Multiplicity of stationary solutions to the Euler–Poisson equations

Yinbin Deng\textsuperscript{a,1}, Tong Yang\textsuperscript{b,*,2}

\textsuperscript{a} Department of Mathematics, Huazhong Normal University, China
\textsuperscript{b} Department of Mathematics, City University of Hong Kong, Hong Kong

Received 9 February 2006
Available online 13 June 2006

Abstract

Consider the system of Euler–Poisson as a model for the time evolution of gaseous stars through the self-induced gravitational force. We study the existence, uniqueness and multiplicity of stationary solutions for some velocity fields and entropy function that solve the conservation of mass and energy a priori. These results generalize the previous works on the irrotational or the rotational gaseous stars around an axis, and then they hold in more general physical settings. Under the assumption of radial symmetry, the monotonicity properties of the radius of the gas with respect to either the strength of the velocity field or the center density are also given which yield the uniqueness under some circumstances.

© 2006 Elsevier Inc. All rights reserved.

\textbf{MSC:} 35J60; 35J65

\textbf{Keywords:} Euler–Poisson equations; Uniqueness; Multiple solutions; Compact support

1. Introduction

The time evolution of the gaseous stars can be modelled by the system of the Euler–Poisson equations which consists of the Euler equations for the conservation of mass, momentum and energy,

\* Corresponding author.
E-mail address: matyang@math.cityu.edu.hk (T. Yang).
\textsuperscript{1} The research was supported by the Natural Science Foundation of China (10471052).
\textsuperscript{2} The research was supported by the Strategic Research Grant of City University of Hong Kong #7001807.
\[ \begin{align*}
\rho_t + \text{div}_x (\rho v) &= 0, \\
(\rho v)_t + \text{div}_x (\rho v \otimes v) + \nabla_x P &= -\rho \nabla_x \Phi, \\
(\rho S)_t + \text{div}_x (\rho Sv) &= 0, \\
\end{align*} \]

(1.1)

and the Poisson equation for the gravitational potential

\[ \Delta_x \Phi = 4 \pi g \rho, \]

(1.2)

where \( t \geq 0, x \in \mathbb{R}^3, \rho = \rho(t, x) \) is the density, \( v = v(t, x) \in \mathbb{R}^3 \) is the velocity, \( S \) is the entropy, \( \Phi = \Phi(t, x) \) is the potential function of the self-gravitational force, \( g \) is the gravitational constant and \( P(\rho, S) \) is the pressure.

This system has been studied extensively since the nineteenth century mainly because of its relation to astrophysics. For example, for the stability of nonmoving stationary solutions with radial symmetry for the barotropic gas, there are well known Chandrasekhar and Eddington principles giving the stability when the adiabatic constant greater than \( 4/3 \) and instability otherwise, cf. [2]. The nonlinear justification of this stability criteria was proved in [11] for the adiabatic constant greater than \( 4/3 \). Recently, there are some works on the existence of stationary solutions with or without rotation around an axis in [11,17]. As a continuation in this direction, in this paper, we will consider the existence of multiple solutions and the exact multiplicity of solutions in a more general setting where the velocity field may not be just a rotation around an axis. The uniqueness and multiplicity of solutions for different velocity fields give rich solution phenomena to this classical system.

In the following discussion, we will concentrate on the barotropic gas, i.e.,

\[ P = k \rho^\gamma e^S, \]

(1.3)

where the constant factor \( k \) will be normalized to 1 in the sequel, and \( \gamma > 1 \) is the adiabatic constant. For the Euler–Poisson system, it is interesting that the stability and existence of stationary solutions crucially depend on the adiabatic constant. In general, the heavier gas corresponds to smaller \( \gamma \). For example, the adiabatic constant \( \gamma \) is \( 5/3 \) for monatomic gas and \( 7/5 \) for diatomic gas. For the significance of the adiabatic constant on the existence, stability, uniqueness and boundary behavior of solutions, please refer to [4–6,11,16,18–20] and references therein.

For a stationary solution to (1.1) and (1.2) with a given velocity field \( v(x) \), the momentum equation can be written as

\[ v \cdot \nabla_x v + \frac{1}{\rho} \nabla_x P = -\nabla_x \Phi. \]

(1.4)

Taking the divergence on both sides gives

\[ \text{div}_x (v \cdot \nabla_x v) + \text{div}_x \left( \frac{1}{\rho} \nabla_x P \right) = -\Delta_x \Phi. \]

Combining this with the Poisson equation (1.2) and (1.3)

\[ \text{div} \left( e^{\alpha S} \nabla_x w \right) + ke^{-\alpha S} w^q - f(x) = 0, \]

(1.5)
where

\[ q = \frac{1}{\gamma - 1}, \quad \alpha = \frac{1}{\gamma}, \quad k = 4\pi g \left( \frac{\gamma - 1}{\gamma} \right)^{\frac{1}{\gamma - 1}}, \quad w = \frac{\gamma}{\gamma - 1} \left( e^{\frac{S}{\gamma}} \rho \right)^{\gamma - 1} \text{ and (1.6)} \]

\[ f(x) = -\text{div}_x (v \cdot \nabla_x v). \quad (1.7) \]

To satisfy the conservation laws of mass and energy, the velocity field cannot be arbitrary. In fact, if \( v(x) \) is a rotation around the \( x_3 \) axis with a prescribed time independent angular velocity \( \Omega = \Omega(\eta) \) as a function of \( \eta(x) = \sqrt{x_1^2 + x_2^2} \), then the functions \( (\rho, v, S, \Phi) \) given by

\[ (\rho, v, S, \Phi)(t, x) = (\rho, v, S)(t, \eta(x), x_3), \quad \text{with} \]

\[ v = (-x_2 \Omega(\eta), x_1 \Omega(\eta), 0), \]

satisfy both the conservation laws of mass and energy, that is,

\[ \text{div}_x (\rho v) = 0, \quad v \cdot \nabla_x S = 0. \]

In this case, the solution to the elliptic equation (1.5) gives a solution to the Euler–Poisson system which was studied in [17]. Moreover, the function \( f(x) \) in this case takes the form

\[ f(x) = -\text{div}_x (v \cdot \nabla_x v) = 2\Omega \left( \Omega + \eta \Omega'(\eta) \right). \]

In this paper, we will consider the case when the function \( f(x) \) is not identically zero.

In other physical situations, the gaseous star may not rotate just around an axis so that the function \( f(x) \) defined in (1.7) can be a general function of space variables. The main purpose of this paper is to study the effect of \( f(x) \) coming from the velocity field on the multiplicity of the solutions. In the following discussion, we assume that the conservations of mass and energy are satisfied by a given velocity field. Thus, we can concentrate on the elliptic equation (1.5). In fact, the problem on the system of (1.5) coupled with the conservation of mass and energy is interesting which is almost open.

By using \( u = e^{\frac{\alpha}{2}S}w \), Eq. (1.5) becomes

\[ \Delta u - a(x)u + K(x)u^q - f(x)e^{-\frac{\alpha}{2}S} = 0, \quad (1.8) \]

where

\[ a(x) = \frac{\alpha}{2} \Delta S + \frac{\alpha^2}{4} |\nabla_x S|, \quad K(x) = ke^{\frac{2-3\gamma}{2(\gamma-1)}}S. \quad (1.9) \]

For a given simply connected open region \( D \) with smooth boundary, we consider the positive solutions to Eq. (1.8) in \( D \) with zero Dirichlet boundary value condition:

\[ u|_{\partial D} = 0. \quad (1.10) \]
For the convenience of discussion, we normalize the function \( f(x) \) defined by the velocity field in \( L^2 \) norm (still denote it by \( f(x) \)) and rewrite Eqs. (1.8) and (1.10) into the following inhomogeneous elliptic boundary value problem with a parameter \( \sigma \) which represents the strength of the velocity field:

\[
\begin{aligned}
\Delta u - a(x)u + K(x)u^q - \sigma f(x)e^{-\frac{\alpha}{2}S} &= 0, \\
u|_{\partial D} = 0, \\u > 0 \text{ in } D.
\end{aligned}
\] (1.11)

Here \( D \subset \mathbb{R}^3 \) is a bounded domain, \( \sigma > 0, q \in (0, 1) \cup (1, \infty) \) and \( f \in C^1(\overline{D}) \setminus \{0\} \). Notice that here \( f(x) \) is allowed to change sign unlike the rotation around a fixed axis.

When \( f(x) \leq 0, f(x) \neq 0, K(x) = 1 \) and \( a(x) = \text{constant} \), it is known that problem (1.11) has at least two positive solutions when \( \sigma \) is sufficiently small, while has no solution when \( \sigma \) is large, cf. [7,8,10]. The solution phenomena become richer when \( f(x) \) has no definite sign. However, in this case, the maximum principle fails. For this, in some particular cases such as \( K(x) = 1, a(x) = 0, \) the existence and nonexistence of solutions to (1.11) were discussed in [8,10].

The multiplicity of solutions depends on the classification of the function \( f(x) \) as well as the properties of positive solutions to the homogeneous problem

\[
\begin{aligned}
\Delta u + a(x)u &= K(x)u^q, \\u|_{\partial D} = 0, \\u > 0 \text{ in } D.
\end{aligned}
\] (1.12)

Problem (1.11) can be viewed as a perturbation of problem (1.12). Furthermore, the existence of positive solutions to (1.11) is also closely related to the solvability of the linear problem

\[
\begin{aligned}
-\Delta \psi + a(x)\psi &= -f(x)e^{-\frac{\alpha}{2}S(x)} \text{ in } D, \\
\psi \geq 0 \text{ in } D, \\\psi|_{\partial D} = 0.
\end{aligned}
\] (1.13)

Set

\[
\mathcal{U} = \{ f(x) \in C^1(\overline{D}) \setminus \{0\}: \text{when (1.13) is solvable} \} \quad \text{and}
\]

\[
\mathcal{N} = \{ C^1(\overline{D}) \setminus \{0\} \} \setminus \mathcal{U}.
\] (1.15)

We will prove that if \( f(x) \in \mathcal{U} \), then (1.11) is solvable for small \( \sigma > 0 \) when \( q \in (1, \infty) \). On the other hand, in some cases, the condition \( f(x) \in \mathcal{U} \) is even necessary for the existence of positive solutions of (1.11). For this, we need the following definition.

**Definition 1.1.** Problem (1.12) is nondegenerate for \( a(x) \) and \( K(x) \) in some smooth domain \( D \) if it has a unique positive solution \( u \) and the corresponding linearized problem

\[
\begin{aligned}
-\Delta \psi + a(x)\psi &= qK(x)u^{q-1}\psi \text{ in } D, \\
\psi|_{\partial D} = 0,
\end{aligned}
\] (1.16)

admits only the trivial solution.

Notice that if \( a(x) \equiv 0, K(x) \equiv 1 \) and \( D \) is a ball in \( \mathbb{R}^3 \), problem (1.12) is nondegenerate, cf. [21].
In what follows, we always assume the following two conditions on the function \( f(x) \) and the entropy function \( S(x) \):

\[(A_1) \quad f(x) \in C^1(D) \setminus \{0\}. \]

\[(A_2) \quad S(x) \in C^3(D) \text{ such that } a(x) \geq 0 \text{ for all } x \in D. \]

We will show that when \( f(x) \in U \), there is a positive solution of (1.11) which bifurcates from the trivial solution of (1.12). However, it is not true when \( f(x) \notin U \), i.e., \( f(x) \in \mathcal{N} \). In fact, when \( f(x) \in \mathcal{N} \), if there is a positive solution \( u_\sigma \) of (1.11) for any \( \sigma \) close to zero, then

\[ ||u_\sigma||_{L^\infty(D)} \not\to 0, \quad \text{as } \sigma \to 0. \]

To further discuss the existence of the positive solution of (1.11) for \( f(x) \notin U \), we denote \( \mathcal{N}(\partial D) \) the intersection of a neighborhood of the boundary \( \partial D \) and \( D \). Set

\[ \mathcal{F}^+ = \{ f(x) \in C^1(D) \setminus \{0\} : \text{there exists } \mathcal{N}(\partial D) \text{ such that } f(x) \leq 0 \text{ for } x \in \mathcal{N}(\partial D) \}. \] (1.17)

Note that \( \mathcal{F}^+ \cap \mathcal{N} \neq \emptyset \). In fact, \( 0 \leq \varphi(x) \in C_0^\infty(D) \) belongs to \( \mathcal{F}^+ \cap \mathcal{N} \).

**Remark 1.1.** From the assumption \((A_2)\), the first eigenvalue \( \lambda_1(D) \) of the operator \(-\Delta + a(x)\) with zero Dirichlet boundary condition is positive, that is,

\[ \int_D |\nabla \varphi|^2 + a(x)\varphi^2 \, dx \geq \lambda_1(D) \int_D \varphi^2 \, dx \quad \text{for all } \varphi \in H_0^1(D) \quad \text{and} \]

\[ \int_D |\nabla \varphi|^2 + a(x)\varphi^2 \, dx \geq \lambda_1^+(D) \int_D |\nabla \varphi|^2 \, dx \quad \text{for all } \varphi \in H_0^1(D), \] (1.18) (1.19)

where \( \lambda_1^+(D) > 0 \) is a constant and \( H_0^1(D) \) is the standard Sobolev space.

The main results of this paper are given as follows.

**Theorem 1.** Assume \( f(x) \in U \) and \( q > 1 \) (i.e., \( 1 < \gamma < 2 \)). There exists a constant \( \sigma_f \in (0, +\infty) \) such that problem (1.11) has a minimal solution \( u_\sigma \) for \( \sigma \in (0, \sigma_f) \). Moreover, it has only one solution for \( \sigma = \sigma_f \) if \( 1 < q < 5 \) (i.e., \( \frac{6}{5} < \gamma < 2 \)), and no solution for \( \sigma > \sigma_f \).

Furthermore, when \( \sigma \in (0, \sigma_f) \), the minimal solution \( u_\sigma \) satisfies \( u_\sigma \geq \sigma \varphi(x) \), and is increasing with respect to \( \sigma \) for all \( x \in D \). Here \( \varphi(x) \) is the solution of (1.13).

**Theorem 2.** Under the assumptions of Theorem 1, problem (1.11) has another solution \( U_\sigma \) if \( 1 < q < 5 \) and \( \sigma \in (0, \sigma_f) \). Furthermore, this solution \( U_\sigma \) and the solution \( u_\sigma \) in Theorem 1 satisfy

(i) \( U_\sigma > u_\sigma \) in \( D \),

(ii) \( \int_D |\nabla u_\sigma|^2 + a(x)u_\sigma^2 \, dx \geq q \int_D K(x)u_\sigma^{q+1} \, dx \),

(iii) \( \int_D |\nabla U_\sigma|^2 + a(x)U_\sigma^2 \, dx \leq q \int_D K(x)U_\sigma^{q+1} \, dx \).
For later use, define the corresponding variational functional of (1.11) by

\[ I_\sigma(u) = \frac{1}{2} \int_D |\nabla u|^2 + a(x)u^2 \, dx - \frac{1}{q + 1} \int_D K(x)(u^+)^{q+1} \, dx + \sigma \int_D f(x)ue^{-\frac{q}{2}S(x)} \, dx, \]

(1.20)

for \( u \in H^1_0(D) \). By using the variational method, the following theorem holds when \( f(x) \notin U \).

**Theorem 3.** Assume \( f(x) \in N \cap F^+ \) and \( 1 < q < 5 \). There exists a positive constant \( \sigma_0 \) such that problem (1.11) has at least one solution \( u_\sigma \) for \( \sigma \in (0, \sigma_0) \). Moreover, \( u_\sigma \) converges uniformly to a positive solution \( u_0 \) of problem (1.12) and the variational functional \( I_\sigma(u_\sigma) \) tends to \( I_0(u_0) \) as \( \sigma \to 0 \).

The following theorem is about the exact number of solutions of (1.11) for \( 1 < q < 5 \).

**Theorem 4.** Assume \( 1 < q < 5 \) and the corresponding homogeneous problem (1.12) is nondegenerate. Then there exists a constant \( \sigma_1 > 0 \) such that problem (1.11) has exactly two solutions if and only if \( f(x) \in U \) for \( \sigma \in (0, \sigma_1) \). Moreover, one of the solutions converges uniformly to 0, and the other one converges uniformly to the unique positive solution of problem (1.12) as \( \sigma \to 0 \).

Furthermore, there exists a constant \( \sigma_2 > 0 \) such that problem (1.11) has exactly one solution for \( \sigma \in (0, \sigma_2) \) if \( f(x) \in F^+ \cap N \). Moreover, this unique solution converges uniformly to the unique positive solution of problem (1.12) as \( \sigma \to 0 \).

**Remark 1.2.** Assume \( S = S(r) = r^\theta \) with \( r = |x|, \theta \in (0] \cup [2, \infty), 1 < q < 5, \) and \( D \) is a ball in \( \mathbb{R}^3 \). By applying the uniqueness result in [9] for the homogeneous problem (1.12), there exists a constant \( \sigma_3 > 0 \) such that problem (1.11) has exactly two solutions for \( \sigma \in (0, \sigma_3) \) if and only if \( f(x) \in U \) and has exact one solution for \( \sigma \in (0, \sigma_3) \) if \( f(x) \in F^+ \cap N \).

The last theorem concerns the existence of solutions to (1.11) for the case when \( q \in (0, 1) \cup (5, +\infty) \) (i.e., \( \gamma \in (2, +\infty) \cup (1, \frac{6}{5}) \)).

**Theorem 5.** Firstly, assume that \( q > 5 \) (i.e., \( 1 < \gamma < \frac{6}{5} \)), \( D \) is star-shaped, \( a(x) \) and \( S(x) \) satisfy

\[
\min_{x \in D} \left\{ 1 - \frac{6}{q + 1} - \frac{2 - 3\gamma}{\gamma^2} (x \cdot \nabla S(x)) \right\} > 0 \quad \text{and} \quad x \cdot \nabla a + 2a(x) \geq 0,
\]

(1.21)

for all \( x \in D \). Then there exists a constant \( \sigma_4 > 0 \) such that (1.11) has at least one solution for all \( \sigma \in (0, \sigma_4) \) if and only if \( f(x) \in U \).

On the other hand, when \( q \in (0, 1) \) (i.e., \( \gamma \in (2, +\infty) \)), (1.11) has a unique solution for all \( \sigma > 0 \) if and only if \( f(x) \in U \).

**Remark 1.3.** For isentropic flow, i.e., \( S = \text{constant}, \) more precise qualitative properties of the solutions can be given when

\[
\sigma f(x) = -\text{div}(v \cdot \nabla_x v) = \mu = \text{constant},
\]

(1.23)
and the domain $D = B_R(0)$ is a ball. Without loss of generality, we can take $S = 0$. In this case, it follows from the important result of Gidas, Ni and Nirenberg [13], that the positive solution of

$$
\begin{align*}
-\Delta u &= u^q - \mu \quad \text{in } D, \\
|u|_{\partial D} &= 0, \quad u > 0 \quad \text{in } D
\end{align*}
$$

must be radially symmetric. That is, the positive solution satisfies

$$
u'' + \frac{2}{r} u' + u^q - \mu = 0.\tag{1.25}
$$

Consider problem (1.25) with initial data

$$u(0) = p > 0, \quad u'(0) = 0. \tag{1.26}
$$

We will again use $u(r, p, \mu)$ to denote the solution to (1.25) and (1.26), and use $R(p, \mu)$ to denote the radius of the support of $u(r, p, \mu)$. That is, $R(p, \mu)$ is the radius of the gaseous stars with the density at the center given by

$$
\rho(0) = \left(\frac{\gamma - 1}{\gamma} p\right)^{\frac{1}{\gamma - 1}}.\tag{1.27}
$$

The qualitative properties of solution to (1.25) and (1.26) when $\mu > 0$ was discussed in [19]. And the case when $\mu < 0$ can be stated as follows. When $1 < q < 5$, we have

(i) If $\mu < 0$, then

$$R(p, \mu) < +\infty \quad \text{for all } p > 0.
$$

(ii) If $\mu_1 < \mu_2 < 0$, then

$$R(p, \mu_1) < R(p, \mu_2).
$$

(iii) For $\mu < 0$, there exist $0 < p_* < +\infty$, such that

$$R(p_1, \mu) > R(p_2, \mu),
$$

provided $0 \leq p_2 < p_1 \leq p_*$. Here $p_* = \left(\frac{\mu}{p^*}\right)^{\gamma - 1}$ and

$$\beta^* = \sup\{\beta \in (-\infty, 0): 2q\beta\lambda'(\beta) + (q - 1)\lambda(\beta) \leq 0\}, \tag{1.28}
$$

where $\lambda(\beta)$ is the radius of the support of the solution $Q(\lambda)$ to the initial value problem

$$
\begin{align*}
Q_{\lambda\lambda} + \frac{2}{\lambda} Q_\lambda + Q^q - \beta &= 0, \\
Q(0) &= 1, \quad Q_\lambda(0) = 0.
\end{align*}
$$

Finally, in the introduction, we will have some discussion in the physical aspects of the theorems given above.
Discussion. The reason for the existence of stationary solutions to the system of Euler–Poisson equations is that the balance of forces from different mechanisms. When the gas has zero velocity, there are only two kinds of forces acting on the gas particles, i.e., forces from the pressure and the gravitational potential. When these two forces are balanced at each point in the support of the gas, we have a stationary solutions. And the existence and stability of this kind of solutions have been extensively studied together with an interesting phenomena, i.e., core collapse, cf. [2,4,11]. On the other hand, if the gas has a nontrivial velocity field like a rotation around an axis, then there is an extra centrifugal force. The balance between these three forces is more subtle so that the phenomena are richer. For the rotation around an axis, there is an interesting mathematical work [17]. And the results in this paper generalize the previous works so that they can be applied to more general physical situations.

From the above mathematical analysis, one can see that the existence of stationary solutions depends on the strength and the sign of the function \( f(x) \) coming from the velocity field besides the adiabatic constant. In general, in a fixed domain, there is no stationary solution if the velocity field is very strong. This can be understood by imaging a gas rotating around an axis. When the angular velocity is large, the gas will expand outwards because of the centrifugal force, so that it may exceed the given domain. On the other hand, in some cases, there are more than one stationary solutions for a given velocity field in a fixed domain. The reason is that the distribution of the gas affects both the pressure and the gravitation potential so that it gives more flexibility for the balance between them and the centrifugal force. One of the time evolution problems is then to study whether these stationary solutions are stable under small perturbation.

Finally, the change of the monotonicity of the radius of the gas with respect to the central density or the strength of the velocity in the radially symmetric case shed some light on the multiplicity of the solutions for \( \mu < 0 \). This excludes the case of the rotation around an axis with constant angular velocity where the monotonicity does not change as discussed in [17].

The rest of the paper will be arranged as follows. In the next section, we will give some preliminaries in the elliptic theory. The existence results on the case when \( f(x) \in \mathcal{U} \) and \( q > 1 \) are given in Section 3. And the multiplicity of the solutions obtained in Section 3 is given in Section 4. In particular, the second solution is constructed when \( q \in (1,5) \) and the existence of solutions for \( f(x) \in \mathcal{P}^+ \cap \mathcal{N} \) and \( 1 < q < 5 \) is proved. The exact number of solutions is given in Section 5. Theorem 5 on the case when \( f(x) \in \mathcal{U} \) and \( q \in (0,1) \cup (5,\infty) \) is proved in the last section.

2. Preliminaries

Besides the estimates for the elliptic equations given in [15,22,23], we need the following estimates. Firstly, we give an estimate on the positive solutions of problem (1.11).

**Lemma 2.1.** There exist \( C > 0 \) and \( \sigma_0 > 0 \), such that for any positive solution \( u_{\sigma} \) of problem (1.11) with \( \sigma \in (0,\sigma_0] \), we have

\[
\|u_{\sigma}\|_{C^2,\sigma(D)} \leq C.
\] (2.1)

**Proof.** By the regularity theory of elliptic equations, it is sufficient to prove

\[
\|u_{\sigma}\|_{L^\infty(D)} \leq C \quad \text{for all } \sigma \in (0,\sigma_0].
\]
We prove it by contradiction as follows. Suppose this is not true. Then there exists a sequence \( \{\sigma_j\} \subset (0, \sigma_0] \) with corresponding positive solutions \( u_j = u_{\sigma_j} \) to problem (1.11), and a sequence of points \( x_j \in D \), such that

\[
M_j = \|u_j\|_{L^\infty(D)} = u_j(x_j) \longrightarrow \infty, \quad \text{as } j \longrightarrow \infty.
\]

Set

\[
V_j(x) = M_j^{-1} u_j(x_j + M_j^{-\frac{q-1}{2}} x),
\]

defined on \( D_j = M_j^{-\frac{q-1}{2}} (D - x_j) \). Note that \( V_j \) satisfies

\[
-\Delta V_j + M_j^{(1-q)} a(x_j + M_j^{-\frac{q-1}{2}} x) V_j = K(x_j + M_j^{-\frac{q-1}{2}} x) V_j^q - \sigma_j M_j^{-q} f(x_j + M_j^{-\frac{q-1}{2}} x) e^{-\frac{q-1}{2} s}, \quad x \in D_j.
\]

In what follows, we use \( H \) to denote the whole space \( \mathbb{R}^3 \) or the half space \( \mathbb{R}^3_+ \). Since \( 0 \leq V_j \leq 1 \), by the standard elliptic estimates, there exists a positive constant \( C \) independent of \( j \) such that for any compact set \( \Theta \subset D_j \), we have

\[
\|V_j\|_{C^{2,\nu}(\Theta)} \leq C.
\]

Hence, up to a subsequence, we can assume that \( V_j \) converges uniformly to a function \( V \) on any compact subset of \( H \), and \( x_j \to x_0 \) as \( j \to \infty \). Since \( f(x) \in C^1(\overline{D}) \) and \( a(x) \in L^\infty(D) \), we have

\[
\sigma_j M_j^{-q} f(x_j + M_j^{-\frac{q-1}{2}} x) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty,
\]

\[
M_j^{1-q} a(x_j + M_j^{-\frac{q-1}{2}} x) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty, \quad \text{and}
\]

\[
K(x_j + M_j^{-\frac{q-1}{2}} x) \longrightarrow K(x_0) > 0, \quad \text{as } j \longrightarrow \infty.
\]

By the standard blowup argument, cf. [14], \( V \) satisfies

\[
\begin{cases}
-\Delta V = K(x_0) V^q, & x \in H, \\
V \geq 0 & \text{in } H, \\
V(0) = 1.
\end{cases}
\]

This contradicts to the fact that when \( q \in (1, 5) \), the only solution of the problem

\[
\begin{cases}
-\Delta V = K(x_0) V^q & \text{in } H, \\
V \geq 0 & \text{in } H \\
V|_{\partial H} = 0 & \text{if } H = \mathbb{R}^3_+
\end{cases}
\]

is \( V \equiv 0 \), cf. [14]. This completes the proof. \( \square \)

The following lemma shows that at least one positive solution of (1.11) bifurcates from the trivial solution of problem (1.12) when (1.11) has at least two positive solutions.
Lemma 2.2. Assume that (1.12) is nondegenerate and (1.11) has at least two positive solutions for some \( \sigma > 0 \). Then one of the solutions, denoted by \( u_\sigma \), satisfies
\[
\|u_\sigma\|_{L^\infty(D)} \rightarrow 0, \quad \text{as } \sigma \rightarrow 0,
\]
when \( q \in (1, 5) \).

Proof. Let \( u_\sigma^{(1)}(x) \) and \( u_\sigma^{(2)}(x) \) be two different positive solutions of problem (1.11). Without loss of generality, assume that
\[
\|u_\sigma^{(1)}\|_{L^\infty(D)} \leq \|u_\sigma^{(2)}\|_{L^\infty(D)}.
\]
We also prove the lemma by contradiction. Suppose that there exists a sequence \( \{\sigma_j\} \) with \( \lim_{j \to \infty} \sigma_j = 0 \) such that
\[
\|u_{\sigma_j}^{(1)}\|_{L^\infty(D)} \rightarrow C > 0, \quad \text{as } j \rightarrow \infty.
\]
By (2.4), up to a subsequence, still denoted by \( \{\sigma_j\} \) for simplicity of notation, we have
\[
\|u_{\sigma_j}^{(2)}\|_{L^\infty(D)} \rightarrow C_1 > 0, \quad \text{as } j \rightarrow \infty.
\]
By Lemma 2.1, we can assume that
\[
u_{\sigma_j}^{(1)} \rightarrow \varphi_1(x), \quad u_{\sigma_j}^{(2)} \rightarrow \varphi_2(x) \quad \text{uniformly in } D, \quad \text{as } j \rightarrow \infty.
\]
Then (2.5) and (2.6) imply that \( \varphi_1(x) \not\equiv 0 \) and \( \varphi_2(x) \not\equiv 0 \) where \( \varphi_1(x) \) and \( \varphi_2(x) \) are positive solutions of problem (1.12). Since (1.12) is nondegenerate, we have
\[
u(x) \equiv \varphi_1(x) \equiv \varphi_2(x) \quad \text{for all } x \in D.
\]
Set \( v_j = u_{\sigma_j}^{(1)} - u_{\sigma_j}^{(2)} \). Then \( v_j \not\equiv 0 \) and satisfies
\[
\begin{cases}
-\Delta v_j + a(x)v_j = \xi_j(x)v_j, & x \in D, \\
v_j|_{\partial D} = 0,
\end{cases}
\]
where
\[
\xi_j(x) = \int_0^1 q K(x)(tu_{\sigma_j}^{(1)}(x) + (1-t)u_{\sigma_j}^{(2)}(x))^{q-1} dt.
\]
Since \( v_j \not\equiv 0 \), we can define \( \tilde{v}_j = \frac{v_j}{\|v_j\|_{H_0^1(D)}} \). Moreover, \( \tilde{v}_j \) satisfies
\[
\begin{cases}
-\Delta \tilde{v}_j + a(x)\tilde{v}_j = \xi_j(x)\tilde{v}_j, & x \in D, \\
\tilde{v}_j|_{\partial D} = 0.
\end{cases}
\]
Up to a subsequence, \( \tilde{v}_j \) converges weakly in \( H^1_0(D) \) and strongly in \( L^2(D) \) to a function \( \tilde{v} \). By Lemma 2.1, we have

\[
\int_D \left[ \int_0^1 q K(x) \left(tu^{(1)}_{\sigma_j} + (1-t)u^{(2)}_{\sigma_j}\right)^{q-1} dt \right] \tilde{v}_j^2 \, dx \longrightarrow q \int_D K(x)u^{q-1}\tilde{v}^2 \, dx, \quad \text{as } j \to \infty.
\]

(2.9)

Then (2.8) and (2.9) imply

\[
1 = \int_D |\nabla \tilde{v}_j|^2 \, dx = \int_D \tilde{v}_j^2 \, dx - \int_D a(x)\tilde{v}_j^2 \, dx
\]

\[
= q \int_D K(x)u^{q-1}\tilde{v}^2 \, dx - \int_D a(x)\tilde{v}_j^2 \, dx + o(1), \quad \text{as } j \to \infty.
\]

By assumption \((A_2)\) and (1.19), we obtain

\[
q \int_D K(x)u^{q-1}\tilde{v}^2 \, dx = \int_D |\nabla \tilde{v}_j|^2 + a(x)\tilde{v}_j^2 \, dx + o(1)
\]

\[
\geq \lambda_1^*(D) \int_D |\nabla \tilde{v}_j|^2 \, dx + o(1) = \lambda_1^*(D) + o(1) > 0, \quad \text{as } j \to \infty,
\]

which implies that \( \tilde{v} \neq 0 \). Taking limit as \( j \to \infty \) in (2.8) yields

\[
\begin{cases}
-\Delta \tilde{v} + a(x)\tilde{v} = qu^{q-1}\tilde{v}, & x \in D, \\
\tilde{v}|_{\partial D} = 0, & \tilde{v} \neq 0 \quad \text{in } D,
\end{cases}
\]

which contradicts to the assumption that (1.12) is the nondegenerate. This completes the proof of the lemma. \( \square \)

**Lemma 2.3.** Under the conditions of Lemma 2.1, there exists a positive constant \( A \) such that problem (1.11) has at most one positive solution \( u \) satisfying \( \|u\|_{L^\infty(D)} \leq A \).

**Proof.** Choose \( A \) so small that

\[
q A^{q-1}\|K(x)\|_{L^\infty(\overline{D})} < \lambda_1(D),
\]

(2.10)

where \( \lambda_1(D) \) is the first eigenvalue of the operator \(-\Delta + a(x)\) with zero Dirichlet boundary condition. Lemma 2.3 can also be proved by contradiction. Suppose that problem (1.11) has two different positive solutions \( u_1(x) \) and \( u_2(x) \) satisfying

\[
\|u_1\|_{L^\infty(D)} \leq A, \quad \|u_2\|_{L^\infty(D)} \leq A.
\]

(2.11)
Then $w = u_1 - u_2 \neq 0$ satisfies
\begin{equation}
\begin{cases}
-\Delta w + a(x)w = qK(x)\xi^{q-1}(x)w & \text{in } D, \\
w|_{\partial D} = 0,
\end{cases}
\end{equation}
where $\xi(x)$ is a nonnegative function between $u_1$ and $u_2$. Multiplying the equation in (2.12) by $w$ and integrating it over $D$ yield
\begin{equation}
\int_D |\nabla w|^2 + a(x)w^2 \, dx = q \int_D K(x)\xi^{q-1}(x)w^2 \, dx.
\end{equation}
By (1.18), we have
\begin{equation}
\lambda_1(D) \int_D w^2 \, dx \leq q \int_D K(x)\xi^{q-1}(x)w^2 \, dx.
\end{equation}
Then (2.11) implies that
\begin{equation}
\|\xi(x)\|_{L^\infty(D)} \leq A.
\end{equation}
Combining this with (2.14) gives
\begin{equation}
\lambda_1(D) \int_D w^2 \, dx \leq q \|K(x)\|_{L^\infty(D)} A^{q-1} \int_D w^2 \, dx,
\end{equation}
i.e.,
\begin{equation}
\lambda_1(D) \leq q A^{q-1} \|K(x)\|_{L^\infty(D)}.
\end{equation}
This contradicts (2.10) and then completes the proof of lemma. \qed

**Lemma 2.4.** When $q \in (1, 5)$, let $u_\sigma$ be a positive solution of problem (1.11) satisfying $\|u_\sigma\|_{L^\infty(D)} \to 0$, as $\sigma \to 0$. Set $u_\sigma = \sigma V_\sigma$, then there exist $\sigma_* > 0$ and constant $C > 0$ independent of $\sigma \in (0, \sigma_*)$ such that
\begin{equation}
\|V_\sigma\|_{L^\infty(D)} \leq C \quad \text{for all } \sigma \in (0, \sigma_*].
\end{equation}

**Proof.** Since $u_\sigma$ is a positive solution of problem (1.11), $V_\sigma$ satisfies
\begin{equation}
\begin{cases}
-\Delta V_\sigma + a(x)V_\sigma = \sigma^{q-1}K(x)V_\sigma^q - f(x)e^{-\frac{q}{2}S} & \text{in } D, \\
V_\sigma|_{\partial D} = 0, \quad V_\sigma > 0 & \text{in } D.
\end{cases}
\end{equation}
Multiplying the equation in (2.16) by $V_\sigma$ and integrating it over $D$ give
\begin{equation}
\int_D |\nabla V_\sigma|^2 + a(x)V_\sigma^2 \, dx = \int_D \left[\sigma^{q-1}K(x)V_\sigma^{q-1}V_\sigma^2 - f(x)V_\sigma e^{-\frac{q}{2}S}\right] \, dx.
\end{equation}
By Hölder’s and Young’s inequalities and (1.18), we have

\[ \int_D f(x)e^{-\frac{q}{2}S} V_\sigma \, dx \leq \left\| e^{-\frac{q}{2}S} \right\|_{L^2(D)} \left\| V_\sigma \right\|_{L^2(D)} \]

\[ \leq \frac{1}{2} \int_D |\nabla V_\sigma|^2 + a(x)V_\sigma^2 \, dx + \frac{1}{2\lambda_1(D)} \left\| e^{-\frac{q}{2}S} \right\|_{L^2(D)}^2. \]

By substituting this into (2.17), we obtain

\[ \int_D |\nabla V_\sigma|^2 + a(x)V_\sigma^2 \, dx \]

\[ \leq 2 \int_D \sigma^{q-1}|K(x)|V_\sigma^{q-1}V_\sigma^2 \, dx + \frac{1}{\lambda_1(D)} \left\| e^{-\frac{q}{2}S} \right\|_{L^2(D)}^2 \]

\[ \leq 2 \left\| K(x) \right\|_{L^\infty(D)} \|u_\sigma\|_{L^\infty(D)} \lambda_1^{-1}(D) \int_D |\nabla V_\sigma|^2 + a(x)V_\sigma^2 \, dx + \frac{1}{\lambda_1(D)} \left\| e^{-\frac{q}{2}S} \right\|_{L^2(D)}^2. \]

Since \( \|u_\sigma\|_{L^\infty(D)} \to 0 \) as \( \sigma \to 0 \), for any \( \varepsilon > 0 \), there exists \( \sigma_* > 0 \) such that

\[ \|u_\sigma\|_{L^\infty(D)} < \varepsilon \quad \text{for all} \quad \sigma \in (0, \sigma_*). \]

Choose \( \varepsilon \) sufficiently small so that

\[ 2 \left\| K(x) \right\|_{L^\infty(D)} \lambda_1^{-1}(D) \varepsilon^{q-1} = \frac{1}{2}, \]

that is,

\[ \varepsilon = \left[ \frac{1}{4} \lambda_1(D) \left\| K(x) \right\|_{L^\infty(D)}^{-1} \right]^{\frac{1}{q-1}}. \]

Then there exists \( \sigma_* > 0 \) such that for all \( \sigma \in (0, \sigma_*] \),

\[ \int_D |\nabla V_\sigma|^2 + a(x)V_\sigma^2 \, dx \leq \frac{2}{\lambda_1(D)} \left\| e^{-\frac{q}{2}S} \right\|_{L^2(D)}^2. \]

Using (1.19) again gives for all \( \sigma \in (0, \sigma_0) \),

\[ \int_D |\nabla V_\sigma|^2 \, dx \leq \frac{1}{\lambda_1^*(D)} \int_D |\nabla V_\sigma|^2 + a(x)V_\sigma^2 \, dx \leq \frac{2}{\lambda_1^*(D)\lambda_1(D)} \left\| e^{-\frac{q}{2}S} \right\|_{L^2(D)}^2. \quad (2.18) \]

Finally, Lemma 2.4 follows from (2.18) by a standard bootstrapping argument. \( \square \)
3. Existence when \( f(x) \in \mathcal{U} \)

In this section, we give the existence and nonexistence of positive solutions to problem (1.11) for \( f(x) \in \mathcal{U} \) and \( q > 1 \). For this, we need the following lemmas.

**Lemma 3.1.** For \( f(x) \in \mathcal{U} \), there exists a positive constant \( \sigma^* > 0 \) such that problem (1.11) has at least one positive solution for all \( q > 1 \) and \( \sigma \in (0, \sigma^*) \).

**Proof.** Let \( \phi(x) \) be the nonnegative solution of (1.13) and set \( V = \sigma \phi(x) \). Then we have

\[
-\Delta V + a(x)V - K(x)V^q + \sigma f(x)e^{-\frac{\alpha}{2}S} = \left(-\Delta \phi + a(x)\phi + f(x)e^{-\frac{\alpha}{2}S}\right)\sigma - \sigma^q K(x)\phi^q = -\sigma^q K(x)\phi^q \leq 0.
\]

Hence, \( V \) is a subsolution of (1.11) for all \( \sigma > 0 \). By subsupersolution method, it is sufficient to find a supersolution \( W_0(x) \) of (1.11) with property \( W_0(x) \geq V(x) \) for \( x \in D \). To this end, let \( u_1(x) \) be the solution of the problem

\[
\begin{cases}
-\Delta u_1(x) + a(x)u_1(x) = 1, & x \in D, \\
u_1(x)|_{\partial D} = 0.
\end{cases}
\]

Then by the strong maximum principle, we have \( u_1 > 0 \) for all \( x \in D \). Set \( W(x) = M u_1(x) \), then

\[
-\Delta W + a(x)W - K(x)W^q + \sigma f(x)e^{-\frac{\alpha}{2}S} = \left(-\Delta u_1 + a(x)u_1\right)M - M^q K(x)u_1^q + \sigma f(x)e^{-\frac{\alpha}{2}S} = M - M^q K(x)u_1^q + \sigma f(x)e^{-\frac{\alpha}{2}S}.
\]

By choosing \( M = M_0 > 0 \) satisfying

\[
M_0 \geq M_0^q \max_{x \in \bar{D}} K(x)u_1^q(x) + M_0^q \max_{x \in \bar{D}} f(x)e^{-\frac{\alpha}{2}S},
\]

when \( \sigma \leq M_0^q \), we have

\[
-\Delta W_0 + a(x)W_0 - K(x)W_0^q + \sigma f(x)e^{-\frac{\alpha}{2}S} \geq M_0^q \max_{x \in \bar{D}} f(x)e^{-\frac{\alpha}{2}S} + \sigma f(x)e^{-\frac{\alpha}{2}S} \geq 0,
\]

where \( W_0 = M_0 u_1(x) \). This implies that \( W_0 \) is a supersolution of problem (1.11). Moreover, if we choose \( \sigma_0 \) so small that \( \sigma_0 \phi(x) \leq M_0 u_1(x) \). The subsupersolution method implies that problem (1.11) has a solution \( u(x) \) satisfying \( V \leq u(x) \leq W_0 \), for all \( 0 < \sigma \leq \sigma_0 \). Moreover, \( u(x) > 0 \) in \( D \). In fact, \( w(x) = u(x) - V(x) = u(x) - \sigma \phi(x) \neq 0 \), satisfying

\[
\begin{cases}
-\Delta w(x) + a(x)w(x) = K(x)u^q \geq 0, \\
w|_{\partial D} = 0.
\end{cases}
\]
Hence, the strong maximum principle guarantees that \( w > 0 \) in \( D \), that is, \( u(x) > V(x) \geq 0 \). And this completes the proof of the lemma by defining
\[
\sigma_* = \sup\{\sigma_0 \in \mathbb{R}^+: \text{problem (1.11) has at least one solution for each } \sigma \in (0, \sigma_0)\}.
\]  
\( \square \) (3.2)

The following lemma shows that \( \sigma_* \) is bounded.

**Lemma 3.2.** Assume \( f(x) \in \mathcal{U} \), and \( q > 1 \). There exists a constant \( C_1 > 0 \) such that problem (1.11) has no positive solution if \( \sigma > C_1 \). Moreover, \( C_1 \) can be chosen as
\[
C_1 = \frac{(q - 1)|D^*|K^*_{\star}^{\frac{q}{q - 1}}}{\int_D f(x)e^{-\frac{q}{2}S\varphi_1}dx},
\]
where \( K^* = \min_{x \in \bar{D}} K(x) \).

**Proof.** We denote by \( \lambda_1(D) \) the first eigenvalue of the eigenvalue problem
\[
\begin{align*}
-\Delta \varphi + a(x)\varphi &= \lambda \varphi, \quad x \in D, \\
\varphi|_{\partial D} &= 0,
\end{align*}
\]  
with the corresponding eigenfunction denoted by \( \varphi_1(x) > 0 \) in \( D \). If (1.11) has a positive solution \( u_\sigma \), then
\[
\int_D (-\Delta u_\sigma \varphi_1 + a(x)u_\sigma \varphi_1)dx = \int_D K(x)u_\sigma^q \varphi_1 dx - \sigma \int_D f(x)e^{-\frac{q}{2}S\varphi_1}dx.
\]  
(3.5)
Thus
\[
\int_D \lambda_1(D)\varphi_1 u_\sigma dx = \int_D K(x)u_\sigma^q \varphi_1 dx - \sigma \int_D f(x)e^{-\frac{q}{2}S\varphi_1}dx, \quad \text{i.e.,}
\]  
(3.6)
\[
\sigma \int_D f(x)e^{-\frac{q}{2}S\varphi_1}dx \geq \int_D (K^* u_\sigma^q - \lambda_1(D)u_\sigma) \varphi_1 dx.
\]  
(3.7)
Since a lower bound of \( K^* u_\sigma^q - \lambda_1(D)u_\sigma \) is given by
\[
K^* u_\sigma^q - \lambda_1(D)u_\sigma \geq (1 - q)K^* \left[ \frac{\lambda_1(D)}{qK^*} \right]^{\frac{q}{q - 1}},
\]
(3.7) implies that
\[
\sigma \int_D f(x)e^{-\frac{q}{2}S\varphi_1}dx \geq (1 - q) \int_D K^* \left[ \frac{\lambda_1(D)}{qK^*} \right]^{\frac{q}{q - 1}} \varphi_1 dx.
\]  
(3.8)
By noticing \( \phi \) is a nonnegative solution of (1.13), we have

\[
- \int_D f(x)e^{-\frac{q}{2}S}\varphi_1\,dx = \int_D (-\Delta \phi + a(x)\phi)\varphi_1\,dx = \int_D (-\Delta \varphi_1 + a(x)\varphi_1)\phi\,dx = \int_D \lambda_1(D)\varphi_1\phi\,dx \geq 0.
\]

(3.9)

From (3.8), we obtain

\[
\sigma \leq \frac{(q-1)|D|K_*[\tilde{\lambda}_2(D)]^\frac{q-1}{q} - \int_D f(x)e^{-\frac{q}{2}S}\varphi_1\,dx}{-\int_D f(x)e^{-\frac{q}{2}S}\varphi_1\,dx} \Delta C_1 < +\infty.
\]

This completes the proof of lemma. \( \square \)

The next lemma shows that \( \sigma_* \) given in Lemma 3.1 can be chosen as a threshold for existence.

**Lemma 3.3.** Assume \( f(x) \in U \) and \( q > 1 \). Problem (1.11) has a positive solution for all \( \sigma \in (0, \Lambda) \) if it has a positive solution when \( \sigma = \Lambda > 0 \).

**Proof.** Let \( u_1(x) \) be a positive solution of (1.11) for \( \sigma = \Lambda \). Set \( w_1 = \Lambda u_1 \). Then \( w_1 \) satisfies

\[
\begin{cases}
-\Delta w_1 + a(x)w_1 = \Lambda^{q-1}K(x)w_1^q - f(x)e^{-\frac{q}{2}S} & \text{in } D, \\
w_1|_{\partial D} = 0, \quad w_1 > 0 & \text{in } D.
\end{cases}
\]

Hence, for any \( 0 < \sigma \leq \Lambda \), \( w_1 \) is a supersolution of the problem

(3.10)

On the other hand, since \( f(x) \in U \), problem (1.13) has a nonnegative solution \( \varphi(x) \) satisfying

\[
\begin{cases}
-\Delta \varphi + a(x)\varphi = -f(x)e^{-\frac{q}{2}S} \leq \sigma^{q-1}K(x)\varphi^q - f(x)e^{-\frac{q}{2}S} & \text{in } D, \\
\varphi(x)|_{\partial D} = 0, \quad \varphi(x) \geq 0 & \text{in } D.
\end{cases}
\]

This implies that \( \varphi(x) \) is a subsolution of (3.10). By the comparison principle, we have

\[
0 \leq \varphi(x) \leq w_1(x) \quad \text{for all } x \in D.
\]

Therefore, there exists a positive solution \( w_\sigma(x) \) of (3.10) for all \( \sigma \in (0, \Lambda] \). Obviously, \( u_\sigma = \sigma w_\sigma(x) \) is a positive solution of problem (1.11). This completes the proof of Lemma 3.3. \( \square \)

By using the above lemma, we have the following theorem.
Theorem 3.4. Assume $f(x) \in \mathcal{U}$ and $q > 1$. There exists a constant $0 < \sigma_f < +\infty$, such that problem (1.11) has at least one positive solution when $\sigma \in (0, \sigma_f)$, while has no solution when $\sigma > \sigma_f$.

Proof. Since $f(x) \in \mathcal{U}$, Lemma 3.1 implies that there exists a positive constant $\sigma_*$ such that problem (1.11) has at least one positive solution for $\sigma \in (0, \sigma_*)$. Let

$$
\sigma_f = \sup \{ \sigma > 0 : \text{problem (1.11) has a positive solution} \}.
$$

It follows immediately from Lemmas 3.2 and 3.3 that there exists $0 < \sigma_f < +\infty$ such that (1.11) has at least one solution for all $\sigma \in (0, \sigma_f)$. The definition of $\sigma_f$ also implies that (1.11) has no solution when $\sigma > \sigma_f$. \hfill \Box

Corollary 3.5. Assume $f(x) \in \mathcal{U}$ and $q > 1$. Problem (1.11) has a minimal positive solution $u_\sigma$ for $\sigma \in (0, \sigma_f)$ satisfying

(i) $u_\sigma \geq \sigma \varphi(x)$, for all $x \in D$ and $\sigma \in (0, \sigma_f)$,

(ii) $u_\sigma(x)$ is increasing with respect to $\sigma$ for all $x \in D$.

Here $\varphi(x)$ is the nonnegative solution of (1.13).

Proof. Set $u_\sigma(x) = \sigma v_\sigma(x)$. Then $v_\sigma(x)$ satisfies

$$
\begin{cases}
-\Delta v_\sigma + a(x)v_\sigma = \sigma^{q-1}v_\sigma^q - f(x)e^{-\frac{q}{2}S} & \text{in } D, \\
v_\sigma|_{\partial D} = 0, \quad v_\sigma \geq 0 & \text{in } D.
\end{cases}
$$

(3.11)

Since $\varphi(x)$ is the nonnegative solution of (1.13), we have

$$
\begin{cases}
-\Delta \varphi + a(x)\varphi = -f(x)e^{-\frac{q}{2}S} & \text{in } D, \\
\varphi|_{\partial D} = 0, \quad \varphi \geq 0 & \text{in } D.
\end{cases}
$$

Set $w = v_\sigma - \varphi$, we have

$$
\begin{cases}
-\Delta w + a(x)w = \sigma^{q-1}v_\sigma^q \geq 0, \\
w|_{\partial D} = 0.
\end{cases}
$$

By maximum principle, we have $w = v_\sigma - \varphi \geq 0$.

Notice that $\sigma \varphi$ is a subsolution of (1.11), and all nonnegative supersolution of (1.11) must be large than or equal to $\sigma \varphi(x)$. Hence, we can find a minimal solution of (1.11) by a monotone iteration starting from $\sigma \varphi(x)$. To prove the second statement, we first let $\sigma_1, \sigma_2 \in (0, \sigma_f)$ with $\sigma_1 < \sigma_2$. Then the corresponding minimal solutions of (1.11) are $u_{\sigma_1}$ and $u_{\sigma_2}$. Set $u_{\sigma_1} = \sigma_1 w_{\sigma_1}$, $u_{\sigma_2} = \sigma_2 w_{\sigma_2}$. Then $w_{\sigma_2}$ must be a supersolution and $\varphi(x)$ is the subsolution of (3.10) when $\sigma = \sigma_1$. By the monotone iteration again, we have $\varphi \leq w_{\sigma_1} \leq w_{\sigma_2}$, that is, $\sigma \varphi \leq u_{\sigma_1} \leq u_{\sigma_2}$. \hfill \Box

Remark 3.1. There is no solution to (1.11) for all $\sigma > 0$ when $\lambda_1(D) \leq 0$ and $f(x) \leq 0$ in $D$. Here $\lambda_1(D)$ is the first eigenvalue of $-\Delta + a(x)$ with zero Dirichlet boundary condition.
In fact, if \( u \) is a solution to (1.11), then

\[
- \Delta u + a(x)u = K(x)u^q - \sigma f(x)e^{-\frac{\alpha}{2}S}
\]

implies

\[
\int_D \lambda_1(D) \varphi_1 u
dx = \int_D K(x)u^q \varphi_1
dx - \sigma \int_D f(x)e^{-\frac{\alpha}{2}S} \varphi_1
dx > 0,
\]

where \( \varphi_1(x) \) is the first eigenfunction corresponding to \( \lambda_1(D) \).

The following lemma concerns an important property of the linearized equation of (1.11) around the minimal solution \( u_\sigma \). It will be used to obtain the second solution of (1.11) when \( \sigma \in (0, \sigma_f) \) and \( 1 < q < 5 \).

**Lemma 3.6.** Assume \( f(x) \in \mathcal{U} \) and \( 1 < q < 5 \), and \( u_\sigma \) is the minimal solution of (1.11) for \( \sigma \in (0, \sigma_f) \). Then the first eigenvalue of the following problem

\[
\begin{aligned}
- \Delta \delta + a(x)\delta - q K(x)u_\sigma^{q-1}\delta &= \beta \delta \quad \text{in } D, \\
\delta|_{\partial D} &= 0,
\end{aligned}
\]

(3.12)

denoted by \( \beta_1 \), is positive. And the corresponding eigenfunction denoted by \( \delta_1(x) \) is also positive in \( D \).

**Proof.** Define

\[
\beta_1 = \inf \left\{ \int_D |\nabla \delta|^2 + a(x)\delta^2 - q K(x)u_\sigma^{q-1}\delta^2
dx : \delta(x) \in H^1_0(D), \int_D |\delta|^2
dx = 1 \right\}.
\]

(3.13)

By standard variational method, the minimum \( \beta_1 \) is attained with some function \( \delta_1(x) \geq 0 \) in \( D \) when \( 1 < q < 5 \). Thus (3.12) has a solution \( (\beta_1, \delta_1(x)) \) and we only need to show that \( \beta_1 > 0 \).

Choose \( \bar{\sigma} > \sigma > 0 \) with \( \bar{\sigma}, \sigma \in (0, \sigma_f) \). Denote corresponding minimal solutions to \( \sigma \) and \( \bar{\sigma} \) of (1.11) by \( u_\sigma \) and \( u_{\bar{\sigma}} \). By Corollary 3.5, we have \( u_\sigma \leq u_{\bar{\sigma}} \). Then (1.11) implies

\[
- \Delta (u_{\bar{\sigma}} - u_\sigma) + a(x)(u_{\bar{\sigma}} - u_\sigma) = K(x)(u_{\bar{\sigma}}^q - u_\sigma^q) + (\sigma - \bar{\sigma}) f(x)e^{-\frac{\alpha}{2}S}
\geq q K(x)u_\sigma^{q-1}(u_{\bar{\sigma}} - u_\sigma) + (\sigma - \bar{\sigma}) f(x)e^{-\frac{\alpha}{2}S}.
\]

(3.14)

By multiplying (3.14) by \( \delta_1(x) \) and using (3.12), we have

\[
\beta_1 \int_D (u_{\bar{\sigma}} - u_\sigma) \delta_1(x)
dx > (\sigma - \bar{\sigma}) \int_D f(x)e^{-\frac{\alpha}{2}S} \delta_1
dx.
\]

(3.15)

By multiplying (3.14) by \( \delta_1(x) \) and using (3.12), we have

\[
\begin{aligned}
- \Delta \varphi + a(x)\varphi &= - f(x)e^{-\frac{\alpha}{2}S} \quad \text{in } D, \\
\varphi|_{\partial D} &= 0, \quad \varphi \geq 0 \quad \text{in } D,
\end{aligned}
\]

(3.16)
has a solution \( \varphi \geq 0 \) in \( D \). Multiplying the equation in (3.16) by \( \delta_1(x) \) and integrating the product over \( D \) give

\[
- \int_D f(x)e^{-\frac{\alpha}{2}S} \delta_1(x) \, dx = \int_D (-\Delta \varphi + a(x)\varphi) \delta_1(x) \, dx = \int_D (-\Delta \delta_1 + a(x)\delta_1)\varphi(x) \, dx
\]

\[
= q \int_D K(x)u_{\sigma}^{q-1}\delta_1\varphi \, dx + \beta_1 \int_D \delta_1(x)\varphi \, dx.
\]

By substituting this into (3.15), we have

\[
\beta_1 \int_D (u_\sigma - u_\sigma) \delta_1(x) \, dx > (\bar{\sigma} - \sigma) \left[ q \int_D K(x)u_{\sigma}^{q-1}\delta_1\varphi \, dx + \beta_1 \int_D \delta_1(x)\varphi \, dx \right].
\]

Thus

\[
\beta_1 \int_D (u_\sigma - u_\sigma - (\bar{\sigma} - \sigma)\varphi) \delta_1(x) \, dx > 0, \quad \text{i.e.,}
\]

\[
\beta_1 \int_D [(u_\sigma - \bar{\sigma}\varphi) - (u_\sigma - \sigma\varphi)] \delta_1(x) \, dx > 0.
\]

(3.17)

Denote \( w_\sigma = u_\sigma - \bar{\sigma}\varphi, \ w_\sigma = u_\sigma - \sigma\varphi. \) Then (1.11) and (3.16) imply

\[
\begin{cases}
-\Delta w_\sigma + a(x)w_\sigma = K(x)u_{\sigma}^q \quad \text{in } D, \\
w_\sigma|_{\partial D} = 0, \quad w_\sigma \geq 0 \quad \text{in } D,
\end{cases}
\]

(3.18)

\[
\begin{cases}
-\Delta w_\sigma + a(x)w_\sigma = K(x)u_{\sigma}^q \quad \text{in } D, \\
w_\sigma|_{\partial D} = 0, \quad w_\sigma \geq 0 \quad \text{in } D.
\end{cases}
\]

(3.19)

Since \( u_\sigma \geq u_\sigma \) when \( \bar{\sigma} > \sigma, \) (3.18) and (3.19) give

\[
\begin{cases}
-\Delta (w_\sigma - w_\sigma) + a(x)(w_\sigma - w_\sigma) = K(x)(u_{\sigma}^q - u_{\sigma}^q) \geq 0 \quad \text{in } D, \\
(w_\sigma - w_\sigma)|_{\partial D} = 0.
\end{cases}
\]

Finally, the maximum principle implies that \( w_\sigma - w_\sigma \geq 0 \) and in \( D. \) Hence, (3.17) yields that \( \beta_1 > 0. \) Moreover, (3.12) and the strong maximum principle give \( \delta_1(x) > 0 \) in \( D. \) \( \square \)

The following theorem is about the existence and uniqueness of the positive solution of (1.11) when \( \sigma = \sigma_f. \)

**Theorem 3.7.** Assume \( f(x) \in \mathcal{U} \) and \( 1 < q < 5. \) Then problem (1.11) has a unique solution when \( \sigma = \sigma_f. \)
Proof. Let
\[ A = \{ u_\sigma : \sigma \in (0, \sigma_f), \ u_\sigma \text{ is the minimal solution of (1.11)} \} . \]

Then there exists a positive constant independent of \( \sigma \in (0, \sigma_f) \) such that
\[ \| u_\sigma \|_{H^2_0(D)} \leq C \quad \text{for all } u_\sigma \in A. \quad (3.20) \]

In fact, for any \( u_\sigma \in A \), Lemma 3.4 implies that
\[ \int_D |\nabla u_\sigma|^2 + a(x)u_\sigma^2 \, dx - \int_D q K(x)u_\sigma^{q+1} \, dx \geq \beta_1 \int_D u_\sigma^2 \, dx. \quad (3.21) \]

By (1.11) and (1.18), we have
\[ \int_D |\nabla u_\sigma|^2 + a(x)u_\sigma^2 \, dx - \int_D K(x)u_\sigma^{q+1} \, dx + \sigma \int_D f(x)e^{-\frac{\alpha}{2}S}u_\sigma \, dx = 0 \quad \text{and} \quad (3.22) \]
\[ \int_D |\nabla u_\sigma|^2 \, dx + \int_D a(x)u_\sigma^2 \, dx \geq \lambda_1(D) \int_D u_\sigma^2 \, dx. \quad (3.23) \]

By (3.21)–(3.23), for any constant \( \delta > 0 \), we obtain
\[ (q - 1)\lambda_1(D) \int_D u_\sigma^2 \, dx \leq -\sigma q \int_D f(x)e^{-\frac{\alpha}{2}S}u_\sigma \, dx \leq \sigma f q \| f(x)e^{-\frac{\alpha}{2}S}\|_{L^2(D)} \| u_\sigma \|_{L^2(D)} \]
\[ \leq \sigma f q \cdot \frac{\delta}{2} \| u_\sigma \|_{L^2(D)}^2 + \sigma f q \cdot \frac{1}{2\delta} \| f e^{-\frac{\alpha}{2}S}\|_{L^2(D)}^2. \]

By choosing \( \delta \) sufficiently small so that
\[ (q - 1)\lambda_1(D) - \sigma f q \cdot \frac{\delta}{2} = \frac{1}{2}, \]
we have
\[ \int_D u_\sigma^2 \, dx \leq \frac{\sigma f q}{\delta} \| f e^{-\frac{\alpha}{2}S}\|_{L^2(D)}^2 \leq C. \quad (3.24) \]

Hence,
\[ (q - 1) \int_D |\nabla u_\sigma|^2 + a(x)u_\sigma^2 \, dx \leq \sigma f q \int_D |f(x)|e^{-\frac{\alpha}{2}S}u_\sigma \, dx. \]

And then (3.24) gives
\[ \int_D |\nabla u_\sigma|^2 \, dx \leq \int_D |a(x)|u_\sigma^2 \, dx + \sigma f q \| u_\sigma \|_{L^2(D)} \| f(x)e^{-\frac{\alpha}{2}S}\|_{L^2(D)} \leq C, \]
where \( C \) is a constant independent of \( \sigma \in (0, \sigma_f) \).
We now turn to the existence of solution for (1.11) when $\sigma = \sigma_f$. Suppose \{\sigma_j\}_{j \geq 1} is an increasing sequence in $(0, \sigma_f)$ satisfying $\lim_{j \to \infty} \sigma_j = \sigma_f$. The corresponding sequence of solutions is denoted by $\{u_{\sigma_j}\} \subset A$. By (3.20), we can choose a subsequence, still denote by $\{u_{\sigma_j}\}$, such that

$$u_{\sigma_j} \rightharpoonup \bar{u}, \quad \text{weakly in } H_0^1(D),$$

here $\bar{u} \in H_0^1(D)$ is a nonnegative function. Since $1 < q < 5$, it is standard to show that $\bar{u}$ is a weak solution of (1.11) with $\sigma = \sigma_f$. Since $\sigma \phi$ is always a subsolution of (1.11) for $\sigma > 0$, we can then find the minimal solution by the monotone iteration. Here $\phi$ is the solution of (3.16).

To prove the uniqueness of the solution of (1.11) when $\sigma = \sigma_f$, first, notice that $(\sigma_f, u_{\sigma_f})$ is a bifurcation point for the mapping:

$$F : \mathbb{R}^1 \times H_0^1(D) \longrightarrow H_0^1(D),$$

$$F(\sigma, u) = -\Delta u + a(x)u - K(x)u^q + \sigma_f(x)e^{-\frac{\alpha}{2}S}.$$

It is easy to verify that the first eigenvalue of (3.12) is 0 when $u_{\sigma} = u_{\sigma_f}$. Assume that there are two different solutions $u_1$ and $u_2$ of (1.11) when $\sigma = \sigma_f$. Without loss of generality, let $u_1$ be the minimal solution so that $u_2 \geq u_1 > 0$ in $D$. Denote $w = u_2 - u_1 \geq 0$. We have

$$-\Delta w + a(x)w = K(x)(u_2^q - u_1^q). \quad (3.25)$$

Multiplying (3.25) by $\delta_1(x)$ and integrating the product over $D$ give

$$\int_D (-\Delta w + a(x)w)\delta_1(x)\,dx = \int_D K(x)(u_2^q - u_1^q)\delta_1\,dx. \quad (3.26)$$

By (3.12) and using $\beta_1 = 0$, we have

$$q \int_D K(x)u_1^{q-1}\delta_1(x)w\,dx = \int_D K(x)(u_2^q - u_1^q)\delta_1(x)\,dx. \quad (3.27)$$

Hence, the Taylor expansion yields

$$\int_D K(x)q(q - 1)(\xi(x))^{q-2}(u_1 - u_2)^2\,dx = 0, \quad (3.28)$$

where $\xi(x)$ is a function between $u_1$ and $u_2$. Then (3.28) implies that $u_1 \equiv u_2$ because $K(x) \geq 0$, $\xi(x) \geq 0$ and $q > 1$ which is a contradiction to the assumption. This completes the proof of the theorem. $\Box$

Finally, by combining Theorems 3.4, 3.7 and Corollary 3.5, we have Theorem 1.
4. Second solution for \( f(x) \in \mathcal{U} \)

In this section, we study the existence of multiple solutions of problem (1.11) when \( f \in \mathcal{U} \), \( \sigma \in (0, \sigma_f) \) and \( 1 < q < 5 \).

Let \( u_\sigma \) be the minimal solution of (1.11) for \( \sigma \in (0, \sigma_f) \). In order to find the second solution of (1.11), we consider the problem

\[
\begin{cases}
-\Delta v + a(x)v = K(x)[(v + u_\sigma)^q - u_\sigma^q] & \text{in } D, \\
v|_{\partial D} = 0, \quad v > 0 & \text{in } D.
\end{cases}
\] (4.1)

Clearly, if (4.1) has a solution \( v_\sigma \), then \( U_\sigma = u_\sigma + v_\sigma \) is the second solution of (1.11). The existence of solution to (4.1) can be proved by mountain pass theorem, cf. [1], as follows.

**Theorem 4.1.** Assume \( f(x) \in \mathcal{U} \) and \( 1 < q < 5 \). Then problem (4.1) has at least one solution for \( \sigma \in (0, \sigma_f) \).

**Proof.** Set

\[
g(x,v) = K(x)[(v + u_\sigma)^q - u_\sigma^q - qu_\sigma^{q-1}v].
\] (4.2)

Then problem (4.1) can be rewritten as

\[
\begin{cases}
-\Delta v + (a(x) - qK(x)u_\sigma^{q-1})v = g(x,v) & \text{in } D, \\
v|_{\partial D} = 0, \quad v > 0 & \text{in } D.
\end{cases}
\] (4.3)

Let \( G(x,v) = \int_0^v g(x,t) \, dt \). From Lemma 3.6 and (4.2), we have

(i) For all \( v \in H_0^1(D) \),

\[
\int_D |\nabla v|^2 + (a(x) - qK(x)u_\sigma^{q-1})v^2 \, dx \geq \beta_1 \int_D v^2 \, dx.
\] (4.4)

(ii) \( \lim_{v \to 0} \frac{g(x,v)}{v} = 0 \), \( \lim_{v \to \infty} \frac{g(x,v)}{v^5} = 0 \). (4.5)

(iii) \( \frac{1}{2}tg(x,t) \geq G(x,t) \) for \( x \in D, t \in \mathbb{R}^+ \) and \( q \geq 1 \).

Denote the variational functional for (4.1) by

\[
J(v) = \frac{1}{2} \int_D |\nabla v|^2 + (a(x) - qK(x)u_\sigma^{q-1})v^2 \, dx - \int_D G(x,v) \, dx, \quad v \in H_0^1(D). \] (4.6)

Since \( q \in (1, 5) \) is subcritical, it is straightforward to show that the functional \( J(v) \) satisfies conditions in the mountain pass theorem, cf. [1,12]. This completes the proof of Theorem 4.1. \( \square \)
Define
\[ \Lambda = \left\{ u \in H^1_0(D) : \int_D |\nabla u|^2 + a(x)u^2 \, dx = \int_D K(x)u^{q+1} \, dx - \sigma \int_D f(x)e^{-\frac{\alpha}{2}Su} \, dx \right\}, \quad (4.7) \]
\[ \Lambda^+ = \left\{ u \in \Lambda : \int_D |\nabla u|^2 + a(x)u^2 \, dx - q \int_D K(x)u^{q+1} \, dx > 0 \right\}, \quad (4.8) \]
\[ \Lambda^0 = \left\{ u \in \Lambda : \int_D |\nabla u|^2 + a(x)u^2 \, dx - q \int_D K(x)u^{q+1} \, dx = 0 \right\}, \quad (4.9) \]
\[ \Lambda^- = \left\{ u \in \Lambda : \int_D |\nabla u|^2 + a(x)u^2 \, dx - q \int_D K(x)u^{q+1} \, dx < 0 \right\}. \quad (4.10) \]

Notice that the variational functional for (1.11) is
\[ I(u) = \frac{1}{2} \int_D |\nabla u|^2 + a(x)u^2 \, dx - \frac{1}{q+1} \int_D K(x)u^{q+1} \, dx + \sigma \int_D f(x)e^{-\frac{\alpha}{2}Su} \, dx, \quad (4.11) \]
for \( u \in H^1_0(D). \) If \( u_\sigma \) is a solution of (1.11) for \( \sigma \in (0, \sigma_f) \), then \( u_\sigma \in \Lambda. \)

**Proposition 4.2.** Assume \( f(x) \in U, \ 1 < q < 5 \) and \( \sigma \in (0, \sigma_f). \) Let \( u_\sigma \) be the minimal solution of (1.11). Then \( u_\sigma \in \Lambda^+ \) and \( u_\sigma \) is a local minimum of the functional \( I(u). \)

**Proof.** By Lemma 3.6, we have
\[ \int_D |\nabla u_\sigma|^2 + a(x)u_\sigma^2 \, dx - q \int_D K(x)u_\sigma^{q+1} \, dx \geq \beta_1 \int_D u_\sigma^2 \, dx > 0, \quad (4.12) \]
which implies \( u_\sigma \in \Lambda^+ \). Here \( \beta_1 \) is the first eigenvalue of (3.12). To prove that \( u_\sigma \) is a local minimum of the functional \( I(u), \) for \( w \in H^1_0(D), \) define
\[ \Phi(t) = I(u_\sigma + tw). \quad (4.13) \]
Since \( u_\sigma \) is a solution of (1.11), \( \Phi'(0) = 0. \) Moreover, by Lemma 3.4, we have \( \Phi''(0) > 0. \) Thus, \( \Phi(t) \) has its minimum at \( t = 0 \) which implies that \( I(u_\sigma) \) is a local minimum of \( I(u). \) \( \square \)

**Proposition 4.3.** Assume \( f(x) \in U, \ 1 < q < 5, \ \sigma \in (0, \sigma_f), \) and \( U_\sigma \) is the second solution of (1.11). Then the first eigenvalue of the eigenvalue problem
\[ \begin{cases} -\Delta \varphi + a(x)\varphi - qK(x)U_\sigma^{q-1}\varphi = \beta \varphi & \text{in } D, \\ \varphi|_{\partial D} = 0 \end{cases} \quad (4.14) \]
denoted by \( \beta_1^* \), is negative.
Proof. We prove this by contradiction. Assume $\beta_1^* \geq 0$, then for any $w \in H_0^1(D)$, we have

$$
\int_D |\nabla w|^2 + a(x)w^2 \, dx - q \int_D K(x)U_{\sigma}^{q-1}w^2 \, dx \geq 0. \tag{4.15}
$$

Since $u_{\sigma}$ is a local minimum of $I(u)$, and $U_{\sigma}$ is a solution of (1.11) obtained by the mountain pass theorem close to $u_{\sigma}$, the functional defined by

$$
\Psi(\lambda) = I((1-\lambda)u_{\sigma} + \lambda U_{\sigma}), \quad \lambda \in [0, 1], \tag{4.16}
$$

has its minimum at $\lambda = 0$ and maximum at $\lambda = 1$. That is,

$$
\Psi'(\lambda)|_{\lambda=1} = 0, \quad \Psi''(\lambda)|_{\lambda=1} < 0. \tag{4.17}
$$

Straightforward calculation gives

$$
\Psi''(\lambda) = \int_D |\nabla(U_{\sigma} - u_{\sigma})|^2 + a(x)(U_{\sigma} - u_{\sigma})^2 \, dx
$$

$$
- q \int_D K(x)[(1-\lambda)u_{\sigma} + \lambda U_{\sigma}]^{q-1}(U_{\sigma} - u_{\sigma})^2 \, dx.
$$

Thus by (4.15), we have

$$
\Psi''(1) = \int_D |\nabla(U_{\sigma} - u_{\sigma})|^2 + a(x)(U_{\sigma} - u_{\sigma})^2 \, dx - q \int_D K(x)U_{\sigma}^{q-1}(U_{\sigma} - u_{\sigma})^2 \, dx > 0.
$$

This contradicts to (4.17) which implies that $\beta_1^* < 0$. \qed

Remark 4.1. By using Ekeland’s variational principle, we can find the second solution $U_{\sigma}$ of (1.11) belongs to $\Lambda^-$, cf. [24].

Proof of Theorem 2. By Theorem 4.1, problem (4.1) has at least one solution $v_{\sigma} > 0$ in $D$. Let $U_{\sigma} = u_{\sigma} + v_{\sigma}$. Then $U_{\sigma}$ is the second solution of (1.11) with $U_{\sigma} > u_{\sigma}$ which gives part (i) in the theorem. The other two parts of Theorem 2 follow from Propositions 4.2 and 4.3. \qed

5. Existence when $f(x) \in F^+ \cap \mathcal{N}$

In this section, we will discuss the existence and nonexistence of solutions of (1.11) when the function $f(x) \in F^+ \cap \mathcal{N}$ and $1 < q < 5$.

Lemma 5.1. Under the assumptions of Theorem 3, there exist positive constants $\tilde{f}$, $R_0$ and $\alpha$ independent of $\sigma$ such that

$$
I_{\sigma}(u)|_{B_{\tilde{f}R_0}} \geq \alpha > 0 \quad \text{for} \quad \sigma \in (0, \tilde{f}). \tag{5.1}
$$

Here $I_{\sigma}(u)$ is the variational functional defined in (1.20).
Proof. Let $\lambda_1(D)$ be the first eigenvalue of $-\Delta + a(x)$ with zero Dirichlet boundary condition, and $\varphi_1(x)$ be the corresponding eigenfunction. Then by Hölder’s and Young’s inequalities and (1.18), we have

$$\left| \sigma \int_D f e^{-\frac{\sigma}{2} S} u \, dx \right| \leq \sigma \left\| f e^{-\frac{\sigma}{2} S} \right\|_{L^2(D)} \| u \|_{L^2(D)} \leq \sigma \left\| f e^{-\frac{\sigma}{2} S} \right\|_{L^2(D)} \cdot \frac{1}{\sqrt{\lambda_1(D)}} \left( \int_D |\nabla u|^2 + a(x)u^2 \, dx \right)^{\frac{1}{2}} \leq \frac{1}{4} \int_D |\nabla u|^2 + a(x)u^2 \, dx + \frac{\sigma^2}{\lambda_1(D)} \| f e^{-\frac{\sigma}{2} S} \|_{L^2(D)}^2.$$  

(5.2)

Notice that there exists a positive constant $C_1(D)$ independent of $\sigma$ such that

$$\| u^+ \|_{L^{q+1}(D)} \leq \| u \|_{L^{q+1}(D)} \leq C_1(D) \| \nabla u \|_{L^2(D)},$$  

(5.3)

By (5.1), (5.2) and (1.18), we obtain

$$I_\sigma (u) \geq \frac{1}{4} \int_D |\nabla u|^2 + a(x)u^2 \, dx - \frac{C_1(D)}{q + 1} \| K(x) \|_{L^\infty} \| \nabla u \|_{L^{q+1}(D)} + \frac{\sigma^2}{\lambda_1(D)} \| f e^{-\frac{\sigma}{2} S} \|_{L^2(D)}^2 \geq \frac{\lambda_1(D)}{4} \| \nabla u \|_{L^2(D)}^2 - \frac{1}{q + 1} C_1(D) \| K(x) \|_{L^\infty} \| \nabla u \|_{L^{q+1}(D)} + \frac{\sigma^2}{\lambda_1(D)} \| f e^{-\frac{\sigma}{2} S} \|_{L^2(D)}^2.$$  

This implies that

$$I_\sigma (u) \big|_{\partial B_R} \geq \frac{\lambda_1(D)}{4} R^2 - \frac{1}{q + 1} C_1(D) \| K(x) \|_{L^\infty} R^{q+1} + \frac{\sigma^2}{\lambda_1(D)} \| f e^{-\frac{\sigma}{2} S} \|_{L^2(D)}^2.$$  

Choose $R_0 > 0$ satisfying

$$\frac{\lambda_1(D)}{4} R_0^2 - \frac{1}{q + 1} C_1(D) \| K(x) \|_{L^\infty} R_0^{q+1} = \frac{\lambda_1(D)}{8} R_0^2, \quad \text{i.e.,} \quad R_0 = \left( \frac{\lambda_1(D)(q + 1)}{8C_1(D)\| K \|_{L^\infty}} \right)^{\frac{1}{q-1}}.$$  

(5.4)

We have

$$I_\sigma \big|_{\partial B_{R_0}} \geq \frac{1}{8} \left[ \frac{\lambda_1(D)(q + 1)}{8C_1(D)\| K \|_{L^\infty}} \right]^{\frac{2}{q-1}} - \frac{\sigma^2}{\lambda_1(D)} \| f e^{-\frac{\sigma}{2} S} \|_{L^2(D)}^2.$$  

(5.6)

Now choose $\tilde{\sigma}_f > 0$ as

$$\tilde{\sigma}_f = \frac{1}{4} \left[ \frac{\lambda_1(D)(q + 1)}{8C_1(D)\| K \|_{L^\infty}} \right]^{\frac{1}{q-1}} \left( \frac{\lambda_1(D)}{\| f e^{-\frac{\sigma}{2} S} \|_{L^2(D)}^2} \right)^{\frac{1}{2}}.$$  

(5.7)
Then, for $\sigma \in (0, \sigma(f))$, we have
\[
I_{\sigma}(u)|_{\partial B_{R_0}} \geq \frac{1}{16} \left[ \frac{\lambda_1(D)(q+1)}{8C_1(D)\|K\|_{L^\infty(D)}} \right]^{\frac{2}{q-1}} = \alpha > 0. \quad \square
\]

**Lemma 5.2.** Under the assumptions of Theorem 3, there exists a constant $t_0 > 0$ independent of $\sigma$ such that for $\sigma \in (0, \bar{\sigma}_f)$ and $t \geq t_0$, we have
\[
I_{\sigma}(t\varphi_1) < 0.
\]

Here, again $\varphi_1$ is the first eigenfunction of $-\Delta + a(x)$ with zero Dirichlet boundary condition.

**Proof.** Since $\varphi_1$ is the first eigenfunction of $-\Delta + a(x)$ with zero Dirichlet boundary condition, we have
\[
I_{\sigma}(t\varphi_1) = \frac{t^2}{2} \int_D |\nabla \varphi_1|^2 + a(x)\varphi_1^2 \, dx - \frac{t^{q+1}}{q+1} \int_D K(x)\varphi_1^{q+1} \, dx + \sigma t \int_D f(x)e^{-\frac{\alpha}{2}S} \varphi_1 \, dx
\]
\[
\leq \frac{t^2}{2} \int_D |\nabla \varphi_1|^2 + a(x)\varphi_1^2 \, dx - \frac{t^{q+1}}{q+1} \int_D K(x)\varphi_1^{q+1} \, dx + \bar{\sigma}_f t \left\| f e^{-\frac{\alpha}{2}S} \right\|_{L^2} \|\varphi_1\|_{L^2}
\]
\[
\rightarrow -\infty, \quad \text{as } t \rightarrow \infty.
\]

Thus, there exists $t_0 > 0$ such that
\[
I_{\sigma}(t\varphi_1) < 0 \quad \text{for } \sigma \in (0, \bar{\sigma}_f) \text{ and } t \geq t_0. \quad \square
\]

We are now ready to prove Theorem 3.

**Proof of Theorem 3.** By Lemmas 5.1, 5.2 and the mountain pass theorem without the (PS) condition from [3], up to a subsequence, there is a strong (PS) sequence of $I_{\sigma}$ which strongly converges in $H^1_0(D)$ for $q \in (1, 5)$. Denote the limit function by $u_{\sigma}$. Then $u_{\sigma}$ satisfies
\[
\begin{align*}
-\Delta u_{\sigma} + a(x)u_{\sigma} &= (u_{\sigma}^+)^q - \sigma f(x)e^{-\frac{\alpha}{2}S}, \quad x \in D, \\
u_{\sigma}|_{\partial D} &= 0,
\end{align*}
\]
and
\[
I_{\sigma}(u_{\sigma}) = c_{\sigma} \geq \alpha > 0, \quad I'_{\sigma}(u_{\sigma}) = 0,
\]
where
\[
c_{\sigma} = \inf_{\gamma \in \Gamma, t \in [0,1]} \sup_{t \in [0,1]} I_{\sigma}(\gamma(t)) \quad \text{and}
\]
\[
\Gamma = \{ \gamma \in C([0,1], H^1_0(D)) : \gamma(0) = 0, \gamma(1) = t_0\varphi_1 \}.
\]

To prove positivity of $u_{\sigma}$, we first show that there exists a positive constant $C$ independent of $\sigma$ such that
\[
\|u_{\sigma}\|_{C^{2,\nu}(D)} \leq C \quad \text{for } \sigma \in (0, \bar{\sigma}_f) \text{ and } \nu \in (0, 1). \quad (5.10)
\]
In fact, since \( I_{\sigma}(t\varphi_1) < 0 \) for \( \sigma \in (0, \bar{\sigma}_f) \) and \( t \geq t_0 \), we have

\[
\sup_{t \geq 0} I_{\sigma}(t\varphi_1) \leq \max_{t \in [0, t_0]} I_{\sigma}(t\varphi_1).
\]

Therefore, there exists a constant \( C > 0 \) independent of \( \sigma \) such that

\[
c_\sigma \leq \max_{t \in [0, t_0]} I_{\sigma}(t\varphi_1) \leq C \quad \text{for all } \sigma \in (0, \bar{\sigma}_f).
\]

By (5.9), we have

\[
\left( \frac{1}{2} - \frac{1}{q+1} \right) \int_D \left| \nabla u_\sigma \right|^2 + a(x)u_\sigma^2 \, dx = c_\sigma + \left( 1 - \frac{1}{q+1} \right) \sigma \int_D f e^{-\frac{q}{2}S} u_\sigma \, dx. \tag{5.11}
\]

Moreover, by Hölder’s and Young’s inequalities and (1.18), we have

\[
\left| \left( 1 - \frac{1}{q+1} \right) \sigma \int_D f e^{-\frac{q}{2}S} u_\sigma \, dx \right| \\
\leq \bar{\sigma}_f \left( 1 - \frac{1}{q+1} \right) \| f e^{-\frac{q}{2}S} \|_{L^2(D)} \| u_\sigma \|_{L^2(D)} \leq \varepsilon \| u_\sigma \|_{L^2(D)}^2 + C_\varepsilon \| f e^{-\frac{q}{2}S} \|_{L^2(D)}^2 \\
\leq \varepsilon \lambda_1^{-1}(D) \int_D \left| \nabla u_\sigma \right|^2 + a(x)u_\sigma^2 \, dx + C_\varepsilon \| f e^{-\frac{q}{2}S} \|_{L^2(D)}^2.
\]

Substituting this into (5.11) yields

\[
\left( \frac{1}{2} - \frac{1}{q+1} - \varepsilon \lambda_1^{-1}(D) \right) \int_D \left| \nabla u_\sigma \right|^2 + a(x)u_\sigma^2 \, dx \leq C_\varepsilon \| f e^{-\frac{q}{2}S} \|_{L^2(D)}^2.
\]

By choosing \( \varepsilon \) sufficiently small so that

\[
\frac{1}{2} - \frac{1}{q+1} - \varepsilon \lambda_1^{-1}(D) = \delta > 0.
\]

Then by (1.19), we have

\[
\lambda_1^*(D) \int_D \left| \nabla u_\sigma \right|^2 \, dx \leq \int_D \left| \nabla u_\sigma \right|^2 + a(x)u_\sigma^2 \, dx \leq \frac{1}{8} C \| f e^{-\frac{q}{2}S} \|_{L^2(D)}^2.
\]

This implies

\[
\| u_\sigma \|_{H^1_0(D)} \leq C \quad \text{for all } \sigma \in (0, \bar{\sigma}_f),
\]

where \( C > 0 \) is a constant independent of \( \sigma \). Furthermore, by a bootstrapping argument, there exists a positive constant \( C \) independent of \( \sigma \) such that

\[
\| u_\sigma \|_{L^\infty(D)} \leq C \quad \text{for } \sigma \in (0, \bar{\sigma}_f).
\]
With this, (5.10) follows from the standard elliptic estimates. We now show that $u_\sigma$ is positive in $D$. Since $f(x) \in \mathcal{F}^+$, there exists a neighborhood $N(\partial D)$ of $\partial D$ such that $f(x) \leq 0$ for $x \in N(\partial D)$. Denote $D_0 = D \setminus N(\partial D)$. We claim that there exists a positive number $\sigma_0$ such that $u_\sigma > 0$ for $x \in D_0$ and $\sigma \in (0, \sigma_0)$.

Otherwise, there exists a sequence $\{\sigma_j\}$ with $\lim_{j \to \infty} \sigma_j = 0$ and a sequence $\{x_j\}$ with $x_j \in D_0$ such that

$$u_{\sigma_j}(x_j) \leq 0 \quad \text{for each } j.$$ 

By (5.10), we assume that

$$u_{\sigma_j} \to u_0 \quad \text{uniformly in } D,$$

and $u_0$ satisfies

$$\begin{cases} -\Delta u_0 + a(x)u_0 = K(x)(u_0^+)q, & \text{in } D, \\ u_0|_{\partial D} = 0. \end{cases}$$

(5.12)

Notice that

$$c_0 = \frac{1}{2} \int_D |\nabla u_0|^2 + a(x)u_0^2 - \frac{1}{q + 1} \int_D K(x)(u_0^+)q + 1 \, dx = \lim_{j \to \infty} c_{\sigma_j} > \alpha > 0.$$ 

This implies that $u_0(x) \not= 0$. Then, by the strong maximum principle, we have $u_0(x) > 0$ in $D$. In particular, $u_0(x) > 0$ in $D_0$. Since $D_0$ is bounded and closed, we can assume that

$$x_j \to x_0 \in D_0, \quad \text{as } j \to \infty, \quad \text{and } u_{\sigma_j}(x_j) \to u_0(x_0), \quad \text{as } j \to \infty.$$ 

So that $u_0(x_0) \leq 0$ which contradicts to $u_0(x) > 0$ in $D_0$.

It remains to show that $u_\sigma(x) > 0$ in $N(\partial D)$ for $\sigma \in (0, \sigma_0)$. Since $u_\sigma$ satisfies

$$\begin{cases} -\Delta u_\sigma + a(x)u_\sigma = K(x)(u_\sigma^+)q + 1 - \sigma f(x)e^{-\frac{q}{2}S}, & x \in N(\partial D), \\ u_\sigma|_{\partial D} = 0, \end{cases}$$

(5.13)

with $f(x) \leq 0$ in $N(\partial D)$, we have

$$\begin{cases} -\Delta u_\sigma + a(x)u_\sigma \geq 0, & x \in N(\partial D), \\ u_\sigma|_{\partial D} = 0, \quad u_\sigma|_{\partial D_0} > 0 \quad \text{for } \sigma \in (0, \sigma_0). \end{cases}$$

The strong maximum principle implies that

$$u_\sigma > 0 \quad \text{for } x \in N(\partial D) \text{ and } \sigma \in (0, \sigma_0).$$

Hence

$$u_\sigma(x) > 0 \quad \text{in } D \text{ for } \sigma \in (0, \sigma_0).$$

This completes the proof of Theorem 3. \qed
6. Exact number of solutions

We will prove Theorem 4 in this section. For this, we first prove the following lemma.

**Lemma 6.1.** Assume \( q \in (1, 5) \) and the corresponding homogeneous problem (1.12) is nondegenerate. Then there exists a positive number \( \sigma_0 \) such that problem (1.11) has at most two solutions for \( \sigma \in (0, \sigma_0) \).

**Proof.** We prove the lemma by contradiction. Assume the statement in Lemma 6.1 fails. Then there exists a sequence \( \{ \sigma_j \} \) with \( \lim_{j \to \infty} \sigma_j = 0 \) such that for each \( \sigma = \sigma_j \), (1.11) has at least three different positive solutions. By Lemma 2.2, there is a solution \( u^{(1)}_{\sigma_j} \) of (1.11) with \( \sigma = \sigma_j \) satisfying

\[
\| u^{(1)}_{\sigma_j} \|_{L^\infty} \to 0, \quad j \to \infty.
\]

Let \( u^{(2)}_{\sigma_j} \) and \( u^{(3)}_{\sigma_j} \) be the other two positive solutions when \( \sigma = \sigma_j \). By Lemma 2.1, we can assume

\[
u^{(2)}_{\sigma_j} \to \varphi_2(x), \quad u^{(3)}_{\sigma_j}(x) \to \varphi_3(x), \quad \text{uniformly on } D, \quad j \to \infty.
\]

Since \( u^{(1)}_{\sigma_j} \to 0 \) as \( \sigma_j \to 0 \), by Lemma 2.3, there exists a positive constant \( A \) such that

\[
\| u^{(i)}_{\sigma_j} \|_{L^\infty} \geq A, \quad i = 2, 3.
\]

Hence \( \varphi_2(x) \not\equiv 0 \) and \( \varphi_3(x) \not\equiv 0 \) in \( D \). By taking \( j \to \infty \) in Eq. (1.11) with \( \sigma = \sigma_j \), we have that \( \varphi_2(x) \) and \( \varphi_3(x) \) are positive solutions of problem (1.12). Since (1.12) is nondegenerate, \( u \equiv \varphi_2 \equiv \varphi_3 \) should be the unique positive solution of problem (2.12). However, since \( w_j = u^{(2)}_{\sigma_j} - u^{(3)}_{\sigma_j} \not\equiv 0 \), the same argument used in the proof of Lemma 2.2 shows that there exists a function \( w \not\equiv 0 \) satisfying

\[
\begin{cases}
-\Delta w + a(x)w = qK(x)u^{q-1}w, & x \in D, \\
w|_{\partial D} = 0.
\end{cases}
\]  

This contradicts to the assumption that (1.12) is nondegenerate and then it completes the proof. \( \square \)

The proof of Theorem 4 can be given as follows.

**Proof of Theorem 4.** When \( f(x) \in \mathcal{U} \), it follows from Theorems 1, 2 and Lemma 6.1 that there exists \( \sigma_0 > 0 \) such that problem (1.11) has exactly two solutions for \( \sigma \in (0, \sigma_0) \). Hence, to complete the proof, we only need to prove that if there exists a positive number \( \sigma_0 \) such that problem (1.11) has at least two positive solutions for each \( \sigma \in (0, \sigma_0) \), then \( f(x) \in \mathcal{U} \).

Suppose that (1.11) has at least two positive solutions for \( \sigma \in (0, \sigma_0) \). It follows from Lemma 2.2 that there exists a positive solution \( u_\sigma \) of (1.11) such that

\[
\| u_\sigma \|_{L^\infty(D)} \to 0, \quad \text{as } \sigma \to 0.
\]
Let \( u_\sigma = \sigma v_\sigma \). Then \( v_\sigma \) satisfies

\[
\begin{aligned}
-\Delta v_\sigma + a(x)v_\sigma &= \sigma^{q-1} K(x) v_\sigma^q - f(x)e^{-\frac{\alpha}{2} S} \quad \text{in } D, \\
v_\sigma |_{\partial D} = 0, \\
v_\sigma > 0 \quad \text{in } D.
\end{aligned}
\] (6.2)

By Lemma 2.4, there exists a positive constant \( C \) independent of \( \sigma \in (0, \sigma_0) \) such that

\[
\|v_\sigma\|_{L^\infty(D)} \leq C.
\] (6.3)

By the regularity theory for elliptic equation, there exists a positive constant \( C \) independent of \( \sigma \in (0, \sigma_0) \) such that

\[
\|v_\sigma\|_{C^{2,\nu}(D)} \leq C \quad \text{for } \sigma \in (0, \sigma_0) \text{ and } \nu \in (0, 1).
\] (6.4)

Hence, we can assume that

\[
v_\sigma \longrightarrow v \geq 0, \quad \text{uniformly in } D, \quad \text{as } \sigma \longrightarrow 0.
\]

Let \( \sigma \rightarrow 0 \), (6.2) gives

\[
\begin{aligned}
-\Delta v + a(x)v &= -f(x)e^{-\frac{\alpha}{2} S} \quad \text{in } D, \\
v |_{\partial D} = 0, \\
v \geq 0 \quad \text{in } D.
\end{aligned}
\]

This implies that \( f(x) \in \mathcal{U} \) and then it completes the proof of the first part of Theorem 4.

For the second part of theorem first prove the following claim.

**Claim.** Assume \( 1 < q < 5 \) and the corresponding homogeneous problem (1.12) is nondegenerate. If \( f(x) \notin \mathcal{U} \), then problem (1.11) has at most one positive solution when \( \sigma \) is sufficient small.

**Proof of the claim.** Suppose that the statement fails. Then there exists a sequence \( \{\sigma_j\} \) with \( \lim_{j \to \infty} \sigma_j = 0 \) such that (1.11) has at least two different positive solutions for each \( \sigma_j \). By Lemma 2.2, one of these solutions tends to zero in \( L^\infty \)-norm when \( j \to \infty \). Similar to the proof of the first part of Theorem 4, we have \( f(x) \in \mathcal{U} \) which contradicts to the assumption \( f(x) \notin \mathcal{U} \). \( \Box \)

Finally, the proof of the second part of Theorem 4 follows directly from Theorem 3 and the above claim. \( \Box \)

**7. More discussion on \( f(x) \in \mathcal{U} \)**

In the last section, we will discuss the existence and nonexistence of solutions to problem (1.11) when \( q \in (0, 1) \cup (5, +\infty) \) stated in Theorem 5. For this, it is sufficient to prove the following three theorems.
Theorem 7.1. Assume $q > 5$, $D$ is star-shaped, $a(x)$ and $K(x)$ satisfy
\begin{align}
\min_{x \in D} & \left( 1 - \frac{6}{q + 1} - \frac{2}{q + 1} (x \cdot \nabla K) K^{-1}(x) \right) > 0 \quad \text{and} \\
(x \cdot \nabla a) + 2a(x) & \geq 0 \quad \text{for all } x \in D.
\end{align}

Then there exists a constant $\sigma_0 > 0$ such that (1.11) has at least one solution for all $\sigma \in (0, \sigma_0)$ if and only if $f(x) \in \mathcal{U}$.

Theorem 7.2. For $q \in (0, 1)$, (1.11) has at least one solution for all $\sigma > 0$ if and only if $f(x) \in \mathcal{U}$.

Theorem 7.3. For $f(x) \in \mathcal{U}$ and $q \in (0, 1)$, the positive solution of (1.11) is unique.

Before proving these theorems, we give the following lemmas. The first lemma is the Pohozaev identity from [12].

Lemma 7.4. Let $D$ be a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$. Suppose that $g : \overline{D} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous mapping and $w \in C^2(D)$ satisfies
\begin{align}
\left\{ \begin{array}{l}
\Delta w + g(x, w(x)) = 0 \quad \text{in } D, \\
w|_{\partial D} = 0.
\end{array} \right.
\end{align}

Let $y \in \mathbb{R}^N$ be a fixed vector and $n(x)$ be the outward unit normal vector on $\partial D$. Then $w$ satisfies
\begin{align}
\int_{\partial D} (x - y) \cdot n(x)|\nabla w|^2 dS &= 2N \int_D G(x, w) \, dx + 2 \int_D (x - y) \cdot \nabla_x G \, dx - (N - 2) \int_D g(x, w) \, dx,
\end{align}
where $G(x, w) = \int_0^w g(x, t) \, dt$, and $\nabla_x G(x, w)$ is the gradient of $G(x, w)$ with respect to the variable $x$.

Lemma 7.5. Under conditions (7.1) and (7.2), if $q > \frac{N + 2}{N - 2}$ and $D$ is star-shaped, then any positive solution $w_\sigma$ of problem (1.11) satisfies
\begin{align}
\|w_\sigma\|_{H^1_0(D)} \rightarrow 0, \quad \text{as } \sigma \rightarrow 0.
\end{align}

Proof. Let $w_\sigma$ be any positive solution of (1.11) and set $w_\sigma = \sigma v_\sigma$. Then $v_\sigma$ satisfies
\begin{align}
\left\{ \begin{array}{l}
-\Delta v_\sigma + a(x)v_\sigma = \sigma^{q-1} K(x)v_\sigma^q - f(x) e^{-\frac{\sigma}{2} S} \quad \text{in } D, \\
v_\sigma|_{\partial D} = 0, \quad v_\sigma > 0 \quad \text{in } D.
\end{array} \right.
\end{align}
By (7.4), (7.5) gives
\[
\int_{\partial D} x \cdot n(x) |\nabla v_\sigma|^2 dS = \left[ \frac{2N}{q+1} - (N - 2) \right] \int_D \sigma^{q-1} K(x) v_\sigma^{q+1} dx - 2 \int_D x \cdot \nabla (f e^{-\frac{q}{2} S}) v_\sigma dx \\
- (2 + N) \int_D f e^{-\frac{q}{2} S} v_\sigma dx - 2 \int_D a(x) v_\sigma^2 dx \\
+ 2 \int_D x \cdot \nabla K(x) \cdot \frac{1}{q+1} \sigma^{q-1} v_\sigma^{q+1} - \frac{1}{2} x \cdot \nabla a(x) v_\sigma^2 dx.
\]

Here, for convenience, we assume $D$ is star-shaped with respect to $y = 0$. Then we have
\[
\int_{\partial D} x \cdot n(x) |\nabla v_\sigma|^2 dS \geq 0.
\]

Thus, by (7.2), we obtain
\[
\sigma^{q-1} \int_D \left[ N - 2 - \frac{2N}{q+1} - \frac{2N}{q+1} x \cdot \nabla K(x) \cdot (K(x))^{-1} \right] K(x) v_\sigma^{q+1} dx \\
\leq - \int_D \left[ 2x \cdot \nabla (f e^{-\frac{q}{2} S}) + (N + 2) f e^{-\frac{q}{2} S} v_\sigma dx \right]. \quad (7.6)
\]

On the other hand, (7.5) gives
\[
\int_D |\nabla v_\sigma|^2 dx = \sigma^{q-1} \int_D K(x) v_\sigma^{q+1} dx - \int_D a(x) v_\sigma^2 dx - \int_D f(x) e^{-\frac{q}{2} S} v_\sigma dx. \quad (7.7)
\]

If
\[
\min_{x \in D} \left\{ (N - 2) - \frac{2N}{q+1} - \frac{2}{q+1} (x \cdot \nabla K(x)) K^{-1}(x) \right\} = k_* > 0, \quad (7.8)
\]

then (7.6) implies
\[
\sigma^{q-1} \int_D K(x) v_\sigma^{q+1} dx \leq - \frac{1}{k_*} \int_D \left[ 2x \cdot \nabla (f e^{-\frac{q}{2} S}) + (N + 2) f e^{-\frac{q}{2} S} \right] v_\sigma dx.
\]

Substituting this into (7.7) yields
\[
\int_D |\nabla v_\sigma|^2 + a(x) v_\sigma^2 dx \leq - \int_D \left[ \frac{2}{k_*} (x \cdot \nabla (f e^{-\frac{q}{2} S})) + \frac{1}{k_*} (N + 2 + k_*) f e^{-\frac{q}{2} S} \right] v_\sigma dx. \quad (7.9)
\]
Set
\[ F(x) = \frac{2}{k^*} (x \cdot \nabla (fe^{-\frac{q}{2}S})) + \frac{1}{k^*} (N + 2 + k_n) fe^{-\frac{q}{2}S}. \]

Then, (7.9) gives
\[
\lambda_1^* (D) \int_D |\nabla v_\sigma|^2 \, dx \leq \int_D |F(x)| \, v_\sigma \, dx
\leq \|F(x)\|_{L^2(D)} \|v_\sigma\|_{L^2(D)} \leq \|F(x)\|_{L^2(D)} \cdot C \|\nabla v_\sigma\|_{L^2(D)}
\leq \frac{C}{2} \|F(x)\|_{L^2(D)}^2 + \frac{\lambda_1^*(D)}{2} \|\nabla v_\sigma\|_{L^2(D)}^2,
\]
where \(\lambda_1^*(D)\) is given in (1.19). This implies that
\[
\int_D |\nabla v_\sigma|^2 \, dx \leq C \|F(x)\|_{L^2(D)}^2 \leq C_1,
\]
where \(C_1\) is a constant independent of \(\sigma\). Noticing that
\[
\|w_\sigma\|_{H_0^1(D)}^2 = \sigma^2 \|\nabla v_\sigma\|_{L^2(D)}^2,
\]
we have
\[
\|w_\sigma\|_{H_0^1(D)} \to 0, \quad \text{as } \sigma \to 0. \tag{7.11}
\]
This completes the proof of the lemma. \(\square\)

We are now ready to prove Theorem 7.1. By Theorem 1, \(f(x) \in \mathcal{U}\) is a sufficient condition for existence. We need to show the necessity of \(f(x) \in \mathcal{U}\) for the existence as stated in the following proposition.

**Proposition 7.6.** Under the assumptions of Theorem 7.1, problem (1.13) has a nonnegative solution if (1.11) has a positive solution for any small parameter \(\sigma\).

**Proof.** Let \(u_\sigma\) be a solution of (1.11). Set \(u_\sigma = \sigma v_\sigma\). Then \(v_\sigma\) satisfies (7.5) and (7.7). Hence
\[
\sigma^{q-1} \int_D K(x) v_\sigma^{q+1} \, dx
\leq \int_D |\nabla v_\sigma|^2 \, dx + \|a\|_{L^\infty(D)} \int_D v_\sigma^2 \, dx + \|fe^{-\frac{q}{2}S}\|_{L^2(D)} \|v_\sigma\|_{L^2(D)}
\leq (1 + \|a\|_{L^\infty(D)} C_1) \int_D |\nabla v_\sigma|^2 \, dx + C_2 \|\nabla v_\sigma\|_{L^2(D)}^2 \leq C_3 \|\nabla v_\sigma\|_{L^2(D)}^2 + C_4.
\]
Since (7.10) implies that
\[
\sigma^{q-1} \int_D K(x)v^{q+1}_\sigma \, dx \leq C,
\]
where \( C \) is a positive constant independent of \( \sigma \), we have
\[
\int_D K(x)v^{q+1}_\sigma \, dx \leq \frac{C}{\sigma^{q-1}}. \tag{7.12}
\]
Hence, up to a subsequence, \( v_\sigma \) converges weakly to some function \( w \in H^1_0(D) \), as \( \sigma \to 0 \). That is,
\[
\int_D \nabla v_\sigma \nabla \varphi \, dx \longrightarrow \int_D \nabla w \cdot \nabla \varphi \, dx, \quad \text{as} \quad \sigma \longrightarrow 0, \quad \text{for} \quad \varphi \in C^\infty_0(D). \tag{7.13}
\]
By Hölder’s inequality and (7.12), we have
\[
\int_D K(x)v^{q}_\sigma \varphi \, dx \leq \left( \int_D (K(x)v^{q}_\sigma)^{\frac{q+1}{q}} \, dx \right)^{\frac{q}{q+1}} \left( \int_D |\varphi|^{q+1} \, dx \right)^{\frac{1}{q+1}}
\]
\[
\leq \|K(x)\|_{L^{q+1}(D)}^{\frac{1}{q+1}} \left[ \int_D K(x)v^{q+1}_\sigma \, dx \right]^{\frac{q}{q+1}} \|\varphi\|_{L^{q+1}}
\]
\[
\leq \|K(x)\|_{L^{q+1}(D)}^{\frac{1}{q+1}} \cdot \sigma^{\frac{q(1-q)}{q+1}} C \|\varphi\|_{L^{q+1}(D)}. \]
Consequently,
\[
\sigma^{q-1} \int_D K(x)v^{q}_\sigma \varphi \, dx \leq C \sigma^{\frac{q-1}{q+1}} \|\varphi\|_{L^{q+1}(D)} \longrightarrow 0, \quad \text{as} \quad \sigma \longrightarrow 0. \tag{7.14}
\]
Moreover,
\[
\int_D a(x)v_\sigma \varphi \, dx \longrightarrow \int_D a(x)w \varphi \, dx, \quad \text{as} \quad \sigma \longrightarrow 0, \quad \text{for} \quad \varphi \in C^\infty_0(D). \tag{7.15}
\]
Taking the weak limit on both sides of (7.5) and using (7.13), (7.14) and (7.15) give
\[
\int_D (\nabla w \nabla \varphi + a(x)w \varphi) \, dx = -\int_D f(x)e^{-\frac{2}{\alpha}S} \varphi \, dx \quad \text{for} \quad \varphi \in C^\infty_0(D). \tag{7.16}
\]
This implies that \( w(x) \) is a weak solution of problem (1.13). By the regularity theorem for elliptic equations, we know that \( w(x) \) is a classical solution of (1.13). Furthermore, \( w(x) \geq 0 \) because \( v_\sigma > 0 \) in \( D \) for \( \sigma > 0 \). This completes the proof of the proposition. \( \square \)
Proof of Theorem 7.2. First, let \( f(x) \in \mathcal{U} \), we will show that (1.11) has a positive solution. Since \( f(x) \in \mathcal{U} \), problem (1.13) has a nonnegative solution \( \varphi(x) \). It is obvious that \( \sigma \varphi(x) \) satisfies
\[
-\Delta (\sigma \varphi) + a(x)(\sigma \varphi) = -\sigma f(x)e^{-\frac{\alpha}{2}S} \leq (\sigma \varphi)^q \cdot K(x) - \sigma f(x)e^{-\frac{\alpha}{2}S} \quad \text{for all } \sigma > 0,
\]
which implies that \( \sigma \varphi(x) \) is a subsolution of (1.11).

On the other hand, let \( v \) be the solution to the problem
\[
\begin{cases}
-\Delta v + a(x)v = 1 & \text{in } D, \\
v|_{\partial D} = 0.
\end{cases}
\]  
(7.17)

It follows from the strong maximum principle that \( v(x) > 0 \) in \( D \). Choose \( M_0 \) sufficient large such that
\[
M_0 \geq M_0^q \cdot \|K(x)\|_{L^\infty(D)} \max_{x \in D} v^q(x) + \sigma \max_{x \in D} |f(x)e^{-\frac{\alpha}{2}S}|.
\]

This is possible because \( 0 < q < 1 \), and
\[
\frac{M^q \|K\|_{L^\infty(D)} \max_{x \in D} v^q(x) + \sigma \max_{x \in D} |f(x)e^{-\frac{\alpha}{2}S}|}{M} \to 0, \quad \text{as } M \to \infty.
\]

Set \( w = M_0 v \). Then \( w \) satisfies
\[
-\Delta w + a(x)w = M_0 \geq K(x)w^q - \sigma f(x)e^{-\frac{\alpha}{2}S},
\]
which implies that \( w \) is a supersolution of (1.11). Moreover, we can choose \( M_0 \) such that \( 0 \leq \sigma \varphi \leq w(x) \). Thus problem (1.11) has at least one nonnegative solution \( u_\sigma \) satisfies
\[
0 \leq \sigma \varphi \leq u_\sigma \leq w \quad \text{in } D.
\]

To prove \( u_\sigma > 0 \) in \( D \), set \( v_\sigma = u_\sigma - \sigma \varphi \). Then \( v_\sigma \) satisfies
\[
\begin{cases}
-\Delta v_\sigma + a(x)v_\sigma = K(x)u_\sigma^q \geq 0 & \text{in } D, \\
v_\sigma|_{\partial D} = 0.
\end{cases}
\]  
(7.18)

The strong maximum principle implies that \( v_\sigma > 0 \) in \( D \) and hence \( u_\sigma > 0 \) in \( D \).

Next, by assuming that problem (1.11) has at least one positive solution for all \( \sigma > 0 \), we will show that \( f(x) \in \mathcal{U} \). To this end, let \( u_\sigma \) be any positive solution of problem (1.11) and set \( u_\sigma = \sigma w_\sigma \). Then \( w_\sigma \) satisfies
\[
\begin{cases}
-\Delta w_\sigma + a(x)w_\sigma = \sigma^{q-1} K(x)w_\sigma^q - f(x)e^{-\frac{\alpha}{2}S} & \text{in } D, \\
w_\sigma > 0 & \text{in } D, \\
w_\sigma|_{\partial D} = 0.
\end{cases}
\]  
(7.19)

Hence, for \( \sigma \geq 1 \), there exists a positive constant \( C \) independent of \( \sigma \) such that
\[
\|w_\sigma\|_{L^\infty(D)} \leq C \quad \text{for } \sigma \in [1, +\infty).
\]  
(7.20)
Multiplying Eq. (7.19) by \( w_\sigma \) and integrating the product over \( D \) give
\[
\int_D |\nabla w_\sigma|^2 + a(x)w_\sigma^2 \, dx = \sigma^{q-1} \int_D K(x)w_\sigma^{q+1} \, dx - \int_D f(x)e^{-\frac{q}{2}S}w_\sigma \, dx \\
\leq |D|^\frac{1-q}{2}\|w_\sigma\|_{L^2(D)}^{q+1}\|K(x)\|_{L^\infty(D)} + C\|f\|_{L^2(D)}\|w_\sigma\|_{L^2(D)} \\
\leq \lambda_1^+(D)\int_D \nabla w_\sigma^2 \, dx + C(D, f, K, \lambda_1^+(D)),
\]
when \( \sigma \geq 1 \) and \( 0 < q < 1 \). Since (1.19) implies
\[
\lambda_1^+(D)\int_D |\nabla w_\sigma|^2 \, dx \leq \int_D |\nabla w_\sigma|^2 + a(x)w_\sigma^2 \, dx \leq \frac{\lambda_1^+(D)}{2}\|\nabla w_\sigma\|_{L^2(D)}^2 + C(D, f, K, \lambda_1^+(D)).
\]
We have
\[
\int_D |\nabla w_\sigma|^2 \, dx \leq C.
\]
(7.22)
Then (7.20) follows from (7.22) and a bootstrapping argument. Furthermore, by the regularity theory for elliptic equations, we can deduce that there exists a constant \( C > 0 \) independent of \( \sigma \) such that
\[
\|w_\sigma\|_{C^{2,\gamma}(D)} \leq C.
\]
Consequently, up to a subsequence,
\[
w_\sigma(x) \longrightarrow \varphi(x) \geq 0 \text{ uniformly on } D, \text{ as } \sigma \longrightarrow \infty.
\]
Let \( \sigma \to \infty \) in (7.19), we have
\[
\begin{cases}
-\Delta \varphi + a(x)\varphi = -f(x)e^{-\frac{q}{2}S} & \text{in } D, \\
\varphi \geq 0 & \text{in } D, \\
\varphi|_{\partial D} = 0.
\end{cases}
\]
(7.23)
This implies that \( f \in \mathcal{U} \). The proof of Theorem 7.2 is then completed. □

Before proving Theorem 7.3, we need one more lemma.

**Lemma 7.7.** For any \( \sigma > 0 \), if \( f \in \mathcal{U} \), then problem (1.11) has a minimal solution \( w_\sigma \).

**Proof.** Let \( \varphi \) be the unique solution of (1.13). Then \( \varphi_\sigma = \sigma \varphi \) satisfies
\[
\begin{cases}
-\Delta \varphi_\sigma + a(x)\varphi_\sigma = -\sigma f(x)e^{-\frac{q}{2}S} & \text{in } D, \\
\varphi_\sigma \geq 0 & \text{in } D, \\
\varphi_\sigma = 0 & \text{on } \partial D.
\end{cases}
\]
(7.24)
Since any positive solution \( u_\sigma \) of problem (1.11) satisfies
\[
-\Delta u_\sigma + a(x)u_\sigma = K(x)u_\sigma^q - \sigma f(x)e^{-\frac{q}{2}S} \geq -\sigma f(x)e^{-\frac{q}{2}S} \text{ in } D,
\]
the strong maximum principle implies that

\[ u_\sigma(x) > \varphi_\sigma(x) \quad \text{in } D. \]

Thus, \( \varphi_\sigma(x) \) is a subsolution of (1.11). By a monotone iteration with initial data \( u_\sigma = \varphi_\sigma(x) \), we can obtain the minimal solution \( w_\sigma \) of (1.11) which satisfies \( \varphi_\sigma(x) \leq w_\sigma(x) \leq u_\sigma(x) \) in \( D \).

**Proof of Theorem 7.3.** Let \( w_\sigma \) be the minimal positive solution of (1.11). We prove the theorem by contradiction. Suppose that (1.11) has a positive solution \( u_\sigma \) which is not equal to \( w_\sigma \). Then \( 0 \leq v_\sigma = u_\sigma - w_\sigma \neq 0 \), and \( v_\sigma \) satisfies

\[
\begin{aligned}
-\Delta v_\sigma + a(x)v_\sigma &= (u_\sigma^q - w_\sigma^q)K(x) \geq 0 \quad \text{in } D, \\
v_\sigma|_{\partial D} &= 0.
\end{aligned}
\]

(7.25)

It follows from the maximum principle that

\[ u_\sigma(x) > w_\sigma(x) \quad \text{for } x \in D. \]

We claim that

\[
\sigma \int_D f(x)e^{-\frac{q}{2}S}(u_\sigma - w_\sigma) \, dx < 0.
\]

(7.26)

In fact, since \( f(x) \in \mathcal{U} \), problem (1.13) has a nonnegative solution \( \varphi(x) \geq 0 \). Hence

\[
-\int_D f(x)e^{-\frac{q}{2}S}(u_\sigma - w_\sigma) \, dx = \int_D \left[ -\Delta \varphi + a(x)\varphi \right](u_\sigma - w_\sigma) \, dx = \int_D (-\Delta v_\sigma + a(x)v_\sigma)\varphi \, dx
\]

\[
= \int_D K(x)(u_\sigma^q - w_\sigma^q)\varphi \, dx > 0,
\]

which gives (7.26).

On the other hand,

\[
-\sigma \int_D f(x)e^{-\frac{q}{2}S}u_\sigma \, dx + \int_D K(x)w_\sigma^q u_\sigma \, dx
\]

\[
= \int_D (-\Delta w_\sigma + a(x)w_\sigma)u_\sigma \, dx = \int_D (-\Delta u_\sigma + a(x)u_\sigma)w_\sigma \, dx
\]

\[
= \int_D K(x)u_\sigma^q w_\sigma \, dx - \sigma \int_D f(x)e^{-\frac{q}{2}S}w_\sigma \, dx.
\]
Thus
\[ -\sigma \int_{\Omega} f(x) e^{-\frac{\alpha}{2} S}(u_\sigma - w_\sigma) = \int_{\Omega} K(x)(u_\sigma^{q-1} - w_\sigma^{q-1}) u_\sigma w_\sigma \, dx. \]

Since \( 0 < q < 1 \) and \( u_\sigma > w_\sigma \) in \( \Omega \), the above equality contradicts to (7.26) and then this completes the proof. \( \Box \)

References