On the Derivation Algebras of Lie Module Triple Systems

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1. INTRODUCTION

We continue here work on the classification of Lie module triple systems over an algebraically closed field $k$ of characteristic zero begun in [4], where we indicated how to decompose a Lie module triple system under certain restrictions into constituents from two relatively simple subclasses. We study these two subclasses in the present paper.

Recall [5] that a Lie module triple system (abbreviated LMTS) is formed from a finite dimensional Lie algebra $\mathcal{L}$ having a nondegenerate symmetric associative (invariant) bilinear form $b$ and a finite dimensional faithful $\mathcal{L}$-module $M$ having a nondegenerate $\mathcal{L}$-invariant bilinear form $\varphi$, that is,

$$\varphi(xl, y) = -\varphi(x, yl)$$

for all $x, y \in M, l \in \mathcal{L}$. A triple product $\{ , , \}$ is defined on $M$ by defining $\{xyz\} := xR(y, z)$, where $R: M \times M \to \mathcal{L}$ is defined by setting

$$b(l, R(y, z)) = \varphi(zl, y)$$

for all $y, z \in M, l \in \mathcal{L}$. The Lie module triple system $(M, \{ , , \})$ is denoted $(M, \{ , , \}, \mathcal{L}, b, \varphi)$ when the ingredients need to be specified.

$(M, \{ , , \}, \mathcal{L}, b, \varphi)$ is a Type I LMTS if $M$ is an irreducible $\mathcal{L}$-module and is a Type II LMTS if $M$ is the direct sum of two irreducible $\mathcal{L}$-submodules such that $\varphi$ restricted to any irreducible $\mathcal{L}$-submodule of $M$ is zero. By Corollary 3.5 of [4], Type I and Type II LMTSs are the constituents out of which can be formed any LMTS $(M, \{ , , \}, \mathcal{L}, b, \varphi)$ for which $M$ is a completely reducible $\mathcal{L}$-module with $\varphi$ symmetric or symplectic.
The goal of this paper is to classify Type I and Type II LMTSs by considering their derivation algebras. In Section 2 we prove that their derivation algebras are reductive and in Section 3 we give a bound on the dimension of toral subalgebras centralizing $\mathcal{L}$ in the derivation algebra of $(M, \{ , , \}, \mathcal{L}, b, \varphi)$. We use these results in Section 4 to obtain the classification.

2. REDUCTIVITY OF DERIVATION ALGEBRAS

Recall from Corollary 2.11 of [5] that in a LMTS $(M, \{ , , \}, \mathcal{L}, b, \varphi)$, $\mathcal{L} = R(M, M) := \{\sum_{i=1}^{n} R(y_i, z_i) | y_i, z_i \in M\}$ and that [5, (2.8.1)]

$$\{\{xyz\ uv\} - \{\{xuw\} yz\} = \{x\{yuv\}z\} + \{xy\{zuv\}\}$$

(2.1)

for all $x, y, z, u, v \in M$. In operator form (2.1) becomes

$$[R(y, z), R(u, v)] = R(yR(u, v), z) + R(y, zR(u, v)).$$

(2.1')

Hence $R(u, v)$ is a derivation of $(M, \{ , , \})$, where $D \in \text{End} M$ is a derivation of $(M, \{ , , \})$ if

$$[R(y, z), D] = R(yD, z) + R(y, zD)$$

(2.2)

for all $y, z \in M$. Elements of $\mathcal{L}$ are called inner derivations. Equation (2.2) implies that $\mathcal{L}$ is an ideal of $\text{Der}(M, \{ , , \})$, the Lie subalgebra of $\text{End} M$ consisting of derivations of $(M, \{ , , \})$.

$\text{End}_{\mathcal{L}} M$ will denote the Lie subalgebra of $\text{End} M$ of $\mathcal{L}$-module endomorphisms of $M$. Note that $\text{End}_{\mathcal{L}} M$ is the centralizer of $\mathcal{L}$ in $\text{End} M$. We need the following technical lemma.

Lemma 2.3. Suppose $(M, \{ , , \}, \mathcal{L}, b, \varphi)$ is a LMTS with $M = \{MMM\}$ and $D \in \text{End}_{\mathcal{L}} M$. Then $D \in \text{Der}(M, \{ , , \})$ if and only if for all $y, z \in M$

$$\varphi(zD, y) = -\varphi(z, yD).$$

(2.3.1)

Proof. If $D \in \text{Der}(M, \{ , , \}) \cap \text{End}_{\mathcal{L}} M$, $R(yD, z) = -R(y, zD)$ and $lD = Dl$ for all $l \in \mathcal{L}$ so by (1.2)

$$\varphi(zl, yD) = b(l, R(yD, z)) = -b(l, R(y, zD))$$

$$= -\varphi(zDl, y) = -\varphi(zlD, y)$$

so (2.3.1) holds since $M$ is spanned by elements of the form $zl$. The converse follows from the same argument in reverse.
THEOREM 2.4. Suppose \((M, \{\cdot, \cdot, \cdot\}, \mathcal{L}, b, \phi)\) is a semisimple LMTS such that \(M\) is a completely reducible \(\mathcal{L}\)-module and \(\phi\) is symmetric or symplectic. Then \(\text{Der}(M, \{\cdot, \cdot, \cdot\})\) is reductive.

Proof: Since \(M\) is a completely reducible \(\mathcal{L}\)-module, \(\mathcal{L}\) is reductive \([7, \text{Theorem 10, p. 811}],\) i.e., \(\mathcal{L} = [\mathcal{L}, \mathcal{L}] \oplus C\) with \([\mathcal{L}, \mathcal{L}]\) semisimple and \(C\) a central ideal whose elements act semisimply on \(M\). Thus, by Schur’s lemma if \(M = M_1 \oplus \cdots \oplus M_n\) as an \(\mathcal{L}\)-module with \(M_i\), irreducible for \(i = 1, \ldots, n\), then for all \(c \in C\) there are scalars \(\alpha_i(c) \in k\) with \(x_i c = \alpha_i(c)x_i\) for all \(x_i \in M_i\).

Define \(\text{ad}: \text{Der}(M, \{\cdot, \cdot, \cdot\}) \to \text{Der} \mathcal{L}\) by \(l(\text{ad} D) = [l, D]\). \(\text{ad}\) is a Lie algebra homomorphism and since \(\text{Der}[\mathcal{L}, \mathcal{L}] \cong \text{ad}[\mathcal{L}, \mathcal{L}] \cong [\mathcal{L}, \mathcal{L}]\) by the semisimplicity of \([\mathcal{L}, \mathcal{L}]\), \([\mathcal{L}, \mathcal{L}]\) \cap \ker \text{ad} = 0 and for all \(D \in \text{Der}(M, \{\cdot, \cdot, \cdot\})\) there is an \(l \in [\mathcal{L}, \mathcal{L}]\) such that for all \(l_1 \in [\mathcal{L}, \mathcal{L}]\), \([l_1, l - D] = 0\). Thus \(\text{Der}(M, \{\cdot, \cdot, \cdot\}) = [\mathcal{L}, \mathcal{L}] \oplus \mathcal{K}\), where \(\mathcal{K} = \text{Der}(M, \{\cdot, \cdot, \cdot\}) \cap \text{End}_{[\mathcal{L}, \mathcal{L}]} M\). \([\mathcal{L}, \mathcal{L}]\) is semisimple so once it is shown that \(\mathcal{K}\) is reductive, we will have the desired result.

Now if \(D \in \mathcal{K}\), by the Jacobi identity \([c, D] \in C\) for all \(c \in C\) since \(\mathcal{L}\) is an ideal of \(\text{Der}(M, \{\cdot, \cdot, \cdot\})\). If \(x \in M_i\) with \(x \neq 0\), \(xD = x_1 + \cdots + x_n\) with \(x_j \in M_j\) for \(1 \leq j \leq n\). Thus for all \(c \in C\),

\[
\alpha_i([c, D])x = x[c, D] = xcD - xDc = (\alpha_i(c) - \alpha_1(c))x_1 + \cdots + 0x_i + \cdots + (\alpha_n(c) - \alpha_n(c))x_n
\]

so \(\alpha_i([c, D]) = 0\) for \(i = 1, \ldots, n\) and hence \([c, D] = 0\) for all \(c \in C\). Hence \(D \in \ker \text{ad} = \text{Der}(M, \{\cdot, \cdot, \cdot\}) \cap \text{End}_\mathcal{L} M\). Now by Lemma 2.3, \(\mathcal{K} = \ker \text{ad}\) is the centralizer of \(\mathcal{L}'\) in \(G := \{A \in \text{End} M \mid \varphi(xA, y) = -\varphi(x, yA)\}\) for all \(x, y \in M\) since the semisimplicity of \((M, \{\cdot, \cdot, \cdot\})\) gives \(M = \{MMM\}\) by Proposition 4.2 of \([4]\). Since \(\varphi\) is symmetric or symplectic, \(G\) is a Lie algebra of type \(B, C,\) or \(D\) and hence is reductive. Thus \(\mathcal{K}\) is reductive by Theorem 7 of \([6]\).

Recall from \([5]\) that \((M, \{\cdot, \cdot, \cdot\})\) is abelian if \(\{xyz\} = 0\) for all \(x, y, z \in M\) and an ideal \(N\) of \((M, \{\cdot, \cdot, \cdot\})\) is central if \(\{xyz\} = \{yoz\} = \{yzx\} = 0\) for all \(x \in N, y, z \in M\).

LEMMA 2.5. If \((M, \{\cdot, \cdot, \cdot\}, \mathcal{L}, b, \phi)\) is a Type I or Type II LMTS, then \((M, \{\cdot, \cdot, \cdot\})\) is either abelian or simple.

Proof: By Proposition 4.2 of \([5]\), \(M = M_1 \oplus Z(M)\), where \((M, \{\cdot, \cdot, \cdot\}, \mathcal{L}, b, \varphi|_{M_1}),\) and \((Z(M), \{\cdot, \cdot, \cdot\}), 0, 0, \varphi|_{Z(M)}\) are ideals of \(M\) with \(M_1\) semisimple and \(Z(M)\) central. Since an ideal of \((M, \{\cdot, \cdot, \cdot\})\) is an \(\mathcal{L}\)-submodule, if \((M, \{\cdot, \cdot, \cdot\})\) is Type I, either \(M = Z(M)\), in which case...
Theorem 4.3 of [5] shows that $(M, \{\ ,\ ,\ \})$ is simple.

**Lemma 2.6.** Suppose $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \varphi)$ is a LMTS and $M_1$ and $M_2$ are irreducible $\mathcal{L}$-submodules of $M$. If $\varphi$ restricted to $M_2 \times M_1$ is nonzero, there is an $a \in k$ with $\varphi(x, y) = a \varphi(y, x)$ for all $x \in M_1, y \in M_2$. In particular, $\varphi$ is symmetric or symplectic if $(M, \{\ ,\ ,\ \})$ is Type I or if $(M, \{\ ,\ ,\ \})$ is Type II and is the direct sum of two isomorphic irreducible $\mathcal{L}$-submodules.

**Proof.** The first statement follows from Section 7.5 of [1]. If $(M, \{\ ,\ ,\ \})$ is Type I, $\varphi \neq 0$ so taking $M_1 = M_2 = M$, $\varphi(x, y) = a \varphi(y, x) = a^2 \varphi(x, y)$ and this gives $a = \pm 1$ so $\varphi$ is symmetric or symplectic.

Suppose $(M, \{\ ,\ ,\ \})$ is Type II and $M = M_1 \oplus M_2$ as an $\mathcal{L}$-module with $M_1 \cong M_2$. Then $\varphi$ restricted to $M_i \times M_i$ is zero for $i = 1, 2$ so $\varphi$ restricted to $M_1 \times M_2$ and $\varphi$ restricted to $M_2 \times M_1$ are both nonzero and hence nondegenerate by irreducibility. In particular, if $x_i \in M_i$ is a nonzero lowest weight vector and $y_j \in M_j$ is a nonzero highest weight vector with $j \neq i$, then $\varphi(x_i, y_j) \neq 0$ by (1.1). Now if $x_i \in M_i$ is a lowest weight vector for $i = 1, 2$, there are $e_1, \ldots, e_m \in \mathcal{L}$ with $y_i := x_i e_1 \cdots e_m$ a nonzero highest weight vector of $M_i$ for $i = 1, 2$. Hence $\varphi(x_1 + x_2, y_1 + y_2) = 0$ since otherwise $\varphi|_P$ would be nonzero for $P$ the $\mathcal{L}$-submodule generated by $x_1 + x_2$. Thus $\varphi(x_1, y_1) + \varphi(x_1, y_2) + \varphi(x_2, y_1) + \varphi(x_2, y_2) = \varphi(x_1, y_2) + \varphi(x_2, y_1) = 0$. Now the one dimensionality of the highest weight space of $M_i$, $i = 1, 2$, gives that there is a scalar $c$ with $x_i e_m \cdots e_1 = cy_i$. Thus

$$
\varphi(x_1, y_2) = -\varphi(x_2, y_1) = -\varphi(x_2, x_1 e_1 \cdots e_m)
$$

$$
= -(-1)^m \varphi(x_2 e_m \cdots e_1, x_1) \quad \text{by (1.1)}
$$

$$
= -(-1)^m c \varphi(y_2, x_1)
$$

$$
= -(-1)^m c \varphi(x_2 e_1 \cdots e_m, x_1)
$$

$$
= -(-1)^{2m} c \varphi(x_2, x_1 e_m \cdots e_1) \quad \text{by (1.1)}
$$

$$
= -c^2 \varphi(x_2, y_1) = c^2 \varphi(x_1, y_2).
$$

Thus $c = \pm 1$ and $a = (-1)^{m+1} c = \pm 1$ so $\varphi$ is symmetric or symplectic.
THEOREM 2.7. If \((M, \{ , , \}, \mathcal{L}, b, \varphi)\) is a Type I or Type II LMTS, then \(\text{Der}(M, \{ , , \})\) is reductive.

Proof. If \((M, \{ , , \})\) is abelian, then \(\text{Der}(M, \{ , , \}) = \text{End } M\), which is reductive. If \((M, \{ , , \})\) is not abelian, it is simple by Lemma 2.5 so if \(\varphi\) is symmetric or symplectic, we are done by Theorem 2.4. By Lemma 2.6 this takes care of Type I LMTSs and Type II's which are the direct sum of two isomorphic irreducible submodules.

The remaining possibility is \((M, \{ , , \})\) is a simple Type II LMTS and \(M = M_1 \oplus M_2\) with \(M_1\) and \(M_2\) nonisomorphic irreducible \(\mathcal{L}\)-submodules. As in the proof of Theorem 2.4, \(\text{Der}(M, \{ , , \}) = [\mathcal{L}, \mathcal{L}] \oplus \mathcal{K}\), where \([\mathcal{L}, \mathcal{L}]\) is semisimple and \(\mathcal{K} = (M, \{ , , \}) \cap \text{End } \mathcal{L} M\). But if \(D \in \mathcal{K}\), \(M_i D \subseteq M_i\) since \(D\) is an \(\mathcal{L}\)-module homomorphism and \(M_1 \not\cong M_2\). Hence by Schur's lemma there are scalars \(\alpha_1, \alpha_2 \in k\) with \((x_1 + x_2) D = \alpha_1 x_1 + \alpha_2 x_2\) for all \(x_i \in M_i\). Hence \(\mathcal{K}\) is a central ideal of \(\text{Der}(M, \{ , , \})\) consisting of semisimple elements and so \(\text{Der}(M, \{ , , \})\) is reductive.

3. THE STRUCTURE OF THE COMPLEMENT \(\mathcal{K}\)

If \((M, \{ , , \}, \mathcal{L}, b, \varphi)\) is a LMTS with \(\text{Der}(M, \{ , , \})\) reductive, \(\mathcal{L}\) is reductive since it is an ideal of \(\text{Der}(M, \{ , , \})\) so, as in the proof of Theorem 2.4, \(\text{Der}(M, \{ , , \}) = [\mathcal{L}, \mathcal{L}] \oplus \mathcal{K}\), where \(\mathcal{K} = \text{Der}(M, \{ , , \}) \cap \text{End } \mathcal{L} M\) is reductive, and the center of \(\mathcal{L}\) centralizes \(\text{Der}(M, \{ , , \})\). Let \(n\) be the number of irreducible summands of \(M\) as an \(\mathcal{L}\)-module, i.e., \(M = M_1 \oplus \cdots \oplus M_n\) with \(M_i\) an irreducible \(\mathcal{L}\)-submodule for \(i = 1, \ldots, n\) [7, Theorem 10, p. 81].

THEOREM 3.1. Suppose \((M, \{ , , \}, \mathcal{L}, b, \varphi)\) is a semisimple LMTS with \(\text{Der}(M, \{ , , \})\) reductive. If \(H\) is a Cartan subalgebra of \(\mathcal{K}\), then \(\dim H < n/2\).

Proof. \(\mathcal{L}_1 = [\mathcal{L}, \mathcal{L}] \oplus H\) is reductive so by Theorem 10, p. 81, of [7], \(M = N_1 \oplus \cdots \oplus N_k\) with \(N_i\) an irreducible \(\mathcal{L}_1\)-submodule for \(i = 1, \ldots, k\). By Schur's lemma for all \(h \in H\) there are scalars \(\alpha_i(h) \in k\) with \(x_i h = \alpha_i(h) x_i\) for all \(x_i \in N_i\). Hence \(N_i\) is an irreducible \(\mathcal{L}\)-submodule and \(k = n\). Now \(\dim H = \dim \langle \alpha_1, \ldots, \alpha_n \rangle\) since \(M\) is a faithful \(H\)-module. By Proposition 4.2 of [5], \(M\) has no abelian ideals so for \(i = 1, \ldots, n\) \(\{N_i, MM\} \neq 0\) by Lemma 3.1 of [5]. Hence \(0 \neq \phi(M, \{N_i, MM\}) = b(R(M, M), R(N_i, M))\) so \(R(N_i, M) \neq 0\) so there is a \(j\) with \(R(N_i, N_j) \neq 0\). Now for \(x_i \in N_i, x_j \in N_j, h \in H, 0 = [R(x_i, x_j), h] = R(x_i h, x_j) + R(x_i, x_j h) = [\alpha_i(h) + \alpha_j(h)] R(x_i, x_j)\) by (2.2). Hence if \(R(N_i, N_j) \neq 0, \alpha_i = 0\) and if \(j \neq i, \alpha_i = -\alpha_j\). Thus \(\dim H \leq n/2\).
This result was shown in [2] for triple systems \((M, \{ , , \})\) which are \(\text{Der}(M, \{ , , \})\) irreducible.

**Corollary 3.2.** Suppose \((M, \{ , , \}, \mathcal{L}, b, \varphi)\) is a simple LMTS and \(\mathcal{K} = \text{Der}(M, \{ , , \}) \cap \text{End}_{\mathcal{L}} M\).

(i) If \((M, \{ , , \})\) is Type I, \(\mathcal{L}\) is semisimple and \(\text{Der}(M, \{ , , \}) = \mathcal{L}\).

(ii) If \((M, \{ , , \})\) is Type II and is the direct sum of nonisomorphic submodules, \(\mathcal{K}\) is one dimensional.

(iii) If \((M, \{ , , \})\) is Type II and is the direct sum of isomorphic submodules, \(\mathcal{K} = sl(2)\).

**Proof.** By Theorem 2.7, \(\text{Der}(M, \{ , , \})\) is reductive if \((M, \{ , , \})\) is Type I or Type II. If \((M, \{ , , \})\) is Type I, \(\dim H = 0\) by Theorem 3.1 for \(H\) a Cartan subalgebra of \(\mathcal{K}\). Hence \(\mathcal{K} = 0\) and so \(\mathcal{L} \subseteq \text{Der}(M, \{ , , \}) = [\mathcal{L}, \mathcal{L}]\), giving \(\mathcal{L}\) semisimple.

Suppose \((M, \{ , , \})\) is Type II and \(M = M_1 \oplus M_2\) with \(M_i\) an irreducible \(\mathcal{L}\)-submodule. In this case \(\dim H = 0\) or \(1\) by Theorem 3.1. Define \(D \in \text{End} M\) by \((x_1 + x_2) D := x_1 - x_2\) for \(x_i \in M_i\). Then (2.3.1) holds so \(D \in \mathcal{K}\) and since \(D\) is semisimple, \(\dim H = 1\). Hence \(\mathcal{K}\) is one dimensional or \(\mathcal{K} = sl(2)\). If \(M_1 \not\cong M_2\), then for any \(E \in \mathcal{K} \subseteq \text{End}_{\mathcal{L}} M\), \(M_i E \subseteq M_i\), so by Schur's lemma \(E\) acts semisimply and hence \(\mathcal{K}\) is one dimensional. So suppose \(M_1 \cong M_2\), in particular suppose \(\tau: M_1 \to M_2\) is an \(\mathcal{L}\)-module isomorphism. \(P := \{x_1 + x_1 \tau \mid x_1 \in M_1\}\) is an irreducible \(\mathcal{L}\)-module of \(M\) so \(\varphi|_P \equiv 0\). Hence if \(x_1, y_1 \in M_1, \varphi(x_1 + x_1 \tau, y_1 + y_1 \tau) = \varphi(x_1, y_1 \tau) + \varphi(x_1 \tau, y_1) = 0\), i.e., \(\varphi(x_1 \tau, y_1) = -\varphi(x_1, y_1 \tau)\). Define \(E \in \text{End}_{\mathcal{L}} M\) by \((x_1 + x_2) E = x_1 \tau\) for \(x_i \in M_i\). Then \(E\) satisfies (2.3.1) since \(\varphi((x_1 + x_2) E, y_1 + y_2) = \varphi(x_1 \tau, y_1) = -\varphi(x_1, y_1 \tau) = -\varphi(x_1 + x_2, (y_1 + y_2) E)\) so \(E \in \mathcal{K}\). \(E\) is nilpotent so \(\mathcal{K} = sl(2)\).

4. Classification

We now prove our classification result:

**Theorem 4.1.** Suppose \((M, \{ , , \}, \mathcal{L}, b, \varphi)\) is a simple LMTS.

(i) If \((M, \{ , , \})\) is Type I, then \(M\) is a self-dual irreducible \(\mathcal{L}\)-module, for \(e = \pm 1, \varphi(x, y) = e\varphi(y, x)\) for all \(x, y \in M\) and \(R(y, z) = -eR(z, y)\) for all \(y, z \in M\), and \(\mathcal{L} = \text{Der}(M, \{ , , \})\) is semisimple.

(ii) Suppose \((M, \{ , , \})\) is Type II so \(M = M_1 \oplus M_2\) with \(M_1\) and \(M_2\) irreducible \(\mathcal{L}\)-modules. Then \(M_1\) and \(M_2\) are dual and there is an \(a \in k^*\)
with \( \varphi(x, y) = a \varphi(y, x) \) for all \( x \in M_1, y \in M_2 \) and \( R(z, y) = -a^{-1}R(y, z) \) for all \( z \in M_1, y \in M_2 \). \( R(M_1, M_1) = R(M_2, M_2) = 0 \) and \( R(M_1, M_2) = R(M_2, M_1) = \mathcal{L} \neq 0 \). There are two possibilities:

(a) \( M_1 \) and \( M_2 \) are isomorphic self-dual \( \mathcal{L} \)-modules, in which case \( a = \pm 1 \), \( \mathcal{L} \) is semisimple, and \( \text{Der}(M, \{ , , \}) = \mathcal{L} \oplus \mathfrak{s}(2) \).

(b) \( M_1 \) and \( M_2 \) are not isomorphic, in which case \( \text{Der}(M, \{ , , \}) \) is reductive with a one dimensional center spanned by \( D \), where \( (x_1 + x_2)D := x_1 - x_2 \) for \( x_i \in M_i \).

Proof: \( \mathcal{L} \) is a self-dual \( \mathcal{L} \)-module since \( x \to \varphi(x, -) \) is an \( \mathcal{L} \)-module isomorphism of \( M \) with \( M^* \). That \( \varphi \) is symmetric or symplectic in (i) and (ii)(a) was shown in Lemma 2.6, as was the existence of \( a \in k \) such that \( \varphi(x, y) = a \varphi(y, x) \) for \( (M, \{ , , \}) \) Type II and \( x \in M_1, y \in M_2 \). By the nondegeneracy of \( \varphi, a \neq 0 \). Hence if \( z \in M_1, y \in M_2, b(l, R(z, y)) = \varphi(yl, z) = a^{-1} \varphi(z, yl) = b(l, -a^{-1}R(y, z)) \), giving \( R(z, y) = -a^{-1}R(y, z) \). Thus \( R(M_1, M_2) = R(M_2, M_1) \). Since \( \varphi|_{M_j} = 0 \) for \( i = 1, 2, b(R(M, M), R(M_i, M_j)) = \varphi(\{ M_i, M, M_j \}, M_i) = \varphi(M_i, M_i) = 0 \) so \( R(M_1, M_1) = R(M_2, M_2) = 0 \), giving \( \mathcal{L} = R(M_1, M_2) = R(M_2, M_1) \) by Corollary 2.11 of [S]. Thus \( M_1 \) and \( M_2 \) are dual since \( x \to \varphi(x, -) \) is an isomorphism of \( M_i \) with \( M_i^* \) for \( i, j = 1, 2, i \neq j \). If \( M_1 \cong M_2 \), they are self-dual. The statements about the derivation algebras were shown in Corollary 3.2 except that \( \mathcal{L} \) is semisimple in (ii)(a). However, in this case \( \text{Der}(M, \{ , , \}) = [\mathcal{L}, \mathcal{L}] \oplus \mathfrak{s}(2) \) is semisimple so since \( \mathcal{L} \) is an ideal of \( (M, \{ , , \}) \), \( \mathcal{L} \) is semisimple.

It remains only to show that all of the possibilities do, in fact, occur. Simple Lie triple systems give examples of Type I LMTSs and the Type II's in Theorem 4.1(ii)(b) for which \( D \in \mathcal{L} \) [8]. Some of the Type II Lie triple systems are direct sums of nonisomorphic \([\mathcal{L}, \mathcal{L}]\)-modules. The \([\mathcal{L}, \mathcal{L}]\)-cutdown (see [4] for the definition) of such a Lie triple system is a Type II for which \( D \notin \mathcal{L} \). Finally we give an example of a Type II LMTS which is the direct sum of isomorphic submodules.

Example 4.2. On the two dimensional space \( V_1 = kx \oplus ky \) define the nondegenerate bilinear form \( \varphi_1 \) by \( \varphi_1(x, x) = \varphi_1(y, y) = 0 \) and \( \varphi_1(x, y) = 8 = -\varphi_1(y, x) \) and on \( V_2 = ka \oplus kb \) define the nondegenerate bilinear form \( \varphi_2 \) by \( \varphi_2(a, a) = \varphi_2(b, b) = 0 \) and \( \varphi_2(a, b) = 8 = -\varphi_2(b, a) \).

Let \( M = V_1 \otimes V_2 \) and define the nondegenerate bilinear form \( \psi \) on \( M \) by \( \psi(w \otimes c, u \otimes d) := \varphi_1(w, u) \varphi_2(c, d) \) for \( w, u \in V_1, c, d \in V_2 \) and define \( \{ , , \} \) on \( M \) by \( \{ w \otimes c, u \otimes d, v \otimes e \} := \frac{1}{2} \varphi_2(e, d) \varphi_1(w, u) v \otimes c + \frac{1}{2} \varphi_2(e, d) \varphi_1(w, v) u \otimes c + \frac{1}{2} \varphi_2(e, c) \varphi_1(w, v) u \otimes c \). Then \( \{ , , \} \) and \( \psi \) satisfy the hypotheses of Corollary 2.8 of [5] so \( (M, \{ , , \}) \) is a LMTS. It is easy to check that \( \mathcal{L} := R(M, M) = \mathfrak{s}(2) \), where \( (w \otimes c) \cdot D = w l \otimes c \) and \( x \).
and \( y \) are respectively the highest and lowest weight vectors. Clearly then \( M = M_1 \oplus M_2 \) as an \( L \)-module with \( M_1 = \{ w \otimes a \mid w \in V_1 \} \), \( M_2 = \{ w \otimes b \mid w \in V_1 \} \) and \( M_1 \cong M_2 \) and both are irreducible. Any lowest weight vector of \( M \) is of the form \( y \otimes e \) for some \( e \in V_2 \). Then the highest weight vector of the \( L \)-submodule \( \Phi \) generated by \( y \otimes e \) is \( x \otimes e \). Now \( \psi(x \otimes e, y \otimes e) = \varphi_1(x, y) \varphi_2(e, e) = 8 \cdot 0 = 0 \) since \( \varphi_2 \) is symplectic so \( \psi_{|\rho} \) is zero and \( (M, \{ , , \}) \) is Type II.

Note that if \( (M, \{ , , \}) \) is a Type II LMTS, then \( (M_1, M_2) \) is a pair algebra as defined by Faulkner in [3].

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**REFERENCES**