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Comparison and regularity results for the fractional Laplacian via symmetrization methods

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ABSTRACT

In this paper we establish a comparison result through symmetrization for solutions to some boundary value problems involving the fractional Laplacian. This allows to get sharp estimates for the solutions, obtained by comparing them with solutions of suitable radial problems. Furthermore, we use such result to prove *a priori* estimates for solutions in terms of the data, providing several regularity results which extend the well-known ones for the classical Laplacian.

1. Introduction and main results

The final goal of this paper is to obtain a comparison principle using symmetrization techniques in order to get sharp estimates for solutions to some elliptic boundary value problems involving the fractional Laplacian operator. If $u : \mathbb{R}^N \to \mathbb{R}$ is a function in the Schwartz space of rapidly decaying functions, the fractional Laplacian $(-\Delta)^{\alpha/2}$ of u, with $\alpha \in (0, 2)$, is defined in a standard sense, that is either by the Riesz potential

$$(-\Delta)^{\alpha/2}u(x) := C_{N,\alpha}\ \mathrm{P.V.} \int\limits_{\mathbb{R}^N} \frac{u(x) - u(\xi)}{|x - \xi|^{N+\alpha}}\, d\xi,$$

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where P.V. denotes the principal value and $C_{N,\alpha}$ is a suitable normalization constant, or as a pseudo differential operator through the Fourier transform on the Schwartz class

$$\mathcal{F}\big[(-\Delta)^{\alpha/2}u\big](\xi) := |\xi|^{\alpha}\mathcal{F}[u](\xi),$$

where $\mathcal{F}[g]$ denotes the Fourier transform of a function g. As for the equivalence between these two notions, as well as a detailed description and properties concerning more general integro-differential operators, we refer to the book of Landkof [29] and the paper [38].

It is well known (see for instance [17]) that to any function u smooth enough on $\mathbb{R}^N$ we can associate its α-harmonic extension, namely a function w defined on the upper half-space $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0, +\infty)$ which is a solution to the local (degenerate or singular) elliptic problem

$$\begin{cases} -\operatorname{div}\big(y^{1-\alpha}\nabla w\big) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w(x,0) = u(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1}$$

Moreover, Caffarelli and Silvestre give in [17] an interpretation of the fractional Laplacian $(-\Delta)^{\alpha/2}$ as a Dirichlet to Neumann map:

$$(-\Delta)^{\alpha/2}u(x) = -\frac{1}{k_\alpha}\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y), \tag{2}$$

where k_α is a suitable constant, whose exact value is

$$\kappa_\alpha = \frac{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})}. \tag{3}$$

In order to define the fractional Laplacian in bounded domains Ω the above characterization has to be suitably adapted. This has been done in the papers [13] and [15], where formula (2) allows to *define* the fractional Laplacian $(-\Delta)^{\alpha/2}$ over a proper function space on Ω, as we shall see in Section 2.

The fractional Laplacian appears in several contexts. For instance, it arises in the study of various physical phenomena, where long-range or anomalous diffusions occur. Just to give few examples, this kind of operator can be found in combustion theory (see [19]), in dislocations processes of mechanical systems (see [25]) or in crystals (see [22]). Moreover, as it is well known in the theory of probability, the fractional Laplacian is the infinitesimal generator of a Lévy process (see for instance [37]). Due to all of that, lots of authors devoted their interest to the subject. We just mention [38,20,16,19] dedicated to the obstacle problem and the free boundaries for the fractional Laplacian, the papers [15,14] regarding some aspects of nonlinear equations involving fractional powers of the Laplacian, the convex–concave problem for the fractional Laplacian described in [13], the work [27] in which a critical exponent problem for the half-Laplacian in an annulus is investigated, the study [41] of a nonlocal energy variational problem, and the papers [8,11,12,9]. Obviously this list is very far from being exhaustive.

In order to describe our main result let us consider the nonlocal Dirichlet problem with homogeneous boundary condition

$$\begin{cases} (-\Delta)^{\alpha/2}u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is an open bounded set of $\mathbb{R}^N$ and f is a smooth function on Ω. Roughly speaking, following for instance [13], a solution to problem (4) is defined as the trace of a suitable Dirichlet–Neumann problem. Namely, if w is a weak solution to the local problem

$$\begin{cases} -\operatorname{div}\left(y^{1-\alpha}\nabla w\right) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ -\dfrac{1}{\kappa_\alpha} \lim\limits_{y\to 0^+} y^{1-\alpha} \dfrac{\partial w}{\partial y} = f(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where $\mathcal{C}_\Omega := \Omega \times (0, +\infty)$ is the cylinder of basis Ω and $\partial_L \mathcal{C}_\Omega := \partial\Omega \times [0, +\infty)$ is its lateral boundary, then its trace on Ω, $tr_\Omega(w) = w(\cdot, 0) =: u$ is a solution to problem (4) (see also Section 3 for precise definitions).

Following [39], the idea is to get sharp estimates for the solution u to (4) by comparing it with a solution ϕ to the radial problem

$$\begin{cases} (-\Delta)^{\alpha/2}\phi = f^\#(x) & \text{in } \Omega^\#, \\ \phi = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{6}$$

where $\Omega^\#$ is the ball centered at 0, having the same measure as Ω and $f^\#$ is the Schwarz rearrangement of f. Since ϕ is the trace on $\Omega^\#$ of a solution v to the problem

$$\begin{cases} -\operatorname{div}\left(y^{1-\alpha}\nabla v\right) = 0 & \text{in } \mathcal{C}_\Omega^\#, \\ v = 0 & \text{on } \partial_L \mathcal{C}_\Omega^\#, \\ -\dfrac{1}{\kappa_\alpha} \lim\limits_{y\to 0^+} y^{1-\alpha} \dfrac{\partial v}{\partial y} = f^\#(x) & \text{in } \Omega^\#, \end{cases} \tag{7}$$

where $\mathcal{C}_\Omega^\# := \Omega^\# \times (0, +\infty)$, $\partial_L \mathcal{C}_{\Omega^\#} := \partial\Omega^\# \times [0, +\infty)$, it makes sense to look for a comparison between concentrations of the functions w and v through their Schwarz rearrangements (see Section 2 for definitions). More precisely, we prove that

$$\int_0^s w^*(\sigma, y)\, d\sigma \leqslant \int_0^s v^*(\sigma, y)\, d\sigma \quad \forall s \in \left[0, |\Omega|\right], \tag{8}$$

where $w^*(\cdot, y)$, $v^*(\cdot, y)$ are the one dimensional rearrangements of w, v respectively, for any fixed $y \in [0, +\infty)$. The achievement of such result looks reasonable because of the nature of problem (5), for which a symmetrization with respect to x keeping the y variable fixed (i.e. Steiner symmetrization with respect to the line $x = 0$) is available. The key role in this framework is played by a second order derivation formula for functions defined by integrals, obtained in [1] for the smooth case and in [21] for less regular functions.

We point out that through inequality (8) we easily get a comparison result between the traces on $\Omega \times \{0\}$ of w and v, namely an integral comparison between u and ϕ:

Theorem 1.1. *Let u and ϕ be the weak solutions to problems* (4) *and* (6), *respectively, and $f \in L^{\frac{2N}{N+\alpha}}(\Omega)$, with $\alpha \in (0, 2)$. Then we have*:

$$\int_0^s u^*(\sigma)\, d\sigma \leqslant \int_0^s \phi^*(\sigma)\, d\sigma \quad \forall s \in \left[0, |\Omega|\right].$$

We emphasize that since for $\alpha = 2$ the fractional Laplacian coincides with the classical Laplacian, for which comparison and regularity results via symmetrization methods are well known (see *e.g.* [39,40,2]), in the following we consider only the case $0 < \alpha < 2$.

In this setting the comparison result proved in Theorem 1.1 allows us to prove *a priori* estimates for solutions of problem (4) in terms of the data f, providing several regularity results which extend the well-known ones for the classical Laplacian (see Section 4).

We also want to remark that the application of symmetrization techniques to general Lévy processes is not new, as it is shown for example in [6] and [4] where several isoperimetric-type issues are investigated. Moreover, our approach is completely "PDE oriented" and it is not based on a probabilistic setting.

This paper is organized as follows. In Section 2 we give some basic definitions and properties concerning the functional setting we are going to work with. In particular, in Subsection 2.1 we recall some fundamental definitions concerning the fractional Laplacian in bounded domains and properties of weak solutions to the related Dirichlet problem. Subsections 2.2 and 2.3 introduce the notion of Schwarz rearrangement and Lorentz space, together with some related properties. In Subsection 2.4 we define the Green function for the fractional Laplacian, an essential tool to write down an explicit integral representation of the solution ϕ to the radial problem (6). In Section 3 we prove the comparison results stated above. Furthermore, in Section 4 we exhibit some regularity results of the solution u in terms of the source data f. Finally in Section 5 we give some comments about the best constant in the L^∞ estimate. In particular, the optimal constant is computed on the unit ball, considering only the case $\alpha = 1$ and $N = 3$.

2. Preliminaries

2.1. Function spaces and definitions

As we pointed out in the introduction, formula (2) given in [17] connects the nonlocal character of $(-\Delta)^{\alpha/2}$ to local problems of the form (1). This interpretation can be extended to the case of bounded domains. To this aim, it is convenient to introduce here a suitable functional setting and basic definitions. For all the details and proofs of the following definitions and properties, we refer to the papers [15,13,14,28].

If Ω is a bounded open subset of $\mathbb{R}^N$, the half-cylinder with base Ω and its lateral boundary will be respectively denoted by

$$\mathcal{C}_\Omega := \Omega \times (0, +\infty) \quad \text{and} \quad \partial_L \mathcal{C}_\Omega := \partial\Omega \times [0, +\infty).$$

We introduce then the weighted energy space

$$X_0^\alpha(\mathcal{C}_\Omega) := \left\{ w \in H^1(\mathcal{C}_\Omega),\ w = 0 \text{ on } \partial_L \mathcal{C}_\Omega\colon \int_{\mathcal{C}_\Omega} y^{1-\alpha} \left|\nabla w(x,y)\right|^2 dx\,dy < \infty \right\}$$

equipped with the norm

$$\|w\|_{X_0^\alpha} := \left(\int_{\mathcal{C}_\Omega} y^{1-\alpha} \left|\nabla w(x,y)\right|^2 dx\,dy \right)^{1/2}.$$

Thus we define the *trace space* by

$$\mathcal{V}_\alpha(\Omega) = \left\{ u = tr_\Omega\, w := w(\cdot, 0)\colon\ w \in X_0^\alpha(\mathcal{C}_\Omega) \right\}, \tag{9}$$

where tr_Ω is the *trace* operator on the space $w \in X_0^\alpha(\mathcal{C}_\Omega)$. Then the fractional Laplacian in Ω is well defined for function in $\mathcal{V}_\alpha(\Omega)$. Indeed it is well known (see *e.g.* [15,13]) that for any function $u \in \mathcal{V}_\alpha(\Omega)$ there exists a unique minimizer w to the problem

$$\inf\left\{\int_{\mathcal{C}_\Omega} y^{1-\alpha}\left|\nabla w(x,y)\right|^2 dx\,dy\colon\ w\in X_0^\alpha(\mathcal{C}_\Omega),\ w(\cdot,0)=u \text{ in } \Omega\right\}.$$

By standard elliptic theory such minimizer w is smooth for $y>0$ and satisfies

$$\begin{cases} -\operatorname{div}\left(y^{1-\alpha}\nabla w\right)=0 & \text{in } \mathcal{C}_\Omega,\\ w=0 & \text{on } \partial_L\mathcal{C}_\Omega,\\ w(\cdot,0)=u & \text{in } \Omega. \end{cases} \tag{10}$$

This yields to consider an *extension operator* in the following sense:

Definition 2.1. Given a function $u\in\mathcal{V}_\alpha(\Omega)$, the solution w to problem (10) will be said the α-harmonic extension of u on the cylinder $\mathcal{C}_\Omega$ and will be denoted by $\operatorname{Ext}_\alpha u$.

Then the fractional Laplacian operator can be defined through the *Dirichlet to Neumann map* as follows (see *e.g.* [15,13]):

Definition 2.2. For any $u\in\mathcal{V}_\alpha(\Omega)$ we define the fractional Laplacian $(-\Delta)^{\alpha/2}$ acting on u as the following limit (in the distributional sense)

$$(-\Delta)^{\alpha/2}u(x):=-\frac{1}{\kappa_\alpha}\lim_{y\to 0} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y),$$

where $w=\operatorname{Ext}_\alpha(u)$ and κ_α is given by (3).

Let $\{\varphi_k\}$ be an orthonormal basis of $L^2(\Omega)$ made by eigenfunctions of $-\Delta$ in Ω with zero Dirichlet boundary conditions and $\{\lambda_k\}$ the corresponding Dirichlet eigenvalues. It is classical that the powers of a positive operator in a bounded domain, evaluated on a certain function u, are defined through the spectral decomposition of u using the powers of the eigenvalues of the original operator. So in the case of the fractional Laplacian $(-\Delta)^{\alpha/2}$, if

$$u=\sum_{k=1}^{\infty}a_k\varphi_k$$

we must have

$$(-\Delta)^{\alpha/2}u=\sum_{k=1}^{\infty}a_k\lambda_k^{\alpha/2}\varphi_k. \tag{11}$$

This definition is coherent with Definition 2.2, since it is possible to give the following characterization of the trace space $\mathcal{V}_\alpha(\Omega)$:

Proposition 2.1. *The space $\mathcal{V}_\alpha(\Omega)$ defined in (9) coincides with the space*

$$H:=\left\{u\in L^2(\Omega)\ \middle|\ u=\sum_{k=1}^{\infty}a_k\varphi_k \text{ satisfying } \sum_{k=1}^{\infty}a_k^2\lambda_k^{\alpha/2}<\infty\right\}. \tag{12}$$

Moreover if $u\in\mathcal{V}_\alpha(\Omega)$ admits the decomposition $u=\sum_{k=1}^{\infty}a_k\varphi_k$, then its α-harmonic has the following explicit representation

$$\operatorname{Ext}_\alpha u(x,y)=\sum_{k=1}^{\infty} a_k\varphi_k(x)\rho\big(\lambda_k^{1/2}y\big) \tag{13}$$

where ρ solves the problem

$$\begin{cases} \rho''(s)+\dfrac{1-\alpha}{s}\rho'(s)=\rho(s), & s>0,\\ \lim\limits_{y\to 0^+} y^{1-\alpha}\rho'(s)=-\kappa_\alpha, \\ \rho(0)=1. \end{cases}$$

Therefore, using (13) and Definition 2.2, equality (11) easily follows.

According to [13], we have the following definition of weak solution to problems of the type (5):

Definition 2.3. Let $f\in L^{\frac{2N}{N+\alpha}}(\Omega)$, where $\alpha\in(0,2)$. We say that $w\in X_0^\alpha(\mathcal{C}_\Omega)$ is the weak solution to problem (5) if for any test function $\varphi\in X_0^\alpha(\mathcal{C}_\Omega)$ the following identity holds:

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha}\nabla w(x,y)\cdot\nabla\varphi(x,y)\,dx\,dy=\kappa_\alpha\int_\Omega f(x)\varphi(x,0)\,dx. \tag{14}$$

We note that for any test function $\varphi\in X_0^\alpha(\mathcal{C}_\Omega)$, by the Sobolev trace inequality (see [13]) it follows that the trace $\varphi(\cdot,0)$ on $\Omega\times\{0\}$ belongs to $L^{\frac{2N}{N-\alpha}}(\Omega)$, hence the integral at the right-hand side of (14) makes sense. Besides, the classical Lax–Milgram theorem ensures that a unique weak solution $w\in X_0^\alpha(\mathcal{C}_\Omega)$ to problem (5) exists.

Then the definition of weak solution to problem (4) is strictly related to the solution of (5) in the following sense:

Definition 2.4. Let $f\in L^{\frac{2N}{N+\alpha}}(\Omega)$, where $\alpha\in(0,2)$. We say that $u\in H$ is the weak solution to (4) if $u=tr_\Omega\, w$, and w is the weak solution to problem (5).

We observe that if u is the weak solution to (4), its α harmonic extension $\operatorname{Ext}_\alpha u$ is smooth for $y>0$ and decays to zero as $y\to\infty$ (see [13]).

Finally we point out that the space H defined in (12) is an interpolation space and it is possible to prove that (see [15] for the case $\alpha=1$ and [31,28] for the general case)

$$H=\begin{cases} H^{\alpha/2}(\Omega) & \text{if } \alpha\in(0,1),\\ H_{00}^{1/2}(\Omega) & \text{if } \alpha=1,\\ H_0^{\alpha/2}(\Omega) & \text{if } \alpha\in(1,2), \end{cases}$$

where $H^{\alpha/2}(\Omega)$ is the usual fractional Sobolev space, $H_0^{\alpha/2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^{\alpha/2}(\Omega)}$ and

$$H_{00}^{1/2}(\Omega):=\left\{u\in H^{1/2}(\Omega)\colon \int_\Omega \frac{u(x)^2}{d(x)}\,dx<\infty\right\},$$

with $d(x):=\operatorname{dist}(x,\partial\Omega)$.

2.2. Basic facts about rearrangements

Let Ω be a bounded open subset of $\mathbb{R}^N$ and u be a real measurable function on Ω. We will denote by $|\cdot|$ the N-dimensional Lebesgue measure. We define the *distribution function* μ_u of u as

$$\mu_u(t) = \left|\left\{x \in \Omega\colon \left|u(x)\right| > t\right\}\right|, \quad t \geqslant 0,$$

and the *decreasing rearrangement* of u as

$$u^*(s) = \sup\left\{t \geqslant 0\colon \mu_u(t) > s\right\}, \quad s \in \left(0, |\Omega|\right).$$

Furthermore, if ω_N is the measure of the unit ball in $\mathbb{R}^N$ and $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin having the same Lebesgue measure as Ω, the function

$$u^\#(x) = u^*\left(\omega_N |x|^N\right), \quad x \in \Omega^\#,$$

is called *spherical decreasing rearrangement* of u. For an exhaustive treatment of rearrangements we refer to [3,26] and to the appendix of [40]. Here we just recall the well-known Hardy–Littlewood inequality (see [24])

$$\int_\Omega \left|u(x)v(x)\right| dx \leqslant \int_0^{|\Omega|} u^*(s)v^*(s)\, ds = \int_{\Omega^\#} u^\#(x)v^\#(x)\, dx \tag{15}$$

where u, v are measurable functions on Ω. We point out that as we will deal with two variable functions of the type

$$u : (x, y) \in \mathcal{C}_\Omega \to u(x, y) \in \mathbb{R} \tag{16}$$

defined on the cylinder $\mathcal{C}_\Omega := \Omega \times (0, +\infty)$, measurable with respect to x, we can define the Steiner symmetrization of $\mathcal{C}_\Omega$ with respect to the variable x, namely the set $\mathcal{C}_\Omega^\# := \Omega^\# \times (0, +\infty)$. In addition, we will denote by $\mu_u(t, y)$ and $u^*(s, y)$ the distribution function and the decreasing rearrangements of (16), with respect to x for y fixed, and we define the function

$$u^\#(x, y) = u^*\left(\omega_N |x|^N, y\right)$$

which is the *Steiner symmetrization* of u, with respect to the line $x = 0$. Obviously $u^\#$ is a spherically symmetric and decreasing function with respect to x for any fixed y.

Now we recall two derivations formulas, that will turn out very useful in the proof of the main result. The following proposition can be found in [32], and it is a generalization of a well-known result by Bandle (see [3]).

Proposition 2.2. *Suppose that $w \in H^1(0, T; L^2(\Omega))$ for some $T > 0$. Then*

$$w^* \in H^1\left(0, T; L^2\left(0, |\Omega|\right)\right)$$

and if $|\{w(x,t)=w^*(s,t)\}|=0$ *for a.e.* $(s,t)\in(0,|\Omega|)\times(0,T)$, *the following derivation formula occurs*

$$\int_{w(x,y)>w^*(s,y)} \frac{\partial w}{\partial y}(x,y)\,dx=\int_0^s \frac{\partial w^*}{\partial y}(s,y)\,ds. \tag{17}$$

Moreover, what follows is a second order derivation formula due to Mercaldo and Ferone (see [21]), which is a suitable generalization of that contained in [1], where only analytic functions are considered.

Proposition 2.3. *Let* $w\in W^{2,\infty}(\mathcal{C}_\Omega)$. *Then for almost every* $y\in(0,+\infty)$ *the following derivation formula holds*:

$$\begin{aligned}
&\int_{w(x,y)>w^*(s,y)} \frac{\partial^2 w}{\partial y^2}(x,y)\,dx\\
&\quad=\frac{\partial^2}{\partial y^2}\int_0^s w^*(\sigma,y)\,d\sigma-\int_{w(x,y)=w^*(s,y)} \frac{(\frac{\partial w}{\partial y}(x,y))^2}{|\nabla_x w|}\,d\mathcal{H}^{N-1}(x)\\
&\qquad+\left(\int_{w(x,y)=w^*(s,y)} \frac{\frac{\partial w}{\partial y}(x,y)}{|\nabla_x w|}\,d\mathcal{H}^{N-1}(x)\right)^2\left(\int_{w(x,y)=w^*(s,y)} \frac{1}{|\nabla_x w|}\,d\mathcal{H}^{N-1}(x)\right)^{-1}.
\end{aligned}$$

2.3. Lorentz and Orlicz spaces

As we will deal with some sharp regularity results of the solution u to (4) in terms of the data f, we introduce here basic notions regarding the functional spaces where f will be supposed to belong to.

Let Ω be a bounded open subset of $\mathbb{R}^N$. We say that a measurable function $u:\Omega\to\mathbb{R}$ belongs to the Lorentz $L^{p,q}(\Omega)$ for $0<p,q\leqslant+\infty$ if the quantity

$$\|u\|_{L^{p,q}(\Omega)}=\begin{cases}\left(\int_0^{+\infty}\left[s^{\frac{1}{p}}u^*(s)\right]^q\frac{ds}{s}\right)^{\frac{1}{q}}, & 0<q<\infty,\\ \sup_{s\in(0,|\Omega|)} s^{\frac{1}{p}}u^*(s), & q=\infty,\end{cases} \tag{18}$$

is finite. We remark that for $p>1$, and $q\geqslant1$, the quantity in (18) can be equivalently defined replacing $u^*(t)$ with

$$u^{**}(s)=\frac{1}{s}\int_0^s u^*(\sigma)\,d\sigma.$$

We stress that the $L^{p,q}$-norm, for every $1<p,q\leqslant+\infty$, is rearrangement invariant, that is

$$\|u\|_{L^{p,q}(\Omega)}=\left\|u^\#\right\|_{L^{p,q}(\Omega^\#)}.$$

Besides, we emphasize that $L^{p,q}(\Omega)=L^p(\Omega)$, $L^{p,\infty}(\Omega)=\mathcal{M}_p$ (the Marcinkiewicz space) for any $1\leqslant p\leqslant\infty$ and, for $1<q<p<r<\infty$ the following inclusion occurs:

$$L^\infty(\Omega)\subset L^r(\Omega)\subset L^{p,1}(\Omega)\subset L^{p,q}(\Omega)\subset L^{p,p}(\Omega)=L^p(\Omega)\subset L^{p,r}(\Omega)\subset L^{p,\infty}(\Omega)\subset L^q(\Omega).$$

Furthermore, for any $r \in (1, \infty]$, if $r' \in [1, \infty)$ is the conjugate exponent of r, we define the Orlicz $L_{\Phi_r}(\Omega)$ generated by the N-function

$$\Phi_r(t) = \exp\bigl(|t|^{r'}\bigr) - 1$$

as the space of all measurable functions u on Ω such that there is a constant $\mathrm{c} = \mathrm{c}(u)$ for which

$$\int_{\Omega} \Phi_r(\mathrm{c}u)\, dx < \infty.$$

According to [5], we can characterize the Orlicz space $L_{\Phi_r}(\Omega)$ in terms of rearrangements, as the space made by all measurable functions u on Ω for which the following norm

$$\sup_{s \in (0, |\Omega|)} \frac{u^{**}(s)}{(1 + \log \frac{|\Omega|}{s})^{\frac{1}{r'}}} \tag{19}$$

is finite.

Now we provide some convolution inequalities due to O'Neil [33], which will play a key role to obtain some *a priori* estimates (hence regularity results) for solutions to problems of the type (4) in terms of data belonging to Lorentz spaces (see Theorems 4.3–4.4):

Lemma 2.1. *If f, g are two measurable functions on Ω, then*

$$(f * g)^{**}(s) \leqslant \int_{s}^{|\Omega|} f^{**}(\sigma) g^{**}(\sigma)\, d\sigma \quad \forall s \in \bigl(0, |\Omega|\bigr).$$

Theorem 2.1. *Suppose that $f \in L^{p_1, q_1}(\Omega)$, $g \in L^{p_2, q_2}(\Omega)$ where*

$$\frac{1}{p_1} + \frac{1}{p_2} > 1.$$

Then $f * g \in L^{p_3, q_3}(\Omega)$ where

$$\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p_3},$$

and $t \geqslant 1$ is any number such that

$$\frac{1}{q_1} + \frac{1}{q_2} \geqslant \frac{1}{q_3}.$$

Moreover

$$\|f * g\|_{L^{p_3, q_3}(\Omega)} \leqslant 3 p_3 \|f\|_{L^{p_1, q_1}(\Omega)} \|g\|_{L^{p_2, q_2}(\Omega)}.$$

2.4. Spectral decomposition of the solution

In this section we highlight some properties concerning the representation of the solution to the fractional Poisson equation by the Green function and the link with its spectral decomposition. According to what we have said in Subsection 2.1, it is always possible to get a spectral decomposition of the solution u to (4) in terms of the Fourier coefficients of the source term f. Indeed, suppose that $\{\varphi_k\}$ is an orthonormal basis of $L^2(\Omega)$ made by eigenfunctions of $-\Delta$ in Ω with zero Dirichlet boundary conditions and $\{\lambda_k\}$ the corresponding Dirichlet eigenvalues. Therefore, if $u \in H$ is the weak solution to problem (4), having the decomposition

$$u = \sum_{k=1}^{\infty} a_k \varphi_k, \tag{20}$$

then the fractional Laplacian of u has the spectral decomposition (11). Thus if

$$f = \sum_{k=1}^{\infty} c_k \varphi_k,$$

where $c_k = (f, \varphi_k)_{L^2(\Omega)}$ are the Fourier coefficient of f, the Fourier coefficients of u are

$$a_k = \frac{c_k}{\lambda_k^{\alpha/2}}. \tag{21}$$

Now, let us denote by $\mathcal{G}_D(x, y)$ the *Green function of a bounded domain* $D \subseteq \mathbb{R}^N$ for the fractional Laplacian $(-\Delta)^{\alpha/2}$. Then we have (see [29,10])

$$-(-\Delta)_x^{\alpha/2} \mathcal{G}_D(x, y) = \delta(x - y) \quad \text{in } \mathcal{D}'(D). \tag{22}$$

Next, suppose that the function $\mathcal{G}_\Omega$ has the following expansion, for any fixed $y \in \Omega$:

$$\mathcal{G}_\Omega(x, y) = \sum_{k=1}^{\infty} c_k(y) \varphi_k(x).$$

Then equality (11) provides the following spectral decomposition for the fractional Laplacian of $\mathcal{G}_\Omega$:

$$(-\Delta)_x^{\alpha/2} \mathcal{G}_\Omega(x, y) = \sum_{k=1}^{\infty} \lambda_k^{\alpha/2} c_k(y) \varphi_k(x). \tag{23}$$

If we multiply both sides of Eq. (22) by φ_m and integrate over Ω with respect to x, Eq. (23) links to

$$\sum_{k=1}^{\infty} \lambda_k^{\alpha/2} c_k(y) \int_\Omega \varphi_k(x) \varphi_m(x)\, dx = -\varphi_m(y)$$

i.e.

$$c_m(y) = -\frac{\varphi_m(y)}{\lambda_m^{\alpha/2}}$$

that is

$$\mathcal{G}_{\Omega}(x,y)=-\sum_{k=1}^{\infty}\frac{\varphi_k(x)\varphi_k(y)}{\lambda_k^{\alpha/2}}. \tag{24}$$

Hence from (20), (21) and (24) we easily infer that

$$u=\sum_{k=1}^{\infty}\frac{\varphi_k(x)}{\lambda_k^{\alpha/2}}\int_{\Omega} f(y)\varphi_k(y)\,dy=-\int_{\Omega}\mathcal{G}_{\Omega}(x,y)f(y)\,dy. \tag{25}$$

When D is a ball $B(0,R)$, we shall frequently use the following explicit expression of the Green function (see [7,36,29,10,7])

$$\mathcal{G}_{B(0,R)}(x,y)=-2^{-\alpha}\frac{\Gamma(\frac{N}{2})}{\pi^{N/2}}\Gamma\left(\frac{\alpha}{2}\right)^{-2}|x-y|^{\alpha-N}\int_0^z\frac{s^{\frac{\alpha}{2}-1}}{(s+1)^{N/2}}\,ds \tag{26}$$

where $x,y\in B(0,R)$ and

$$z=\frac{(R^2-|x|^2)(R^2-|y|^2)}{|x-y|^2}.$$

We stress that (26) coincides with the Green function of classical Laplacian for $\alpha=2$. Clearly we have

$$\left|\mathcal{G}_{B(0,R)}(x,y)\right|\leqslant\frac{\mathsf{a}\mathsf{b}}{|x-y|^{N-\alpha}} \tag{27}$$

for $x,y\in B(0,R)$ s.t. $x\neq y$, where

$$\mathsf{a}:=2^{-\alpha}\frac{\Gamma(\frac{N}{2})\Gamma(\frac{\alpha}{2})^{-2}}{\pi^{N/2}},\qquad \mathsf{b}:=\int_0^{\infty}\frac{s^{\frac{\alpha}{2}-1}}{(s+1)^{N/2}}\,ds. \tag{28}$$

3. Comparison result

The aim of this section is to obtain a comparison result between the solutions of problems (5) and (7). The symmetrization method allows to obtain a priori estimates which are the main tools to obtain regularity results.

Theorem 3.1. *Let w and v be the weak solutions to problems (5) and (7), respectively, and $f\in L^{\frac{2N}{N+\alpha}}(\Omega)$, with $\alpha\in(0,2)$. Then we have:*

$$\int_0^s w^*(\sigma,z)\,d\sigma\leqslant\int_0^s v^*(\sigma,z)\,d\sigma\quad\forall s\in\left[0,|\Omega|\right] \tag{29}$$

for any fixed $z\in[0,+\infty)$.

Proof. We first observe that actually there is a clever way to rewrite equation in problem (5), that is

$$\Delta_x w + \frac{1-\alpha}{y}\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} = 0.$$

As a matter of fact, if we follow [18] and make the change of variable

$$z = \left(\frac{y}{\alpha}\right)^{\alpha},$$

we find that problem (5) is equivalent to the Cauchy–Dirichlet problem

$$\begin{cases} z^{\beta}\dfrac{\partial^2 w}{\partial z^2} + \Delta_x w = 0 & \text{in } \mathcal{C}_{\Omega}, \\ w = 0 & \text{on } \partial_L \mathcal{C}_{\Omega}, \\ -\dfrac{\partial w}{\partial z}(x,0) = \kappa_{\alpha}\alpha^{\alpha-1} f(x) & \text{in } \Omega, \end{cases} \tag{30}$$

where $\beta := 2(\alpha-1)/\alpha$. The aim is to compare problem (30) with the corresponding symmetrized one:

$$\begin{cases} z^{\beta}\dfrac{\partial^2 v}{\partial z^2} + \Delta_x v = 0 & \text{in } \mathcal{C}_{\Omega}^{\#}, \\ v = 0 & \text{on } \partial_L \mathcal{C}_{\Omega}^{\#}, \\ -\dfrac{\partial v}{\partial z}(x,0) = \kappa_{\alpha}\alpha^{\alpha-1} f^{\#}(x) & \text{in } \Omega^{\#}. \end{cases} \tag{31}$$

Now we recall that w is smooth for any $z > 0$, so if for a *fixed* $z > 0$ we consider the test function

$$\varphi_h^z(x) = \begin{cases} \operatorname{sign}(w(x,z)) & \text{if } |w(x,z)| \geqslant t+h, \\ \frac{|w(x,z)|-t}{h}\operatorname{sign}(w(x,z)) & \text{if } t < |w(x,z)| < t+h, \\ 0 & \text{if } |w(x,z)| \leqslant t, \end{cases}$$

we can multiply the first equation in (30) by $\varphi_h^z(x)$ and integrate over Ω. A simple integration by parts yields the identity

$$\frac{1}{h}\int\limits_{t<|w|<t+h} |\nabla_x w|^2\,dx - z^{\beta}\frac{1}{h}\int\limits_{|w|>t+h} \frac{\partial^2 w}{\partial z^2}\,dx \\ - z^{\beta}\frac{1}{h}\int\limits_{t<|w|<t+h} \frac{\partial^2 w}{\partial z^2}\left(\frac{|w|-t}{h}\operatorname{sign}(w)\right)dx = 0.$$

Letting $h \to 0$ and using the isoperimetric inequality, by standard arguments (see *e.g.* [39]) we get

$$-z^{\beta}\int\limits_{w(x,z)>t} \frac{\partial^2 w}{\partial z^2}\,dx - \left(\frac{\partial \mu_w}{\partial t}\right)^{-1} N^2 \omega_N^{\frac{2}{N}} \big(\mu_w(t)\big)^{2-\frac{2}{N}} \leqslant 0.$$

Now if we set

$$U(s,z)=\int_0^s w^*(\sigma,z)\,d\sigma,$$

using the second order derivation formula of Proposition 2.3, we find that U verifies the following differential inequality

$$-z^\beta \frac{\partial^2 U}{\partial z^2}-p(s)\frac{\partial^2 U}{\partial s^2}\leqslant 0 \tag{32}$$

for a.e. $s\in(0,|\Omega|)$ and for any $z\in(0,+\infty)$, where $p(s)=N^2\ \omega_N^{\frac{2}{N}}s^{2-\frac{2}{N}}$. Moreover, the first order derivation formula (17) implies

$$\frac{\partial U}{\partial z}=\frac{\partial}{\partial z}\int_0^s w^*(\sigma,z)\,d\sigma=\frac{\partial}{\partial z}\int_{w(x,z)>w^*(s,z)} w(x,z)\,dx=\int_{w(x,z)>w^*(s,z)}\frac{\partial w}{\partial z}(x,z)\,dx,$$

hence making use of the Hardy–Littlewood inequality (15), we easily get

$$\begin{aligned}\frac{\partial U}{\partial z}(s,0)&=\int_{w(x,0)>w^*(s,0)}\frac{\partial w}{\partial z}(x,0)\,dx=-\alpha^{\alpha-1}\kappa_\alpha\int_{u(x)>u^*(s)} f(x)\,dx\\&\geqslant -\alpha^{\alpha-1}\kappa_\alpha\int_0^s f^*(\sigma)\,d\sigma,\quad s\in\big(0,|\Omega|\big).\end{aligned}$$

So the function U satisfies the following boundary conditions

$$\begin{aligned}&U(0,z)=0\quad \forall z\in[0,+\infty),\\&\frac{\partial U}{\partial s}\big(|\Omega|,z\big)=0\quad \forall z\in[0,+\infty),\\&\frac{\partial U}{\partial z}(s,0)\geqslant -\alpha^{\alpha-1}\kappa_\alpha\int_0^s f^*(\sigma)\,d\sigma,\quad s\in\big(0,|\Omega|\big).\end{aligned}$$

Now if v is the solution of the symmetrized problem (31), being v radially decreasing with respect to x, we obtain

$$-z^\beta \frac{\partial^2 V}{\partial z^2}-p(s)\frac{\partial^2 V}{\partial s^2}=0 \tag{33}$$

where

$$V(s,z)=\int_0^s v^*(\sigma,z)\,d\sigma.$$

Concerning the boundary conditions, we remark that in this case one has

$$\begin{aligned}\frac{\partial V}{\partial z}(s,0) &= -\alpha^{\alpha-1}\kappa_\alpha \int\limits_{v(|x|)>v^*(s)} f^{\#}(x)\,dx \\ &= -N\omega_N\alpha^{\alpha-1}\kappa_\alpha \int\limits_0^{(s/\omega_N)^{1/N}} f^*\big(\omega_N r^N\big) r^{N-1}\,dr \\ &= -\alpha^{\alpha-1}\kappa_\alpha \int\limits_0^s f^*(\sigma)\,d\sigma, \quad s\in\big(0,|\Omega|\big),\end{aligned}$$

therefore V satisfies the conditions

$$\begin{aligned}&V(0,z)=0 \quad \forall z\in[0,+\infty),\\ &\frac{\partial V}{\partial s}\big(|\Omega|,z\big)=0 \quad \forall z\in[0,+\infty),\\ &\frac{\partial V}{\partial z}(s,0) = -\alpha^{\alpha-1}\kappa_\alpha \int\limits_0^s f^*(\sigma)\,d\sigma, \quad s\in\big(0,|\Omega|\big).\end{aligned}$$

If we put

$$Z(s,z)=U(s,z)-V(s,z)=\int\limits_0^s \big[w^*(\sigma,z)-v^*(\sigma,z)\big]\,d\sigma$$

by (32) and (33), one has

$$L[Z] := -z^\beta \frac{\partial^2 Z}{\partial z^2} - p(s)\frac{\partial^2 Z}{\partial s^2} \leqslant 0$$

for a.e. $(s,z)\in D := (0,|\Omega|)\times(0,+\infty)$ and the following boundary conditions hold

$$\begin{aligned}&Z(0,z)=0 \quad \forall z\in[0,+\infty),\\ &\frac{\partial Z}{\partial s}\big(|\Omega|,z\big)=0 \quad \forall z\in[0,+\infty),\\ &\frac{\partial Z}{\partial z}(s,0)\geqslant 0, \quad s\in\big(0,|\Omega|\big).\end{aligned} \tag{34}$$

In particular

$$\frac{\partial Z}{\partial \nu}(s,0) = -\frac{\partial Z}{\partial z}(s,0)\leqslant 0, \quad s\in\big(0,|\Omega|\big), \tag{35}$$

where ν is the outward normal to the line segment $(0,|\Omega|)$. We observe that the operator L is elliptic in any point $(s,z)\in D$ hence by Hopf's maximum principle (see [35]), Z attains its maximum on the boundary of D, and in the points where the maximum is attained we get

$$\frac{\partial Z}{\partial \nu} > 0.$$

Hence by (34), (35), this ensures that

$$Z(s,z) \leqslant 0, \quad s \in \bigl[0, |\Omega|\bigr],$$

that is

$$\int_0^s w^*(\sigma, z)\, d\sigma \leqslant \int_0^s v^*(\sigma, z)\, d\sigma, \quad s \in \bigl[0, |\Omega|\bigr],$$

for any $z \in [0, +\infty)$. $\square$

Obviously, since ϕ the trace on $\Omega^\#$ of the solution v of (7) and u the trace on Ω of the solution w of (5), by Theorem 3.1 we get Theorem 1.1.

4. Regularity results

In this section we are interested in regularity results for solution u of problem (4). Using Theorems 1.1 and 3.1, we are able to prove some regularity results of the solution u in terms of the data f. In the following we will use the integral form (25) for the solution ϕ to the symmetrized problem (6), namely

$$\phi(x) = -\int_{\Omega^\#} \mathcal{G}_{\Omega^\#}(x,y) f^\#(y)\, dy. \tag{36}$$

We start by generalizing a well-known result for the classical Laplacian:

Theorem 4.1. *Let u be the solution to problem (4), where $f \in L^{\frac{N}{\alpha},1}(\Omega)$ with $0 < \alpha < 2$. Then $u \in L^\infty(\Omega)$.*

Proof. Let us consider the solution ϕ to problem (6). Since ϕ is radially decreasing, using (36) and (27) we obtain that, for some constant C,

$$\begin{aligned}
\|\phi\|_{L^\infty(\Omega^\#)} = \phi(0) &= \int_{\Omega^\#} f^\#(y)\bigl|\mathcal{G}_{\Omega^\#}(0,y)\bigr|\, dy \leqslant \mathsf{a}\,\mathsf{b} \int_{\Omega^\#} \frac{f^\#(y)}{|y|^{N-\alpha}}\, dy \\
&= \mathsf{a}\,\mathsf{b} \int_0^{R_\Omega} r^{\alpha-1} f^*\bigl(\omega_N r^N\bigr)\, dr = \mathsf{a}\,\mathsf{b} \int_0^{|\Omega|} s^{\frac{\alpha-N}{N}} f^*(s)\, ds = \mathsf{a}\,\mathsf{b} \|f\|_{L^{\frac{N}{\alpha},1}(\Omega)}.
\end{aligned}$$

On the other hand, Theorem 1.1 gives

$$\|u\|_{L^\infty(\Omega)} \leqslant \|\phi\|_{L^\infty(\Omega^\#)}$$

and the result follows. $\square$

Remark 4.1. We stress that if $f \in L^p(\Omega)$, for some $p > N/\alpha$, then according to Lorentz embedding (see Subsection 2.3) by Theorem 4.1 we get $u \in L^\infty(\Omega)$.

A consequence of the comparison result of Theorem 3.1 is the boundedness of the α-extension w of u in $\bar{\mathcal{C}}_\Omega$ when $f \in L^{\frac{N}{\alpha},1}(\Omega)$, for $0 < \alpha < 2$. To prove this result, we first compute the solution v to the radial problem (31) by using the separation of variable method. We look for a function v, radial with respect to x, such that

$$v(x,z) = X\big(|x|\big)W(z).$$

Putting v inside the first equation of (31), we find there must be a value λ such that

$$z^\beta \frac{W''(z)}{W(z)} = -\Delta_x X = \lambda \tag{37}$$

that is the function $X(x) = X(|x|)$ solves the classical eigenvalue problem for the Laplacian

$$\begin{cases} -\Delta_x X = \lambda X & \text{in } \Omega^\#, \\ X = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{38}$$

while $W(z)$ verifies the problem

$$\begin{cases} z^\beta W''(z) - \lambda W(z) = 0, \\ \lim\limits_{z\to+\infty} W(z) = 0. \end{cases} \tag{39}$$

Therefore $(\lambda, X) = (\lambda_k, X_k)$, for some k, where $\{\lambda_k\}$ and $\{X_k(|x|)\}$ are the eigenvalues and the *radial* eigenfunctions of the Laplace operator in $\Omega^\#$ with zero Dirichlet boundary values on $\partial\Omega^\#$, namely

$$\lambda_k = \left(\frac{\theta_k}{R_\Omega}\right)^2, \quad k = 1, 2, \ldots, \tag{40}$$

where

$$R_\Omega = \left(\frac{|\Omega|}{\omega_N}\right)^{1/N}$$

is the radius of the ball $\Omega^\#$, θ_k are the zeros of the Bessel function $J_{(N-2)/2}(z)$ of order $(N-2)/2$, and

$$X_k(r) = \frac{1}{R_\Omega |J_{\frac{N}{2}}(\theta_k)|}\left(\frac{2}{N\omega_N}\right)^{1/2} r^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}\left(\frac{\theta_k}{R_\Omega} r\right), \quad k = 1, 2, \ldots, \tag{41}$$

where $r := |x|$. We recall that the system $\{X_k(|x|)\}$ forms an orthonormal basis of the space $L^2_{rad}(\Omega^\#)$ made by all radial functions in L^2.

Then, since the solution ϕ of (6) is radially decreasing, we can represent it by

$$\phi(r) = \sum_{k=1}^\infty a_k X_k(r), \tag{42}$$

where the a_k are given by (21), and c_k are the Fourier coefficients of $f^{\#}$ with respect to (41), i.e.

$$c_k = N\omega_N \int_0^{R_\Omega} r^{N-1} f^*(\omega_N r^N) X_k(r)\,dr.$$

Now, to each eigenvalue λ_k we associate a solution W_k to problem (39). The first equation in (39) is a modified Bessel equation (see [34,30]), whose solutions are combinations of Bessel functions of the third kind. According to the asymptotic behavior at infinity of the Bessel functions (see [30]), we have that

$$W_k(z) = C_k H_k(z), \tag{43}$$

where

$$H_k(z) := \sqrt{z} K_{\frac{1}{2-\beta}}\left(\frac{2}{2-\beta}\sqrt{\lambda_k} z^{\frac{2-\beta}{2}}\right),$$

$\beta = 2(\alpha-1)/\alpha$, the C_k are constants and $K_\nu(t)$ is a Bessel function of the third kind. We also notice that

$$H_k'(z) = -z^{\frac{1-\beta}{2}} \sqrt{\lambda_k} K_{\frac{1-\beta}{2-\beta}}\left(\frac{2}{2-\beta}\sqrt{\lambda_k} z^{\frac{2-\beta}{2}}\right). \tag{44}$$

Finally, using the boundary condition of problem (31), we can write the following explicit expression of v (here $r = |x|$):

$$v(r,z) = \sum_{k=1}^{\infty} X_k(r) W_k(z) = \frac{1}{R_\Omega}\left(\frac{2}{N\omega_N}\right)^{1/2} r^{-\frac{N-2}{2}} \sum_{k=1}^{\infty} \frac{C_k}{|J_{\frac{N}{2}}(\theta_k)|} J_{\frac{N-2}{2}}\left(\frac{\theta_k}{R_\Omega} r\right) H_k(z), \tag{45}$$

with coefficients

$$C_k H_k'(0) = -\frac{(2N\omega_N)^{1/2}\alpha^{\alpha-1}\kappa_\alpha}{R_\Omega |J_{\frac{N}{2}}(\theta_k)|} \int_0^{R_\Omega} r^{\frac{N}{2}} J_{\frac{N-2}{2}}\left(\theta_k \frac{r}{R_\Omega}\right) f^*(\omega_N r^N)\,dr. \tag{46}$$

Of course the trace $v(r,0)$ given in (45) coincides with the solution ϕ represented by (42). Indeed by the asymptotic behavior (see [30])

$$K_\nu(t) \approx \frac{2^{\nu-1}\Gamma(\nu)}{t^\nu}, \quad t \to 0, \tag{47}$$

then by (44), (46) and (42) we find

$$\begin{aligned} v(r,0) &= \sum_{k=1}^{\infty} X_k(r) C_k H_k(0) \\ &= \frac{(2N\omega_N)^{1/2}}{R_\Omega} \sum_{k=1}^{\infty} \frac{\lambda_k^{-\frac{\alpha}{2}} X_k(r)}{|J_{\frac{N}{2}}(\theta_k)|} \int_0^{R_\Omega} t^{N/2} J_{\frac{N-2}{2}}\left(\frac{\theta_k}{R_\Omega} t\right) f^*(\omega_N t^N)\,dt \\ &= \phi(x). \end{aligned}$$

Now we are able to prove the following result.

Theorem 4.2. *Let w the solution to problem (5), where $f \in L^{\frac{N}{\alpha},1}(\Omega)$ with $0<\alpha<2$. Then $w \in L^{\infty}(\bar{\mathcal{C}}_{\Omega})$.*

Proof. Let v be the solution of (7) as in (45). By the asymptotic behavior of the Bessel functions K_{ν} at infinity (see [30]) we deduce that $H_k(z) \to 0$ as $z \to \infty$, therefore $v(r,z) \to 0$ as $z \to \infty$. Besides, since $v(x,0) = \phi(x)$, by Theorem 4.1 we find that $v \in L^{\infty}(\bar{\mathcal{C}}_{\Omega^{\#}})$. Moreover, Theorem 3.1 ensures that

$$\left\| w(\cdot,z)\right\|_{L^{\infty}(\Omega)} \leqslant \left\| v(\cdot,z)\right\|_{L^{\infty}(\Omega^{\#})} \quad \forall z \in [0,+\infty), \tag{48}$$

hence $w \in L^{\infty}(\bar{\mathcal{C}}_{\Omega})$. □

We emphasize that the result of Theorem 4.2 is not new (see for instance [13,14]), although our techniques make us able to achieve the sharper L^{∞} estimate (48).

Now we provide *new* regularity results when f belongs to Lorentz spaces $L(p,r)$ for $p < N/\alpha$, obtaining the generalization of the corresponding classical regularity result for the Laplacian.

Theorem 4.3. *Let u be the solution to problem (4), where $f \in L^{p,r}(\Omega)$ with*

$$\frac{2N}{N+\alpha} \leqslant p < \frac{N}{\alpha}$$

and $r \geqslant 1$. Then $u \in L^{q,r}(\Omega)$ with

$$q := \frac{Np}{N-\alpha p}.$$

Proof. Inserting inequality (27) into (36) we find

$$\left|\phi(x)\right| \leqslant \mathrm{a}\,\mathrm{b}\left(f^{\#} * |x|^{\alpha-N}\right).$$

Then applying Theorem 2.3 with the choices $g = |x|^{\alpha-N}$, $p_1 = p$, $p_2 = N/(N-\alpha)$, $q_1 = r$, $q_2 = \infty$, we have $p_3 = q = Np/(N-\alpha p)$, $q_3 = r$ and

$$\begin{aligned}\|\phi\|_{L^{q,r}(\Omega^{\#})} &\leqslant \mathrm{a}\,\mathrm{b}\left\| f^{\#} * |x|^{\alpha-N}\right\|_{L^{q,r}(\Omega^{\#})}\\ &\leqslant 3\mathrm{a}\,\mathrm{b}\,q\|f\|_{L^{p,r}(\Omega^{\#})}\left\||x|^{\alpha-N}\right\|_{L^{N/(N-\alpha),\infty}(\Omega^{\#})}\\ &= 3\mathrm{a}\,\mathrm{b}\,q\|f\|_{L^{p,r}(\Omega^{\#})}.\end{aligned} \tag{49}$$

Finally by Theorem 1.1 we get

$$\|u\|_{L^{q,r}(\Omega)} \leqslant \|\phi\|_{L^{q,r}(\Omega^{\#})}$$

and inequality (49) allows to conclude. □

Just like in the classical case $\alpha = 2$, it is possible to show that whenever we choose the source term f into the Lorentz space $L^{\frac{N}{\alpha},r}(\Omega)$, the solution u to (4) belongs to a suitable Orlicz space. Indeed, we have the following result:

Theorem 4.4. *Let u be the solution to problem (4), where $f \in L^{\frac{N}{\alpha},r}(\Omega)$, with $r \in (1,\infty]$ and $0<\alpha<2$. Then $u \in L_{\Phi_r}(\Omega)$.*

Proof. According to what explained in Subsection 2.3, we may interpret $L_{\Phi_r}(\Omega)$ as made by all the functions u measurable on Ω, for which the quantity (19) is finite. Therefore, we can use Theorem 1.1, Lemma 2.1 and identity (36) to obtain

$$\begin{aligned} u^{**}(s) &\leqslant \phi^{**}(s) \\ &\leqslant C\big(f^{\#} * |\cdot|^{\alpha-N}\big)^{**} \\ &\leqslant C\int_s^{|\Omega|} \sigma^{\frac{\alpha}{N}} f^{**}(\sigma)\frac{d\sigma}{\sigma}. \end{aligned} \tag{50}$$

Now, if $r=\infty$ we have from (50)

$$\frac{u^{**}(s)}{1+\log\frac{|\Omega|}{s}} \leqslant \frac{C}{1+\log\frac{|\Omega|}{s}}\log\frac{|\Omega|}{s} \leqslant C$$

and we have done. If instead $r \in (1,\infty)$, using Hölder inequality in (50) we easily obtain

$$\begin{aligned} \frac{u^{**}(s)}{(1+\log\frac{|\Omega|}{s})^{1/r'}} &\leqslant \frac{C(\log\frac{|\Omega|}{s})^{1/r'}}{(1+\log\frac{|\Omega|}{s})^{1/r'}}\left(\int_0^{|\Omega|}\big[\sigma^{\frac{\alpha}{N}} f^{**}(\sigma)\big]^r\frac{d\sigma}{\sigma}\right)^{1/r} \\ &= C\|f\|_{L^{\frac{N}{\alpha},r}(\Omega)} \end{aligned}$$

and the result follows for all $r \in (1,\infty]$. □

5. Best constant in L^∞ estimate

In virtue of Theorem 4.1, if $f \in L^{N/\alpha,1}(\Omega)$ there is a constant C such that

$$\|u\|_{L^\infty} \leqslant \mathsf{C}\|f\|_{L^{\frac{N}{\alpha},1}(\Omega)}.$$

Due to the form of the Green function in (26), it seems quite difficult to face the problem of finding the *best* value for C in (51). Nevertheless, we remark that this becomes reasonably easy when one seeks the best C in the following inequality (see Remark 4.1)

$$\|u\|_{L^\infty} \leqslant \mathsf{C}\|f\|_{L^p(\Omega)} \tag{51}$$

where $f \in L^p(\Omega)$ for some $p > N/\alpha$. In fact, since the solution ϕ is radially decreasing, in order to get an L^∞ estimate of ϕ it is enough to look for a sharp upper bound of $\phi(0)$. To this end, we first observe that (36) yields

$$\phi(0) = -\int_0^{|\Omega|} \psi\big((s/\omega_N)^{1/N}\big) f^*(s)\,ds, \tag{52}$$

where

$$\psi(t) := -2^{-\alpha} \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}} \Gamma\left(\frac{\alpha}{2}\right)^{-2} t^{\alpha-N} \int_0^{\frac{R_\Omega^2}{t^2}(R_\Omega^2 - t^2)} \frac{s^{\frac{\alpha}{2}-1}}{(s+1)^{N/2}}\, ds. \tag{53}$$

We remark that it is possible to write an explicit form of the integral at the right-hand side of (53). Indeed, we know that (*e.g.* see [23])

$$\int_0^w \frac{s^{\frac{\alpha}{2}-1}}{(s+1)^{N/2}}\, ds = 2\frac{w^{\alpha/2}}{\alpha} {}_2F_1\left(\frac{N}{2}; \frac{\alpha}{2}; 1 + \frac{\alpha}{2}; -w\right)$$

where ${}_2F_1(\cdot\,;\cdot\,;\cdot\,;z)$ denotes the *Gauss hypergeometric function*. Therefore by (53)

$$\begin{aligned}
&\psi\big((s/\omega_N)^{1/N}\big) \\
&\quad = \mathcal{B}_{N,\alpha} s^{\frac{\alpha-N}{N}} \left[\frac{1}{s^{\alpha/N}} \big(|\Omega|^{2/N} - s^{2/N}\big)^{\alpha/2} {}_2F_1\left(\frac{N}{2}; \frac{\alpha}{2}; 1 + \frac{\alpha}{2}; \frac{|\Omega|^{2/N}}{\omega_N^{2/N} s^{2/N}} \big(s^{2/N} - |\Omega|^{2/N}\big)\right)\right]
\end{aligned}$$

where

$$\mathcal{B}_{N,\alpha} := -\frac{2^{1-\alpha}\Gamma(\frac{N}{2})|\Omega|^{\frac{\alpha}{N}}}{\alpha\Gamma(\frac{\alpha}{2})^2 \pi^{N/2} \omega_N^{2\frac{\alpha}{N}-1}}.$$

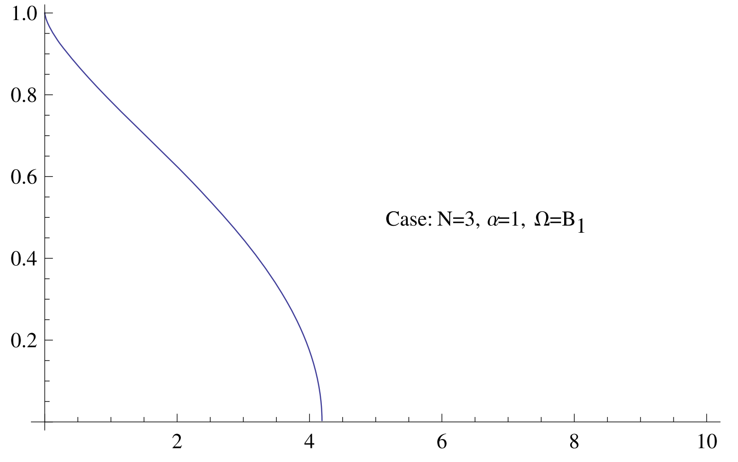
So if we set

$$\varphi(s) := \frac{1}{s^{\alpha/N}} \big(|\Omega|^{2/N} - s^{2/N}\big)^{\alpha/2} {}_2F_1\left(\frac{N}{2}; \frac{\alpha}{2}; 1 + \frac{\alpha}{2}; \frac{|\Omega|^{2/N}}{\omega_N^{2/N} s^{2/N}} \big(s^{2/N} - |\Omega|^{2/N}\big)\right)$$

using Hölder inequality in (52) we have

$$\big|\phi(0)\big| \leqslant |\mathcal{B}_{N,\alpha}| \|f\|_{L^p(\Omega)} \left(\int_0^{|\Omega|} s^{\frac{\alpha-N}{N}p'} \big[\varphi(s)\big]^{p'}\, ds\right)^{1/p'}.$$

We point out that the function φ is bounded in $[0, |\Omega|]$ (see also the picture below), so the integral at the right-hand side of the last inequality converges if and only if $p > N/\alpha$.



The best constant $\mathsf{C}(N, p, \alpha, \Omega)$ in (51) is then

$$\mathsf{C}(N, p, \alpha, \Omega) := |\mathcal{B}_{N,\alpha}| \left(\int_0^{|\Omega|} s^{\frac{\alpha-N}{N} p'} \varphi(s)^{p'} \, ds \right)^{1/p'}.$$

Example 5.1. Let us calculate the best constant C in the case of the square root of the Laplacian $\sqrt{-\Delta}$ (i.e. the case $\alpha = 1$), when $N = 3$ and $\Omega = B(0, 1)$. In this case, we have the following, explicit form of the Gauss hypergeometric function:

$${}_2F_1\left(\frac{3}{2}; \frac{1}{2}; \frac{3}{2}, -z\right) = \frac{1}{\sqrt{z+1}}.$$

Then

$$\varphi(s) = \left(\frac{3}{4\pi}\right)^{1/3} \sqrt{\left(\frac{4}{3}\pi\right)^{2/3} - s^{2/3}}$$

and we have, by a change of variable,

$$\begin{aligned} \mathsf{C}(p) &= \frac{(2\pi)^{1/p'}}{2\pi^2} \left(\int_0^1 t^{\frac{1}{2} - p'} (1-t)^{\frac{p'}{2}} \, dt \right)^{1/p'} \\ &= \frac{(2\pi)^{1/p'}}{2\pi^2} B\left(\frac{p-3}{2(p-1)}, \frac{3p-2}{2(p-1)} \right)^{(p-1)/p}, \end{aligned}$$

where $B(\cdot,\cdot)$ is the Euler beta function.

References

[1] A. Alvino, G. Trombetti, J.I. Diaz, P.L. Lions, Elliptic equations and Steiner symmetrization, Comm. Pure Appl. Math. 49 (1996) 217–236.

[2] A. Alvino, G. Trombetti, P.-L. Lions, Comparison results for elliptic and parabolic equations via Schwarz symmetrization, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990) 37–65.
[3] C. Bandle, Isoperimetric Inequalities and Applications, Monogr. Stud. Math., vol. 7, Pitman (Advanced Publishing Program), Boston, MA, 1980.
[4] R. Bañuelos, P.J. Méndez-Hernández, Symmetrization of Lévy processes and applications, J. Funct. Anal. 258 (2010) 4026–4051.
[5] C. Bennett, R. Sharpley, Interpolation of Operators, Pure Appl. Math., vol. 129, Academic Press Inc., Boston, MA, 1988.
[6] D. Betsakos, Symmetrization, symmetric stable processes, and Riesz capacities, Trans. Amer. Math. Soc. 356 (2004) 735–755 (electronic).
[7] R.M. Blumenthal, R.K. Getoor, D.B. Ray, On the distribution of first hits for the symmetric stable processes, Trans. Amer. Math. Soc. 99 (1961) 540–554.
[8] K. Bogdan, T. Byczkowski, Potential theory of Schrödinger operator based on fractional Laplacian, Probab. Math. Statist. 20 (2000) 293–335.
[9] K. Bogdan, T. Grzywny, Heat kernel of fractional Laplacian in cones, Colloq. Math. 118 (2010) 365–377.
[10] K. Bogdan, T. Jakubowski, Estimates of the Green function for the fractional Laplacian perturbed by gradient, http://arxiv.org/abs/1009.2472.
[11] K. Bogdan, T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, Comm. Math. Phys. 271 (2007) 179–198.
[12] K. Bogdan, P. Sztonyk, Estimates of the potential kernel and Harnack's inequality for the anisotropic fractional Laplacian, Studia Math. 181 (2007) 101–123.
[13] C. Brändle, E. Colorado, A. de Pablo, A concave–convex elliptic problem involving the fractional Laplacian, arXiv:1006.4510v2.
[14] X. Cabré, Y. Sire, Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates, arXiv:1012.0867v2.
[15] X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, Adv. Math. 224 (2010) 2052–2093.
[16] L. Caffarelli, Free boundary problems for fractional powers of the Laplacian, in: A Great Mathematician of the Nineteenth Century. Papers in Honor of Eugenio Beltrami (1835–1900), in: Ist. Lombardo Accad. Sci. Lett. Incontr. Studio, vol. 39, LED–Ed. Univ. Lett. Econ. Diritto, Milan, 2007, pp. 273–286 (in Italian).
[17] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.
[18] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.
[19] L.A. Caffarelli, J.-M. Roquejoffre, Y. Sire, Variational problems for free boundaries for the fractional Laplacian, J. Eur. Math. Soc. (JEMS) 12 (2010) 1151–1179.
[20] L.A. Caffarelli, S. Salsa, L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, Invent. Math. 171 (2008) 425–461.
[21] V. Ferone, A. Mercaldo, A second order derivation formula for functions defined by integrals, C. R. Acad. Sci. Paris Sér. I Math. 326 (1998) 549–554.
[22] A. Garroni, S. Müller, Γ-limit of a phase-field model of dislocations, SIAM J. Math. Anal. 36 (2005) 1943–1964 (electronic).
[23] I.S. Gradstein, I.M. Ryshik, Summen-, Produkt- und Integraltafeln, Bands 1, 2, language ed., Verlag Harri Deutsch, Thun, 1982, translation from the Russian edited by Ludwig Boll, based on the second German–English edition translated by Christa Berg, Lothar Berg and Martin Strauss, incorporating the fifth Russian edition edited by Yu.V. Geronimus and M.Yu. Tseĭtlin.
[24] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, 2nd ed., University Press, Cambridge, 1952.
[25] C. Imbert, R. Monneau, E. Rouy, Homogenization of first order equations with (u/ϵ)-periodic Hamiltonians, II. Application to dislocations dynamics, Comm. Partial Differential Equations 33 (2008) 479–516.
[26] B. Kawohl, Rearrangements and Convexity of Level Sets in PDE, Lecture Notes in Math., vol. 1150, Springer-Verlag, Berlin, 1985.
[27] A.C. Kort, Solutions of a pure critical exponent problem involving the half-Laplacian in annular-shaped domains, preprint, arXiv:1004.3800v1 [math.AP].
[28] A.C. Kort, J. Dávila, L. Dupaigne, Y. Sire, Regularity of radial extremal solutions for some non local semilinear equations, preprint, arXiv:1004.3800v1 [math.AP].
[29] N.S. Landkof, Foundations of Modern Potential Theory, Grundlehren Math. Wiss., vol. 180, Springer-Verlag, New York, 1972, translated from the Russian by A.P. Doohovskoy.
[30] N.N. Lebedev, Special Functions and Their Applications, revised English edition, Prentice Hall Inc., Englewood Cliffs, NJ, 1965, translated and edited by Richard A. Silverman.
[31] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, vol. 1, Trav. Rech. Math., vol. 17, Dunod, Paris, 1968.
[32] J. Mossino, J.-M. Rakotoson, Isoperimetric inequalities in parabolic equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 13 (1986) 51–73.
[33] R. O'Neil, Convolution operators and $L(p,q)$ spaces, Duke Math. J. 30 (1963) 129–142.
[34] A.D. Polyanin, V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, second ed., Chapman & Hall/CRC, Boca Raton, FL, 2003.
[35] M.H. Protter, H.F. Weinberger, Maximum Principles in Differential Equations, Prentice Hall Inc., Englewood Cliffs, NJ, 1967.
[36] M. Riesz, Intégrales de Riemann–Liouville et potentiels, Acta Sci. Math. (Szeged) 9 (1938/1940) 1–42.

[37] K.-i. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge Stud. Adv. Math., vol. 68, Cambridge University Press, Cambridge, 1999, translated from the 1990 Japanese original, revised by the author.
[38] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. 60 (2007) 67–112.
[39] G. Talenti, Elliptic equations and rearrangements, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 3 (1976) 697–718.
[40] G. Talenti, Linear elliptic p.d.e.'s: level sets, rearrangements and a priori estimates of solutions, Boll. Unione Mat. Ital. B (6) 4 (1985) 917–949.
[41] E. Valdinoci, O. Savin, Density estimates for a nonlocal variational model via the Sobolev inequality, arXiv:1103.6205v1.