2)}) \cdot g_{(3)}_{(2)} \\ &= \varepsilon_A(a) S_A(f(S_H(g_{(1)}_{(1)}), g_{(2)})) f(S_H(g_{(1)}_{(2)}_{(1)}), g_{(3)}_{(1)}) \times S_H(g_{(1)}_{(2)}_{(2)}) \cdot g_{(3)}_{(2)} \\ &= \varepsilon_A(a) S_A(f(S_H(g_{(1)}_{(1)}), g_{(2)})) f(S_H(g_{(1)}_{(2)}), g_{(3)}_{(1)}) \times S_H(g_{(1)}_{(3)}) \cdot g_{(3)}_{(2)} \\ &= \varepsilon_A(a) S_A(f(S_H(g_{(1)}_{(1)}), g_{(2)}_{(1)})) f(S_H(g_{(1)}_{(2)}), g_{(2)}_{(2)}_{(1)}) \times S_H(g_{(1)}_{(3)}) \cdot g_{(2)}_{(2)}_{(2)} \\ &= \varepsilon_A(a) S_A(f(S_H(g_{(1)}_{(1)}), g_{(2)}_{(1)})) f(S_H(g_{(1)}_{(2)}), g_{(2)}_{(2)}) \times S_H(g_{(1)}_{(3)}) \cdot g_{(2)}_{(3)} \\ &= \varepsilon_A(a) S_A(f(S_H(g_{(1)}_{(1)}_{(1)}), g_{(2)}_{(1)}_{(1)})) f(S_H(g_{(1)}_{(1)}_{(2)}), g_{(2)}_{(1)}_{(2)}) \\ &\quad \times S_H(g_{(1)}_{(2)}) \cdot g_{(2)}_{(2)} \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(3)}{=} \varepsilon_A(a) S_A(f(S_H(g(1))_{(1)}, g_{(2)(1)})_{(1)}) f(S_H(g(1))_{(1)}, g_{(2)(1)})_{(2)} \rtimes S_H(g(1))_{(2)} \cdot g_{(2)(2)} \\
 &= \varepsilon_A(a) \varepsilon_A(f(S_H(g(1))_{(1)}, g_{(2)(1)})) 1_A \rtimes S_H(g(1))_{(2)} \cdot g_{(2)(2)} \\
 &\stackrel{(3)}{=} \varepsilon_A(a) \varepsilon_H(S_H(g(1))_{(1)}) \varepsilon_H(g_{(2)(1)}) \rtimes S_H(g(1))_{(2)} \cdot g_{(2)(2)} \\
 &= \varepsilon_A(a) 1_A \rtimes S_H(g(1)) \cdot g_{(2)} \\
 &= \varepsilon_A(a) \varepsilon_H(g) 1_A \rtimes 1_H
 \end{aligned}$$

Thus S is a left inverse of the identity in the convolution algebra $\text{Hom}(A \rtimes H, A \rtimes H)$. By similar computations one can show that S is also a right inverse of the identity, thus is an antipode of $A \rtimes H$. \square

In Proposition 2.8 we imposed the condition for S_H to be a coalgebra antimorphism because the algebra structure on H is not an associative one and for this reason a k -linear map S_H which satisfies the antipode condition (35) is not necessarily a coalgebra antimorphism as in the classical case of Hopf algebras.

3. The classification of unified products

In this section we prove the classification theorem for unified products: as a special case, a classification theorem for bicrossed products of Hopf algebras is obtained. Our view point is inspired from Schreier’s classification theorem for extensions of an abelian group K by a group Q [15, Theorem 7.34]: they are classified by the second cohomology group $H^2(Q, K)$. Let $\varphi : G \rightarrow G'$ be a morphism between two extensions of a group K by a group Q , i.e. φ is a morphism of groups such that the diagram

$$\begin{array}{ccccc}
 k[K] & \xrightarrow{i} & k[G] & \xrightarrow{\pi} & k[Q] \\
 \text{Id} \downarrow & & \downarrow \varphi & & \downarrow \text{Id} \\
 k[K] & \xrightarrow{i'} & k[G'] & \xrightarrow{\pi'} & k[Q]
 \end{array}$$

is commutative (we wrote the diagram for the induced morphism for group algebras). Then φ is an isomorphism [15, Theorem 7.32]. Now, the left-hand square of the diagram is commutative if and only if φ is a left $k[K]$ -module map while the right-hand square of the diagram is commutative if and only if φ is a morphism of right $k[Q]$ -comodules. This motivates the way of considering the classification of unified products up to an isomorphism of Hopf algebras that is also a left A -module map and a right H -comodule map.

Let $\Omega(A) = (H, \triangleleft, \triangleright, f)$ be a bialgebra extending structure of A . The unified product $A \rtimes H$ is a right H -comodule via the coaction $a \rtimes h \mapsto a \rtimes h_{(1)} \otimes h_{(2)}$, for all $a \in A$ and $h \in H$ and a left A -module via the restriction of scalars map $i_A : A \rightarrow A \rtimes H$.

From now on the Hopf algebra structure on A and the coalgebra structure on H will be set. First, we need the following.

Lemma 3.1. *Let A be a Hopf algebra, $\Omega(A) = (H, \triangleleft, \triangleright, f)$ and $\Omega'(A) = (H, \triangleleft', \triangleright', f')$ two Hopf algebra extending structures of A . Then a k -linear map $\varphi : A \rtimes H \rightarrow A \rtimes' H$ is a left A -module, a right H -comodule and a coalgebra morphism if and only if there exists a unique morphism of coalgebras $u : H \rightarrow A$ such that*

$$h_{(1)} \otimes u(h_{(2)}) = h_{(2)} \otimes u(h_{(1)}) \tag{37}$$

for all $h \in H$ and φ is given by

$$\varphi(a \rtimes h) = au(h_{(1)}) \rtimes' h_{(2)} \tag{38}$$

for all $a \in A$ and $h \in H$. Furthermore, any such a morphism $\varphi : A \ltimes H \rightarrow A \ltimes' H$ is an isomorphism with the inverse given by

$$\psi : A \ltimes' H \rightarrow A \ltimes H, \quad \psi(a \ltimes' h) = aS_A(u(h_{(1)})) \ltimes h_{(2)}$$

for all $a \in A$ and $h \in H$.

Proof. Let $\varphi : A \ltimes H \rightarrow A \ltimes' H$ be a left A -module, a right H -comodule and a coalgebra morphism. We shall adopt the notation $\varphi(1_A \ltimes h) = \sum h^A \otimes h^H \in A \otimes H$, for all $h \in H$. Since φ is a left A -module map we have

$$\varphi(a \ltimes h) = a\varphi(1_A \ltimes h) = a \sum h^A \otimes h^H$$

for all $a \in A$ and $h \in H$. As φ is also a right H -comodule map we have:

$$\sum ah^A \otimes (h^H)_{(1)} \otimes (h^H)_{(2)} = \varphi(a \ltimes h_{(1)}) \otimes h_{(2)}$$

By applying ε_H on the second position of the above identity we obtain:

$$\varphi(a \ltimes h) = \sum a(h_{(1)})^A \varepsilon_H((h_{(1)})^H) \otimes h_{(2)}$$

for all $a \in A$ and $h \in H$. Now, if we define $u : H \rightarrow A$ by:

$$u(h) = (Id \otimes \varepsilon_H) \circ \varphi(1_A \ltimes h) = \sum h^A \varepsilon_H(h^H)$$

for all $h \in H$, it follows that (38) holds. We shall prove now that φ given by (38) is a coalgebra map if and only if u is a coalgebra map and (37) holds. First we observe that $\varepsilon_{A \ltimes' H} \circ \varphi = \varepsilon_{A \ltimes H}$ if and only if $\varepsilon_A \circ u = \varepsilon_H$. Now, the fact that φ is comultiplicative is equivalent to:

$$u(h_{(1)})_{(1)} \otimes h_{(2)} \otimes u(h_{(1)})_{(2)} \otimes h_{(3)} = u(h_{(1)}) \otimes h_{(2)} \otimes u(h_{(3)}) \otimes h_{(4)} \tag{39}$$

for all $h \in H$. By applying $Id \otimes \varepsilon_H \otimes Id \otimes \varepsilon_H$ to this relation we obtain that u is a coalgebra map; using this fact and then applying $\varepsilon_A \otimes Id \otimes Id \otimes \varepsilon_H$ in relation (39) we obtain relation (37). Conversely, if u is a coalgebra map such that relation (37) holds, then (39) follows straightforward, i.e. φ is a coalgebra map. The fact that ψ is an inverse for ϕ is also straightforward. \square

Remark 3.2. At this point we should remark the perfect similarity with the theory of extensions from the groups case. If $\varphi : A \ltimes H \rightarrow A \ltimes' H$ is a left A -module, a right H -comodule and a coalgebra morphism between two unified products then the following diagram

$$\begin{CD} A @>i_A>> A \ltimes H @>\pi_H>> H \\ @VId_AVV @VV\varphi V @VVId_HV \\ A @>i_A>> A \ltimes' H @>\pi_H>> H \end{CD}$$

is commutative and φ is an isomorphism.

For $A = k$ and a Hopf algebra H the group $H_1^1(H, k)$ of all unitary algebra maps $u : H \rightarrow k$ satisfying the compatibility condition (40) below was called in [4, Definition 1.1] the *first lazy cohomology group* of H with coefficients in k . We shall now define the coalgebra version of lazy 1-cocycles.

Definition 3.3. Let A be a Hopf algebra and H a coalgebra, unitary not necessarily associative algebra. A morphism of coalgebras $u : H \rightarrow A$ is called a *coalgebra lazy 1-cocycle* if $u(1_H) = 1_A$ and the following compatibility holds:

$$h_{(1)} \otimes u(h_{(2)}) = h_{(2)} \otimes u(h_{(1)}) \tag{40}$$

for all $h \in H$. We denote by $H^1_{l,c}(H, A)$ the group of all coalgebra lazy 1-cocycles of H with coefficients in A .

$H^1_{l,c}(H, A)$ is a group with respect to the convolution product. We have to prove that if u and $v \in H^1_{l,c}(H, A)$, then $u * v \in H^1_{l,c}(H, A)$. Indeed, is straightforward to prove that $u * v$ satisfy (40). Let us show that $u * v$ is a morphism of coalgebras. First, if we apply v on the first position in (40) we obtain $v(h_{(1)}) \otimes u(h_{(2)}) = v(h_{(2)}) \otimes u(h_{(1)})$, for all $h \in H$. Using this relation we obtain:

$$\begin{aligned} \Delta_A(u(h_{(1)})v(h_{(2)})) &= u(h_{(1)})v(h_{(3)}) \otimes u(h_{(2)})v(h_{(4)}) \\ &= u(h_{(1)})v(h_{(2)}) \otimes u(h_{(3)})v(h_{(4)}) \\ &= u * v(h_{(1)}) \otimes u * v(h_{(2)}) \end{aligned}$$

for all $h \in H$, hence $u * v$ is also a coalgebra map.

The main theorem of this section now follows:

Theorem 3.4. Let A be a Hopf algebra, $\Omega(A) = (H, \triangleleft, \triangleright, f)$ and $\Omega'(A) = (H, \triangleleft', \triangleright', f')$ two Hopf algebra extending structures of A . Then there exists $\varphi : A \ltimes' H \rightarrow A \ltimes H$ a left A -module, a right H -comodule and a Hopf algebra map if and only if $\triangleleft' = \triangleleft$ and there exists a coalgebra lazy 1-cocycle $u \in H^1_{l,c}(H, A)$ such that:

$$h \triangleright' c = u(h_{(1)})(h_{(2)} \triangleright c_{(1)})S_A(u(h_{(3)} \triangleleft c_{(2)})) \tag{41}$$

$$f'(h, g) = u(h_{(1)})(h_{(2)} \triangleright u(g_{(1)}))f(h_{(3)} \triangleleft u(g_{(2)}), g_{(3)})S_A(u(h_{(4)} \cdot' g_{(4)})) \tag{42}$$

$$h \cdot' g = (h \triangleleft u(g_{(1)})) \cdot g_{(2)} \tag{43}$$

for all $h, g \in H$ and $c \in A$. In this case φ is given by (38) and it is an isomorphism.

Proof. We already proved in Lemma 3.1 that $\varphi : A \ltimes' H \rightarrow A \ltimes H$ is a left A -module, a right H -comodule and a coalgebra map if and only if $\varphi(a \ltimes' h) = au(h_{(1)}) \ltimes h_{(2)}$, for all $a \in A, h \in H$ and for a unique coalgebra map $u : H \rightarrow A$ such that the (40) holds. Of course, $\varphi(1_A \ltimes' 1_H) = 1_A \ltimes 1_H$ if and only if u is unitary. Moreover, as u is a morphism of coalgebras it is invertible in convolution with the inverse $u^{-1} = S_A \circ u$.

In what follows we shall prove, in the hypothesis that φ is a coalgebra map and u is unitary, that φ is an algebra map (thus a map of bialgebras) if and only if $\triangleleft' = \triangleleft$ and the compatibility conditions (41)–(43) hold. By a straightforward computation we can show that φ is an algebra map if and only if

$$\begin{aligned} (C)(h_{(1)} \triangleright' c_{(1)})f'(h_{(2)} \triangleleft' c_{(2)}, g_{(1)})u((h_{(3)} \triangleleft' c_{(3)}) \cdot' g_{(2)}) \times (h_{(4)} \triangleleft' c_{(4)}) \cdot' g_{(3)} \\ = u(h_{(1)})(h_{(2)} \triangleright c_{(1)}u(g_{(1)}))f(h_{(3)} \triangleleft c_{(2)}u(g_{(2)}), g_{(4)}) \times (h_{(4)} \triangleleft c_{(3)}u(g_{(3)})) \cdot g_{(5)} \end{aligned}$$

for all $h, g \in H$ and $c \in A$. We shall prove that the compatibility (C) is equivalent to (41)–(43).

Indeed, by considering $g = 1_H$ in (C) and then by applying $\varepsilon_A \otimes Id$ we obtain $h \triangleleft' c = h \triangleleft c$, for all $h \in H$ and $c \in A$. If we consider again $g = 1_H$ we obtain, after applying first $Id \otimes \varepsilon_H$ and then inverting u , that (41) holds. Relation (43) is obtained by considering $c = 1_A$ in (C), applying $Id \otimes \varepsilon_H$

and finally inverting u . To end with, relation (43) follows by considering $c = 1_A$ and by applying $\varepsilon_A \otimes Id$ in (C).

Conversely, suppose that $h \triangleleft' c = h \triangleleft c$, for all $h \in H$ and $c \in A$ and there exists a coalgebra lazy 1-cocycle u such that relations (41)–(43) are fulfilled. We then have (we denote by LHS the left-hand side of (C)):

$$\begin{aligned}
 \text{LHS} &\stackrel{(41)-(43)}{=} u(h_{(1)(1)})(h_{(1)(2)} \triangleright c_{(1)(1)})S_A(u(h_{(1)(3)} \triangleleft c_{(1)(2)}))u(h_{(2)(1)} \triangleleft c_{(2)(1)}) \\
 &\quad \times ((h_{(2)(2)} \triangleleft c_{(2)(2)}) \triangleright u(g_{(1)(1)}))f((h_{(2)(3)} \triangleleft c_{(2)(3)}) \triangleleft u(g_{(1)(2)}), g_{(1)(3)}) \\
 &\quad \times S_A(u(h_{(2)(4)} \triangleleft c_{(2)(4)}) \cdot' g_{(1)(4)})u((h_{(3)} \triangleleft c_{(3)}) \cdot' g_{(2)}) \times (h_{(4)} \triangleleft c_{(4)}) \cdot' g_{(3)} \\
 &= u(h_{(1)})(h_{(2)} \triangleright c_{(1)})S_A(u(h_{(3)} \triangleleft c_{(2)}))u(h_{(4)} \triangleleft c_{(3)})(h_{(5)} \triangleleft c_{(4)}) \triangleright u(g_{(1)}) \\
 &\quad \times f((h_{(6)} \triangleleft c_{(5)}) \triangleleft u(g_{(2)}), g_{(3)})S_A(u(h_{(7)} \triangleleft c_{(6)}) \cdot' g_{(4)})u((h_{(8)} \triangleleft c_{(7)}) \cdot' g_{(5)}) \\
 &\quad \times (h_{(9)} \triangleleft c_{(8)}) \cdot' g_{(6)} \\
 &= u(h_{(1)})(h_{(2)} \triangleright c_{(1)})(h_{(3)} \triangleleft c_{(2)}) \triangleright u(g_{(1)})f((h_{(4)} \triangleleft c_{(3)}) \triangleleft u(g_{(2)}), g_{(3)}) \\
 &\quad \times (h_{(5)} \triangleleft c_{(4)}) \cdot' g_{(4)} \\
 &\stackrel{(2d)}{=} u(h_{(1)})(h_{(2)} \triangleright (c_{(1)}u(g_{(1)})))f((h_{(3)} \triangleleft c_{(2)}) \triangleleft u(g_{(2)}), g_{(3)}) \\
 &\quad \times (h_{(4)} \triangleleft c_{(3)}) \cdot' g_{(4)} \\
 &\stackrel{(43)}{=} u(h_{(1)})(h_{(2)} \triangleright (c_{(1)}u(g_{(1)})))f((h_{(3)} \triangleleft c_{(2)}) \triangleleft u(g_{(2)}), g_{(3)}) \\
 &\quad \times (h_{(4)} \triangleleft c_{(3)}u(g_{(4)})) \cdot g_{(5)} \\
 &\stackrel{(40)}{=} u(h_{(1)})(h_{(2)} \triangleright (c_{(1)}u(g_{(1)})))f((h_{(3)} \triangleleft c_{(2)}) \triangleleft u(g_{(2)}), g_{(4)}) \\
 &\quad \times (h_{(4)} \triangleleft c_{(3)}u(g_{(3)})) \cdot g_{(5)}
 \end{aligned}$$

where the third equality holds by using the antipode conditions and the fact that u is a coalgebra map. Thus (C) holds and the proof is now finished. \square

Even if for the classification problem we only set the Hopf algebra structure of A and the coalgebra structure of H , Theorem 3.4 tells us that we can set also the coalgebra map $\triangleleft : H \otimes A \rightarrow H$. We shall phrase Theorem 3.4 as a description of the skeleton for the category $\mathcal{C}(A, H, \triangleleft)$ defined below.

Let A be a Hopf algebra, H a coalgebra with a fixed group-like element $1_H \in H$ and $\triangleleft : H \otimes A \rightarrow H$ a morphism of coalgebras. Let $\mathcal{ES}(A, H, \triangleleft)$ be the set of all triples $(\cdot, \triangleright, f)$ such that $((H, 1_H, \cdot), \triangleleft, \triangleright, f)$ is a Hopf algebra extending structure of A . The next definition is the Hopf algebra version for unified product [15, Definition 7.31] given for extensions of groups.

Definition 3.5. Two elements $(\cdot, \triangleright, f), (\cdot', \triangleright', f')$ of $\mathcal{ES}(A, H, \triangleleft)$ are called *cohomologous* and we denote this by $(\cdot, \triangleright, f) \approx (\cdot', \triangleright', f')$ if there exists a coalgebra lazy 1-cocycle $u \in H_{l,c}^1(H, A)$ such that the compatibility conditions (41)–(43) are fulfilled.

It follows from Theorem 3.4 that $(\cdot, \triangleright, f) \approx (\cdot', \triangleright', f')$ if and only if there exists $\varphi : A \rtimes' H \rightarrow A \rtimes H$ a left A -module, a right H -comodule and a Hopf algebra map. Moreover, from Lemma 3.1 we obtain that any such map $\varphi : A \rtimes' H \rightarrow A \rtimes H$ is an isomorphism, thus \approx is an equivalence relation on the set $\mathcal{ES}(A, H, \triangleleft)$. We denote by $H_{l,c}^2(H, A, \triangleleft)$ the quotient set $\mathcal{ES}(A, H, \triangleleft) / \approx$.

Let $\mathcal{C}(A, H, \triangleleft)$ be the category whose class of objects is the set $\mathcal{ES}(A, H, \triangleleft)$. A morphism $\varphi : (\cdot, \triangleright, f) \rightarrow (\cdot, \triangleright', f')$ in $\mathcal{C}(A, H, \triangleleft)$ is a Hopf algebra morphism $\varphi : A \rtimes H \rightarrow A \rtimes' H$ that is a left A -module and a right H -comodule map. Thus we obtain the categorical version of Theorem 3.4:

Corollary 3.6 (Schreier theorem for unified products). *Let A be a Hopf algebra, H a coalgebra with a group-like element 1_H and $\triangleleft : H \otimes A \rightarrow A$ a morphism of coalgebras. There exists a bijection between the set of objects of the skeleton of the category $\mathcal{C}(A, H, \triangleleft)$ and the quotient set $H^2_{l,c}(H, A, \triangleleft)$.*

$H^2_{l,c}(H, A, \triangleleft)$ is for the classification of the unified products the counterpart of the second cohomology group for the classification of an extension of an abelian group by a group [15, Theorem 7.34].

We can apply Theorem 3.4 to obtain classification theorems for various special cases of the unified products: for instance, Doi’s results on the classification of crossed products [7] is obtain as a special case if we let $\triangleleft' = \triangleleft$ be the trivial actions. Now, we shall indicate the classification of bicrossed product of Hopf algebras.

Corollary 3.7 (Schreier theorem for bicrossed products). *Let A and H be two Hopf algebras and $(A, H, \triangleleft, \triangleright)$, $(A, H, \triangleleft', \triangleright')$ two matched pairs of Hopf algebras. Then $A \rtimes H \cong A \rtimes' H$ (isomorphism of Hopf algebras, left A -modules and right H -comodules) if and only if $\triangleleft' = \triangleleft$ and there exists a coalgebra lazy 1-cocycle $u \in H^1_{l,c}(H, A)$ such that:*

$$\begin{aligned} h \triangleright' c &= u(h_{(1)})(h_{(2)} \triangleright c_{(1)})S_A(u(h_{(3)} \triangleleft c_{(2)})) \\ u(h_{(1)})(h_{(2)} \triangleright u(g_{(1)}))S_A(u(h_{(3)}g_{(2)})) &= \varepsilon_H(g)\varepsilon_H(h)1_A \\ h \triangleleft u(g) &= h\varepsilon_H(g) \end{aligned}$$

for all $h, g \in H$ and $c \in A$.

Proof. We apply Theorem 3.4 for the case when f and f' are the trivial cocycles. As the multiplication on the algebra H is the same (i.e. $\cdot = \cdot'$), the condition (43) in Theorem 3.4 takes the equivalent form $h \triangleleft u(g) = h\varepsilon_H(g)$, for all $h, g \in H$. \square

The construction of unified products is a challenging problem considering the number of compatibilities that need to be fulfilled. In particular, an example of an unified product which is neither a crossed product nor a bicrossed product is interesting in the picture. We provide such an example below: it is a Hopf algebra $k[A_4] \rtimes k[S] \cong k[A_6]$, where A_n is the alternating group on a set with n elements and S is a set with thirty elements.

Example 3.8. Let G be a group and $(X, 1_X)$ a pointed set. We consider the group Hopf algebra $A := k[G]$ and the group-like coalgebra $H := k[X]$. We note that coalgebra morphisms between two group-like coalgebras are in one to one correspondence with the maps between the corresponding sets. Thus, any bialgebra extending structure $(k[X], \triangleleft, \triangleright, f)$ of the Hopf algebra $k[G]$ is induced by an extending structure $(X, \triangleleft', \triangleright', f')$ of the group G in the sense of [1, Definition 2.3]. Moreover there exists a canonical isomorphism of bialgebras

$$k[G] \rtimes k[X] \cong k[G \times X]$$

where $G \times X$ is the unified product at the level of groups (see [1] for further details). This generalizes the fact that a bicrossed product of two group Hopf algebras is isomorphic to the group Hopf algebra of the bicrossed product of the corresponding groups [10, Example 1, p. 207]. The same type of isomorphism holds also for crossed products of Hopf algebras between two group Hopf algebras.

Now, let A_6 be the alternating group on a set with six elements. A_6 is the simple group of smallest order that cannot be written as a bicrossed product of two proper subgroups [16]. Being a simple

group it can not be written neither as a crossed product of two proper subgroups. On the other hand, A_6 can be written as an unified product between any of its subgroups and an extending structure. For instance, we can write

$$A_6 \cong A_4 \ltimes S$$

for an extending structure $(S, 1_S, \alpha, \beta, f, *, i)$ of A_4 , where S is a set of representatives for the right cosets of A_4 in A_6 with 30 elements such that $1 \in S$. Thus there exists an example of an unified product for Hopf algebras $k[A_4] \ltimes k[S] \cong k[A_6]$ which is neither a crossed product nor a bicrossed product of two Hopf algebras.

Two general methods for constructing unified products are proposed in [2]. One of them constructs an unified product starting with a minimal set of data: a Hopf algebra A , a unitary not necessarily associative bialgebra H which is a right A -module coalgebra and a unitary coalgebra map $\gamma : H \rightarrow A$ satisfying four technical compatibility conditions [2, Theorem 2.9].

Acknowledgments

A.L. Agore is "Aspirant" Fellow of the Fund for Scientific Research–Flanders (Belgium) (F.W.O.–Vlaanderen). G. Militaru was supported from CNSIS grant 24/28.09.07 of PN II "Groups, quantum groups, corings and representation theory".

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