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The Λ_S-Householder matrices

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ABSTRACT

Let $A, S \in M_n(\mathbb{C})$ be given. Suppose that S is nonsingular and Hermitian. Then A is Λ_S-orthogonal if $A^*SA = S$. Let $u \in \mathbb{C}^n$ be such that $u^*Su \neq 0$. The Λ_S-Householder matrix of u is $S_u \equiv I - tuu^*S$, where $t = \frac{2}{u^*Su}$. We show that $\det(S_u) = -1$, so that products of Λ_S-Householder matrices have determinant ± 1. Let $n \geqslant 2$ and let k be positive integers with $k \leqslant n$. Set $L_k \equiv I_k \oplus -I_{n-k}$. We show that every Λ_{L_k}-orthogonal matrix having determinant ± 1 can be written as a product of at most $2n + 2$ Λ_{L_k}-Householder matrices. We also determine the possible Jordan Canonical Forms of products of two Λ_{L_k}-Householder matrices.

1. Introduction

We denote by $M_{m,n}(\mathbb{F})$ the set of m-by-n matrices with entries in $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. When $m = n$, we set $M_n(\mathbb{F}) \equiv M_{n,n}(\mathbb{F})$. We denote by $\mathbb{F}^n$ the set of column vectors with entries in $\mathbb{F}$. For $x \in \mathbb{F}^n$, we set $\langle x \rangle = \{\alpha x : \alpha \in \mathbb{F}\}$.

Let an integer $n \geqslant 2$ and a unit vector $v \in \mathbb{C}^n$ be given. The Householder matrix of v is $H_v \equiv I - 2vv^*$. One checks that H_v is Hermitian, unitary, and an involution. Let $v_1 = v$ and extend this to an orthonormal basis of $\mathbb{C}^n$, say $\{v_1, \ldots, v_n\}$. Set $V = [v_1 \ \cdots \ v_n]$, and notice that $V^*H_vV = [-1] \oplus I_{n-1}$. Hence, $\det(H_v) = -1$.

Consider $B \equiv \operatorname{diag}\left(e^{i\theta}, -e^{-i\theta}\right)$, where $\theta \in \mathbb{R}$ and $\theta \neq k\pi$ with k an integer. Then $B - I$ is nonsingular, so that B is not a Householder matrix. Suppose that B is a product of Householder matrices. Because $\det(B) = -1$, we must have that B is a product of an odd number of Householder matrices. That is, if B can be written as a product of Householder matrices, then B can be written as a product of

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at least three Householder matrices. This contradicts Theorem 1 in [4], which says that every unitary $U \in M_n(\mathbb{C})$ can be written as a product of at most n Householder matrices.

Let $e_i^{(n)} \in \mathbb{C}^n$ be the vector whose ith entry is 1 and 0 elsewhere. When the context is clear, we drop the superscript.

Let $e_1, e_2 \in \mathbb{C}^2$. Set $a = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and set $b = \frac{1}{\sqrt{2}}\left(e_1 + e^{i\theta} e_2\right)$. Notice that $C = \mathrm{diag}\left(e^{i\theta},\ e^{-i\theta}\right) = H_a H_b$. Moreover, $B = H_{e_2} C$ is a product of three Householder matrices. Suppose that $n \geqslant 3$. Let $V = \mathrm{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)$, where $\theta_1, \ldots, \theta_n \in \mathbb{R}$ and $\theta_1 + \cdots + \theta_n = k\pi$ for some integer k. Let $C = \mathrm{diag}\left(e^{i\theta},\ e^{-i\theta}\right) \oplus I_{n-2}$ and let $D = \mathrm{diag}\left(1,\ e^{i(\theta_1+\theta_2)},\ e^{i\theta_3}, \ldots,\ e^{i\theta_n}\right)$. Then C can be written as a product of two Householder matrices. An easy induction argument now shows that V can be written as a product of at most $2n-1$ Householder matrices. This confirms Theorem 3 in [4]. In fact, if $\mathrm{rank}(V - I) = k$, then V can be written as a product of at most $2k-1$ Householder matrices.

Let $Q \in M_n(\mathbb{C})$ be unitary and let $v \in \mathbb{C}^n$ be a unit vector. Then $QH_vQ^* = H_{Qv}$. If $U \in M_n(\mathbb{C})$ is unitary with $\det(U) = \pm 1$, then there exists a unitary Q such that $QUQ^* = \mathrm{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)$. Hence, U can be written as a product of at most $2k-1$ Householder matrices, where $k = \mathrm{rank}(U - I)$. In particular, U can be written as a product of at most $2n-1$ Householder matrices.

For more discussion on Householder matrices and related topics, see [3–6].

2. Λ_S-Householder matrices

Definition 1. Let $S \in M_n(\mathbb{C})$ be nonsingular. Let $\Lambda_S : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be given by $\Lambda_S(A) = S^{-1}A^*S$ for every $A \in M_n(\mathbb{C})$. A given $A \in M_n(\mathbb{C})$ is called Λ_S-*symmetric* if $\Lambda_S(A) = A$; and A is called Λ_S-*orthogonal* if $\Lambda_S(A) = A^{-1}$.

Notice that $\Lambda_S(AB) = \Lambda_S(B)\Lambda_S(A)$ and that $\Lambda_S(I) = I$. Hence, if A is nonsingular, then $\Lambda_S(A)$ is nonsingular and $(\Lambda_S(A))^{-1} = \Lambda_S\left(A^{-1}\right)$. When S is Hermitian, then $\Lambda_S(\Lambda_S(A)) = A$ for every $A \in M_n(\mathbb{C})$.

Let $A \in M_n(\mathbb{C})$ be Λ_S-symmetric. Then, [1, Theorem 4.1.7] guarantees that A is similar to a real matrix. Hence, the trace and the determinant of A are both real. If k is a positive integer and if $\alpha \in \mathbb{R}$ is given, then αA^k is Λ_S-symmetric. It follows that if $p(x)$ is a polynomial with real coefficients, then $p(A)$ is also Λ_S-symmetric. If S is Hermitian, then for any $A \in M_n(\mathbb{C})$, the matrices $\Lambda_S(A)A$, $A\Lambda_S(A)$ and $A + \Lambda_S(A)$ are all Λ_S-symmetric.

Let $A \in M_n(\mathbb{C})$ be Λ_S-orthogonal. Then $A^*SA = S$, so that $|\det(A)| = 1$. If $x \in \mathbb{C}^n$ and if $\langle x, x\rangle_S \equiv x^*Sx$, then $\langle Ax, Ax\rangle_S = \langle x, x\rangle_S$. Moreover, if $\alpha \in \mathbb{C}$ is such that $|\alpha| = 1$, then αA is also Λ_S-orthogonal. Notice that $S = A^{-*}SA^{-1}$, so that A^{-1} is also Λ_S-orthogonal. In addition, the product of two Λ_S-orthogonal matrices is Λ_S-orthogonal. We denote by $\mathcal{O}_S$ the set all Λ_S-orthogonal matrices, and by $\mathcal{SO}_S$ the set of all Λ_S-orthogonal matrices having determinant ± 1.

Definition 2. Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. Let $0 \neq v \in \mathbb{C}^n$ be given. Then v is *isotropic* with respect to S (or S-*isotropic*) if $v^*Sv = 0$. If $v^*Sv \neq 0$, then v is *nonisotropic* with respect to S (or S-*nonisotropic*).

Take $S = \mathrm{diag}(1, -1) \in M_2(\mathbb{C})$ and take $u = \left[ae^{i\alpha}\ ae^{i\beta}\right]^T$. For any $a \in \mathbb{C}$, and for any $\alpha, \beta \in \mathbb{R}$, notice that u is S-isotropic.

Definition 3. Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. Let $x, y \in \mathbb{C}^n$ be given. Then x and y are S-perpendicular if $x^*Sy = 0$. Two subspaces V and W (of $\mathbb{C}^n$) are S-perpendicular if $v^*Sw = 0$ for all $v \in V$ and all $w \in W$.

If $x, y \in \mathbb{C}^n$ are S-perpendicular, then $\langle x\rangle$ and $\langle y\rangle$ are S-perpendicular. Take $S = \mathrm{diag}(1, -1) \in M_2(\mathbb{C})$, take $v = [a\ b]^T$, and take $w = [b\ a]^T$ with $a, b \in \mathbb{R}$. Then v and w are S-perpendicular.

Definition 4. Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. Let $v \in \mathbb{C}^n$ be S-nonisotropic. The Λ_S-*Householder* matrix of v is $S_v \equiv I - tvv^*S$, where $t = \frac{2}{v^*Sv}$.

Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. If $n = 1$, then for any $0 \neq v \in \mathbb{C}$, we have $S_v = [-1]$. Conversely, if $S_v = -I$, then $I = \frac{1}{v^*Sv}vv^*S$. Hence, $n = \text{rank}(I) = \text{rank}\left(\frac{1}{v^*Sv}vv^*S\right) = 1$. Thus, $S_v = -I$ if and only if $n = 1$.

Let $n \geqslant 2$ be a given integer. Let $v \in \mathbb{C}^n$ be S-nonisotropic. Then $\langle v\rangle_S^{\perp} \equiv \{x \in \mathbb{C}^n : x^*Sv = 0\}$ has dimension $n-1$. Now, if $x \in \langle v\rangle$, then $S_vx = -x$. If $x \in \langle v\rangle_S^{\perp}$, then $S_vx = x$.

Proposition 5. *Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. Let $u, v \in \mathbb{C}^n$ be S-nonisotropic.*

1. *S_u is Λ_S-symmetric, is Λ_S-orthogonal, and is an involution.*
2. *S_u is diagonalizable and S_u is similar to $I_{n-1} \oplus [-1]$. Hence, $tr(S_u) = n-2$ and $det(S_u) = -1$. If $n = 1$, then the minimal polynomial of S_u is $x+1$. If $n \geqslant 2$, then the minimal polynomial of S_u is x^2-1.*
3. *$S_u = S_v$ if and only if $\langle u\rangle = \langle v\rangle$.*
4. *If u and v are S-perpendicular, then $S_uS_v = S_vS_u$.*
5. *If $P \in \mathcal{O}_S$, then $PS_uP^{-1} = S_{Pu}$.*
6. *If $n \geqslant 2$, then the singular values of S_u are 1 (with multiplicity $n-2$) and $\sqrt{\frac{\mu \pm \sqrt{\mu^2-4}}{2}}$, where $\mu = \frac{4(u^*S^2u)u^*u}{(u^*Su)^2} - 2$.*

Proof. The first claim can be verified by direct computation. For the second claim, if $n = 1$, then $S_u = -1$. If $n \geqslant 2$, then notice that 1 is an eigenvalue of S_u, with an eigenspace of dimension $n-1$, and -1 is an eigenvalue of S_u with an eigenspace of dimension 1.

To show the third claim, suppose $S_u = S_v$. Then $-u = S_uu = S_vu = (I - tvv^*S)\,u = u - t\,(v^*Su)\,v$. Notice that $v^*Su \neq 0$, otherwise, $u = 0$. Now, $u = \frac{tv^*Su}{2}v$ and $\langle u\rangle = \langle v\rangle$. If $\langle u\rangle = \langle v\rangle$, then $v = \alpha u$ for some $0 \neq \alpha \in \mathbb{C}$. Now, $S_v = S_{\alpha u} = I - \frac{2}{(\alpha u)^*S(\alpha u)}\,(\alpha u)\,(\alpha u)^*\,S = S_u$.

The next two claims can be shown by direct computation.

For the last claim, we have

$$\text{rank}\left(S_uS_u^* - I\right) = \text{rank}\left(S_u\left(S_u^* - S_u\right)\right) = \text{rank}\left(S_u^* - S_u\right).$$

Now, $S_u^* - S_u = tuu^*S - tSuu^*$ has rank at most 2. Hence, 1 is an eigenvalue of $S_uS_u^*$ with geometric multiplicity at least $n-2$. Let α and β be the (possibly) other two eigenvalues of $S_uS_u^*$. Then $1 = \det\left(S_uS_u^*\right) = \alpha\beta$. A direct computation now shows that $S_uS_u^* = I - tuu^*S - tSuu^* + \frac{4(u^*S^2u)}{(u^*Su)^2}uu^*$. Taking the trace of both sides, we get $n-2+\alpha+\frac{1}{\alpha} = n-4+\frac{4(u^*S^2u)u^*u}{(u^*Su)^2}$. Setting $\mu \equiv \frac{4(u^*S^2u)u^*u}{(u^*Su)^2} - 2$, we get $\alpha^2 - \mu\alpha + 1 = 0$, and $\alpha = \frac{\mu \pm \sqrt{\mu^2-4}}{2}$. The remaining singular values of S_u are $\sqrt{\frac{\mu \pm \sqrt{\mu^2-4}}{2}}$. $\square$

2.1. $*$-Congruence

Let $S, T \in M_n(\mathbb{C})$ be Hermitian matrices. Then S and T are $*$-congruent ($S = P^*TP$ for some nonsingular P) if and only if they have the same inertia, that is, they have the same number of positive, negative, and zero eigenvalues. If S is nonsingular, then its spectrum contains only positive and negative eigenvalues, that is, S is $*$-congruent to $L_k \equiv I_k \oplus -I_{n-k}$ for some $k = 0, 1, \ldots, n$, and where we make the convention that I_0 is not present.

Let $S, T \in M_n(\mathbb{C})$ be nonsingular Hermitian matrices and suppose that $S = P^*TP$ for some nonsingular P. Suppose that $C \in M_n(\mathbb{C})$ is Λ_S-symmetric, that is, $S^{-1}C^*S = C$. Then, $P^{-1}T^{-1}P^{-*}C^*P^*TP =$

C and $T^{-1}\left(PCP^{-1}\right)^{*}T = \left(PCP^{-1}\right)$, so that PCP^{-1} is Λ_T-symmetric. Conversely, if PCP^{-1} is Λ_T-symmetric, then C is Λ_S-symmetric.

A similar calculation also shows that $C \in M_n(\mathbb{C})$ is Λ_S-orthogonal if and only if $PCP^{-1}$is Λ_T-orthogonal.

Theorem 6. *Let $S, T \in M_n(\mathbb{C})$ be nonsingular Hermitian matrices. Suppose that $S = P^*TP$ for some nonsingular $P \in M_n(\mathbb{C})$.*

1. *$C \in M_n(\mathbb{C})$ is Λ_S-symmetric if and only if PCP^{-1} is Λ_T-symmetric.*
2. *$C \in M_n(\mathbb{C})$ is Λ_S-orthogonal if and only if PCP^{-1} is Λ_T-orthogonal.*
3. *Let $v \in \mathbb{C}^n$ be S-nonisotropic. Then Pv is T-nonisotropic and $PS_vP^{-1} = T_{Pv}$.*

Proof. Suppose that v is S-nonisotropic. Then $v^*P^*TPv = v^*Sv \neq 0$. Now, $PS_vP^{-1} = P\left(I - \frac{2}{v^*Sv}vv^*S\right) P^{-1} = P\left(I - \frac{2}{v^*P^*TPv}vv^*P^*TP\right)P^{-1} = I - \frac{2}{(Pv)^*T(Pv)}(Pv)(Pv)^*T = T_{Pv}$. □

Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. Then there exists an integer k, with $0 \leqslant k \leqslant n$ such that S is $*$-congruent to L_k. Now, $-L_k$ and L_{n-k} have the same inertia. In fact, if we set $P = \begin{bmatrix} 0 & I_{n-k} \\ I_k & 0 \end{bmatrix}$, then $P^{-1} = P^*$ and $-L_k = P^*L_{n-k}P$. Let $A \in M_n(\mathbb{C})$ be given. Then $\Lambda_{L_k}(A) = L_k^{-1}A^*L_k = (-L_k)^{-1}A^*(-L_k) = \Lambda_{-L_k}(A)$.

Lemma 7. *Let $n \geqslant 2$ be a given integer. Suppose that $0 \leqslant k \leqslant n$ is an integer.*

1. *If $C \in M_n(\mathbb{C})$ is Λ_{L_k}-symmetric , then C is permutation similar to a $\Lambda_{L_{n-k}}$-symmetric matrix.*
2. *If $C \in M_n(\mathbb{C})$ is Λ_{L_k}-orthogonal , then C is permutation similar to a $\Lambda_{L_{n-k}}$-orthogonal matrix.*

Proof. Suppose that $C \in M_n(\mathbb{C})$ is Λ_{L_k}-symmetric. Then C is Λ_{-L_k}-symmetric. Theorem 6 (1) now guarantees that PCP^{-1} is $\Lambda_{L_{n-k}}$-symmetric.

The second claim can be proven similarly. □

3. Product of Λ_S-Householder matrices

Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. Then there exist an integer k, with $0 \leqslant k \leqslant n$, and a nonsingular $P \in M_n(\mathbb{C})$ such that $S = P^*L_kP$. Suppose that $Q = Q_1Q_2$ is a product of Λ_S-orthogonal matrices Q_1 and Q_2. Theorem 6 guarantees that $PQP^{-1} = \left(PQ_1P^{-1}\right)\left(PQ_2P^{-1}\right)$ is a product of Λ_{L_k}-orthogonal matrices.

For now, we let $n \geqslant 2$, we fix k, and we drop the subscript, that is, we say that $L = L_k$.

Let $Q \in \mathcal{O}_S$ be given. Then $|\det Q| = 1$. Hence, there exists $\alpha \in \mathbb{C}$ such that $\alpha Q \in \mathcal{SO}_S$, that is, $\det(\alpha Q) = \pm 1$. Our goal is to determine which elements of $\mathcal{SO}_S$ can be written as a product of Λ_S-Householder matrices. We are also interested in finding the least number of Λ_S-Householder matrices necessary to form such a product. Our approach is to study Λ_L-orthogonal matrices.

3.1. Λ_L-orthogonal matrices

We begin with the following observation.

Lemma 8. *Let $S \in M_n(\mathbb{C})$ be nonsingular and Hermitian. Suppose that $S = P^*LP$, where $P \in M_n(\mathbb{C})$ is nonsingular and $L = I_k \oplus -I_{n-k}$ for some integer k with $0 \leqslant k \leqslant n$. Then $A \in \mathcal{O}_S$ can be written as a product of Λ_S-Householder matrices if and only if PAP^{-1} can be written as a product of Λ_L-Householder matrices. Moreover, the minimum number used in both cases are the same.*

Let $A \in M_n(\mathbb{C})$ be Λ_L-orthogonal. If A can be written as a product of Λ_L-Householder matrices, then necessarily, $\det(A) = \pm 1$. We show that the converse holds, as well.

Suppose that $A = L_{u_1} L_{u_2} \cdots L_{u_m}$ and suppose that such a factorization is minimal. If $\langle u_k \rangle = \langle u_{k+1} \rangle$, then $L_{u_k} = L_{u_{k+1}}$, and since Proposition 5 guarantees that L_u is an involution, we can take away $L_{u_k} L_{u_{k+1}}$ (which is I). If $\langle u_k \rangle = \langle u_{k+t} \rangle$ for $t \geqslant 2$, then $L_{u_k} L_{u_{k+1}} \cdots L_{u_{k+t}} = L_{u_k} L_{u_{k+1}} \cdots L_{u_k} = \left(L_{u_k} L_{u_{k+1}} L_{u_k} \right) \cdots \left(L_{u_k} L_{u_{k+t-1}} L_{u_k} \right)$. Now, for each $l = 1, \ldots, t-1$, Proposition 5 (5) guarantees that $L_{u_k} L_{u_{k+l}} L_{u_k} = L_{w_l}$, where each $w_l = L_{u_k} u_{k+l}$. This apparent contradiction shows the following.

Lemma 9. *Let $u_i \in \mathbb{C}^n$ for $i = 1, \ldots, m$. Let $A \in M_n(\mathbb{C})$ be Λ_L-orthogonal. Suppose that $A = L_{u_1} L_{u_2} \cdots L_{u_m}$. If m is minimal, then $\langle u_i \rangle \neq \langle u_j \rangle$ for $i \neq j$.*

Let $x, y \in \mathbb{R}^n$ be given, and let D be a nonsingular diagonal matrix in $M_n(\mathbb{R})$. If $x^T D x = y^T D y \neq 0$, then either $(x+y)^T D (x+y) \neq 0$ or $(x-y)^T D (x-y) \neq 0$ [4, Lemma 3]. The following is an analog in the complex case and using $D = L$.

Lemma 10. *Let $x, y \in \mathbb{C}^n$ be such that $x^* L x = y^* L y \neq 0$. Then $x + y$ is L-nonisotropic or $x - y$ is L-nonisotropic.*

Proof. The assumption assures that $x^* L x + y^* L y \neq 0$. Suppose now that both $x + y$ and $x - y$ are L-isotropic. Then $(x+y)^* L (x+y) = 0$, so that $\mathrm{Re}(x^* L y) = -\frac{1}{2}(x^* L x + y^* L y)$. Now, we also have $(x-y)^* L (x-y) = 0$, so that $\mathrm{Re}(x^* L y) = \frac{1}{2}(x^* L x + y^* L y)$, a contradiction. □

Let $x, y \in \mathbb{C}^n$ be such that $x^* L x = y^* L y \neq 0$. Say, $w = x + y$ is L-nonisotropic. Suppose that $x^* L y \in \mathbb{R}$. Now, compute: $L_w x = x - \frac{2}{w^* L w} w w^* L x$. Notice that $w^* L w = 2(x^* L x + x^* L y)$. Also, $w^* L x = x^* L x + y^* L x = x^* L x + x^* L y$. Hence, $L_w x = -y$.

Conversely, if $L_w x = -y$, then $w \left(1 - \frac{2 w^* L x}{w^* L w} \right) = 0$, so that $w^* L w - 2 w^* L x = 0$. Now, $w^* L w = 2 x^* L x + 2 \mathrm{Re}(y^* L x)$ and $w^* L x = x^* L x + y^* L x$. Hence, $\mathrm{Re}(y^* L x) = y^* L x$. Consequently, $x^* L y \in \mathbb{R}$.

If $v = x - y$ is L-nonisotropic, then a similar calculation shows that $L_v x = y$ if and only if $x^* L y \in \mathbb{R}$.

Lemma 11. *Let $x, y \in \mathbb{C}^n$ be such that $x^* L x = y^* L y \neq 0$. If $w = x + y$ is L-nonisotropic, then $L_w x = -y$ if and only if $x^* L y \in \mathbb{R}$. If $v = x - y$ is L-nonisotropic, then $L_v x = y$ if and only if $x^* L y \in \mathbb{R}$.*

It is known that if $x, y \in \mathbb{C}^n$ have the same Euclidean norm, then there exists a unitary U such that $Ux = y$ [1, Problem 4 on page 77]. The following is an analog.

Lemma 12. *Let $x, y \in \mathbb{C}^n$ be such that $x^* L x = y^* L y \neq 0$. Then there exists a Λ_L-orthogonal P such that $Px = y$.*

Proof. Suppose that $x^* L y = r e^{i\theta}$, where $r, \theta \in \mathbb{R}$. Set $u = e^{i\theta} x$. Then $u^* L u = y^* L y \neq 0$. Moreover, $u^* L y \in \mathbb{R}$. Let $w = u + y$ and let $v = u - y$. Lemma 10 guarantees that w or v is L-nonisotropic. If w is L-nonisotropic, then Lemma 11 guarantees that $L_w u = -y$. We take $P = -e^{i\theta} L_w$. If v is L-nonisotropic, then Lemma 11 guarantees that $L_v u = y$. We take $P = e^{i\theta} L_v$. □

Let $x \in \mathbb{C}^n$ be L-nonisotropic. Write $x = [x_i]$. Then $x^* L x = \sum_{i=1}^{k} |x_i|^2 - \sum_{i=k+1}^{n} |x_i|^2 \in \mathbb{R}$. Suppose that $x^* L x = \alpha^2$, with $\alpha > 0$. Set $e \equiv \alpha e_1$. Then $e^* L e = \alpha^2$. Lemma 12 guarantees that there exists a Λ_L-orthogonal P such that $x = Pe$. Now, $L_x = L_{Pe} = P L_e P^{-1}$. Notice that $L_e = L_{\alpha e_1} = L_{e_1}$ is diagonal.

Suppose that $x^* L x = -\alpha^2$, with $\alpha > 0$. Set $e \equiv \alpha e_{k+1}$. Then $e^* L e = -\alpha^2$. Lemma 12 guarantees that there exists a Λ_L-orthogonal P such that $x = Pe$. Now, $L_x = L_{Pe} = P L_e P^{-1}$. Notice that $L_e = L_{\alpha e_{k+1}} = L_{e_{k+1}}$ is diagonal.

Theorem 13. *Let $x \in \mathbb{C}^n$ be L-nonisotropic. There exists a Λ_L-orthogonal P such that $P L_x P^{-1}$ is diagonal.*

Let $0 \neq x \in \mathbb{C}^n$ be L-isotropic. Write $x = \left[x_1^T\, x_2^T\right]^T$, with $x_1 \in \mathbb{C}^k$. Notice that $0 = x^*Lx = x_1^*x_1 - x_2^*x_2$, so that $||x_1||_2 = ||x_2||_2$. Because $x \neq 0$, we have $x_1 \neq 0$ (and also $x_2 \neq 0$). Let $U_1 \in M_k(\mathbb{C})$ be a unitary such that $U_1x_1 = ||x_1||_2\, e_1^{(k)}$ and let $U_2 \in M_{n-k}(\mathbb{C})$ be a unitary such that $U_2x_2 = ||x_1||_2\, e_1^{(n-k)}$. Set $U = U_1 \oplus U_2$. Then U is Λ_L-orthogonal and $Ux = ||x_1||_2\, e_1^{(n)} + ||x_1||_2\, e_{k+1}^{(n)}$. Set $H_\beta \equiv \begin{bmatrix} [\cosh\beta] \oplus I_{k-1} & [\sinh\beta] \oplus 0 \\ [\sinh\beta] \oplus 0 & [\cosh\beta] \oplus I_{n-k-1} \end{bmatrix}$, and notice that H_β is Λ_L-orthogonal. Moreover, $H_\beta Ux = e^\beta\, ||x_1||_2\, e_1^{(n)} + e^\beta\, ||x_1||_2\, e_{k+1}^{(n)}$. Choosing $\beta = -\ln(||x_1||_2)$, we have $H_\beta Ux = e_1^{(n)} + e_{k+1}^{(n)}$.

Lemma 14. *Let $0 \neq x \in \mathbb{C}^n$ be L-isotropic. Then there exists a Λ_L-orthogonal P such that $Px = e_1^{(n)} + e_{k+1}^{(n)}$.*

Let $U \in M_n(\mathbb{C})$ be unitary. If U is block upper triangular, then in fact, U is block diagonal.

Lemma 15. *Let $A \in M_n(\mathbb{C})$ be Λ_L-orthogonal. If A is block upper triangular, then A is block diagonal.*

Proof. Suppose that $A = \begin{bmatrix} W & X \\ 0 & Y \end{bmatrix}$. Because A is Λ_L-orthogonal, W and Y are both nonsingular. Write $L = D_1 \oplus D_2$ conformal to A. Looking at the $(1, 2)$ entries of the equation $A^*LA = L$, we have $W^*D_1X = 0$. Since both W and D_1 are nonsingular, we have $X = 0$. □

3.2. Product of Λ_L-Householder matrices

Let $L = \operatorname{diag}(1, -1)$ and let $A \in M_2(\mathbb{C})$ be Λ_L-orthogonal and suppose that $\det(A) = \pm 1$. Let $u = [u_1\, u_2]^T$ be the first column of A with $u_1 = re^{i\theta}$ and $r, \theta \in \mathbb{R}$. Then $u^*Lu = 1$. Set $e = e^{i\theta}e_1$, so that $u^*Le = r$ and $e^*Le = 1$. Lemma 10 guarantees that $w = u + e$ is L-nonisotropic or that $v = u - e$ is L-nonisotropic. If w is L-nonisotropic, then Lemma 11 guarantees that $L_wu = -e$ and if v is L-nonisotropic, then Lemma 11 guarantees that $L_vu = e$.

Suppose that w is L-nonisotropic. Then $L_wA = \begin{bmatrix} -e^{i\theta} & b \\ 0 & c \end{bmatrix}$. Lemma 15 ensures that $b = 0$, and since $\det(A) = \pm 1$, we must have $c = \pm e^{-i\theta}$.

If v is L-nonisotropic, then $L_vA = \begin{bmatrix} e^{i\theta} & b \\ 0 & c \end{bmatrix}$. Lemma 15 ensures that $b = 0$, and since $\det(A) = \pm 1$, we must have $c = \pm e^{-i\theta}$.

We look at the number of Λ_L-Householder factors of $X_1 \equiv \operatorname{diag}\left(e^{i\theta}, e^{-i\theta}\right)$ and $X_2 \equiv \operatorname{diag}\left(e^{i\theta}, -e^{-i\theta}\right)$. First, notice that for many values of θ, we have $X_1 - I$ and $X_2 - I$ are both nonsingular. Hence, for these values of θ, neither X_1 nor X_2 is Λ_L-Householder. Moreover, since $\det(X_1) = 1$, if X_1 can be written as a product of Λ_L-Householder matrices, then the number of such factors must be even. Now, if X_2 can be written as a product of Λ_L-Householder matrices, then the number of factors must be odd and that number must be bigger than or equal to 3.

When $\theta = 0$, we have $X_1 = I = L_t^2$ for any L-nonisotropic vector t and we have $X_2 = L_{e_2}$.

When $\theta = \pi$, we have $X_1 = -I = L_{e_1}L_{e_2}$ and we have $X_2 = L_{e_1}$.

Suppose now that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We show that X_1 and X_2 can be written as a product of Λ_L-Householder matrices. Let $u \in \mathbb{C}^2$ be L-nonisotropic. Suppose that $u^*Lu > 0$. Set $v = \frac{1}{\sqrt{u^*Lu}}u$ and notice that $v^*Lv = 1$. Suppose that $u^*Lu < 0$. Set $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u$ and notice that $w^*Lw = -u^*Lu > 0$. Hence, we may assume that $u^*Lu = 1$.

Let $u = \left[re^{i\alpha}\ se^{i\beta}\right]^T$, where $r, s, \alpha, \beta \in \mathbb{R}$. Then $w = e^{-i\alpha}u \in \langle u\rangle$ and $L_u = L_w$. Hence, we may further assume that $u = \left[r\ se^{i\theta}\right]^T$, where $r^2 - s^2 = 1$. Now, notice that $L_u = \begin{bmatrix} 1-2r^2 & 2rse^{-i\theta} \\ -2rse^{i\theta} & 1+2s^2 \end{bmatrix}$. Set $a \equiv 1-2r^2$ and $b \equiv -2rs$. Then we have $a^2 - b^2 = 1$, we have $1 + 2s^2 = 2r^2 - 1 = -a$, and we have

$$L_u = \begin{bmatrix} a & -be^{-i\theta} \\ be^{i\theta} & -a \end{bmatrix}.$$

Let $u_1 = \left[r\ se^{i\alpha}\right]^T$ and let $u_2 = \left[r\ se^{i\beta}\right]^T$. We look at the product $L_{u_1}L_{u_2}$. A direct computation shows that

$$L_{u_1}L_{u_2}e_1 = \begin{bmatrix} a^2 - b^2e^{i(\beta-\alpha)} \\ ab\left(e^{i\alpha} - e^{i\beta}\right) \end{bmatrix}.$$

Let $x = \beta - \alpha$ and note that $d \equiv a^2 - b^2e^{ix} = a^2 - b^2\cos x - ib^2\sin x$. Write $d = c\cos y + ic\sin y$, with $c, y \in \mathbb{R}$ and $c \geqslant 0$. Then $\tan y = \frac{-b^2\sin x}{a^2-b^2\cos x} = \frac{-b^2\sin x}{1+b^2-b^2\cos x}$, so that $\cot y = -\frac{1-\cos x}{\sin x} - \frac{1}{b^2\sin x} = -\tan\left(\frac{x}{2}\right) - \frac{1}{b^2\sin x}$. For each b, the range of $f(x, b) = -\tan\left(\frac{x}{2}\right) - \frac{1}{b^2\sin x}$ is $\mathbb{R}\setminus\{0\}$. Now choose $y = \cot^{-1}\left(-\tan\left(\frac{x}{2}\right) - \frac{1}{b^2\sin x}\right)$ so that $y \in \left(-\frac{\pi}{2},\ 0\right) \cup \left(0,\ \frac{\pi}{2}\right)$, and notice that for each b, the function $g(y) = \cot y$ is a bijection from $\left(-\frac{\pi}{2},\ 0\right) \cup \left(0,\ \frac{\pi}{2}\right)$ to $\mathbb{R}\setminus\{0\}$.

Now, $(L_{u_1}L_{u_2}e_1)^* L(L_{u_1}L_{u_2}e_1) = e_1^*Le_1 = 1$. Set $e = e^{iy}e_1$, so that $e^*L(L_{u_1}L_{u_2}e_1) = c \in \mathbb{R}$. Moreover, $e^*Le = 1$. Hence, $w = L_{u_1}L_{u_2}e_1 + e$ is L-nonisotropic or $v = L_{u_1}L_{u_2}e_1 - e$ is L-nonisotropic.

If w is L-nonisotropic, then $L_wL_{u_1}L_{u_2}e_1 = -e$, so that $L_wL_{u_1}L_{u_2} = \begin{bmatrix} -e^{iy} & h \\ 0 & j \end{bmatrix}$. Since $L_wL_{u_1}L_{u_2}$ is Λ_L-orthogonal, we have $h = 0$ and $j = \pm e^{-iy}$. That is, $Y \equiv \begin{bmatrix} -e^{iy} & 0 \\ 0 & \pm e^{-iy} \end{bmatrix}$ can be written as a product of three Λ_L-Householder matrices. However, since the determinant of an odd number of Λ_L-Householder matrices is -1, we have $Y = \begin{bmatrix} -e^{iy} & 0 \\ 0 & e^{-iy} \end{bmatrix}$. Now, $X_1 = L_{e_1}Y$ is a product of four Λ_L-Householder matrices. Also, X_2 is a product of five Λ_L-Householder matrices.

Notice that $Y^2 = \mathrm{diag}\left(e^{2iy}, e^{-2iy}\right) = (L_wL_{u_1}L_{u_2})^2$. Proposition 5 (5) guarantees that

$$(L_wL_{u_1}L_{u_2})^2 = L_{L_wu_1}L_{L_wu_2}L_{u_1}L_{u_2}$$

is a product of four Λ_L-Householder matrices. Moreover, $\mathrm{diag}\left(e^{2iy},\ -e^{-2iy}\right) = L_{e_2}Y^2$ is a product of five Λ_L-Householder matrices.

If v is L-nonisotropic, then $L_vL_{u_1}L_{u_2}e_1 = e$, so that $L_vL_{u_1}L_{u_2} = \begin{bmatrix} e^{iy} & h \\ 0 & j \end{bmatrix}$. Since $L_vL_{u_1}L_{u_2}$ is Λ_L-orthogonal with determinant -1, we have $h = 0$ and $j = -e^{-iy}$. That is, $Y = \begin{bmatrix} e^{iy} & 0 \\ 0 & -e^{-iy} \end{bmatrix}$. Thus, $X_2 = L_vL_{u_1}L_{u_2}$ can be written as a product of three Λ_L-Householder matrices, and $X_1 = L_{e_2}X_2$ is a product of four Λ_L-Householder matrices.

Notice that $X_2^2 = \operatorname{diag}\left(e^{2iy}, e^{-2iy}\right) = \left(L_v L_{u_1} L_{u_2}\right)^2$ is a product of four Λ_L-Householder matrices. Moreover, $\operatorname{diag}\left(e^{2iy}, -e^{-2iy}\right) = L_{e_2} X_2^2$ is a product of five Λ_L-Householder matrices.

Suppose that $-\pi < \theta < \pi$. Set $\theta = 2y$. Then $\operatorname{diag}\left(e^{2iy}, e^{-2iy}\right)$ can be written as a product of four Λ_L-Householder matrices, while $\operatorname{diag}\left(e^{2iy}, -e^{-2iy}\right)$ can be written as a product of five Λ_L-Householder matrices.

Lemma 16. *Let $L = \operatorname{diag}(1, -1)$. Let $\theta \in \mathbb{R}$ be given. Then $\operatorname{diag}\left(e^{i\theta}, \pm e^{-i\theta}\right)$ can be written as a product of at most five Λ_L-Householder matrices in $M_2(\mathbb{C})$.*

We summarize our results.

Lemma 17. *Let $A \in M_2(\mathbb{C})$ be Λ_L-orthogonal. If $\det(A) = \pm 1$, then A can be written as a product of at most 6 Λ_L-Householder matrices.*

Let $A \in M_n(\mathbb{C})$ be Λ_L-orthogonal with $\det(A) = \pm 1$. Suppose that $k = 1$. We look at the first column of A, say u, and suppose that the first entry of u is $ce^{i\theta}$, with $c, \theta \in \mathbb{R}$. Let $w = u + e^{i\theta} e_1$ and let $v = u - e^{i\theta} e_1$. As before, either w is L-nonisotropic or v is L-nonisotropic. Moreover, $L_w u = -e^{i\theta} e_1$ or $L_v u = e^{i\theta} e_1$. For $j = 1, 2$, we let $B_j = \begin{bmatrix} 1 & 0 \\ 0 & A_j \end{bmatrix}$ and let $C(a, b, n) \equiv \left(\begin{bmatrix} e^{i(a+b)} & 0 \\ 0 & e^{-i(a+b)} \end{bmatrix} \oplus I_{n-2} \right)$. If w is L-nonisotropic, then we have $L_w A = B_1 C(\theta, \pi, n)$. If v is L-nonisotropic, then we have $L_v A = B_2 C(\theta, 0, n)$. Notice that B_1 and B_2 have the same forms, and that $C(\theta, \pi, n)$ and $C(\theta, 0, n)$ have the same forms. Hence, it is without loss of generality to assume that v is L-nonisotropic. Now, $A_1 \in M_{n-1}(\mathbb{C})$ is a unitary matrix having determinant ± 1, and hence a product of at most $2(n-1) - 1 = 2n - 3$ Householder matrices [4, Theorem 1]. Let $H_x = I - 2xx^* \in M_{n-1}(\mathbb{C})$ be a Householder matrix. Set $y = \left[0 \; x^T\right]^T \in \mathbb{C}^n$. Set $L_y = I - \frac{2}{y^* L y} y y^* L$. Then $L_y = \begin{bmatrix} 1 & 0 \\ 0 & H_x \end{bmatrix}$. Hence, $\begin{bmatrix} 1 & 0 \\ 0 & A_1 \end{bmatrix}$ is a product of at most $2n - 3$ Λ_L-Householder matrices. Let $C = C(\theta, \pi, n)$ or $C = C(\theta, 0, n)$ so that $\det(C) = 1$. Notice that we can write C as a product of at most 4 Λ_L-Householder matrices. Hence, A is a product of at most $2n + 2$ Λ_L-Householder matrices.

Suppose $k \geqslant 2$. We look at the first column of A, say u and suppose that the second entry of u is $ce^{i\theta}$, with $c, \theta \in \mathbb{R}$. Let $w = u + e^{i\theta} e_2$ and let $v = u - e^{i\theta} e_2$. Then, either w is L-nonisotropic or v is L-nonisotropic. Moreover, $L_w u = -e^{i\theta} e_2$ or $L_v u = e^{i\theta} e_2$.

Suppose that w is L-nonisotropic. Then $L_w A = \begin{bmatrix} 0 & b^T \\ -e^{i\theta} & c^T \\ 0 & B \end{bmatrix}$, where $b, c \in \mathbb{C}^{n-1}$ and $B \in M_{(n-2),(n-1)}(\mathbb{C})$. Let $p = \frac{1}{\sqrt{2}} \left(e_1 + e^{i\theta} e_2\right)$. Then $L_p = \begin{bmatrix} 0 & -e^{-i\theta} \\ -e^{i\theta} & 0 \end{bmatrix} \oplus I_{n-2}$. Hence, we have $L_p L_w A = \begin{bmatrix} 1 & d^T \\ 0 & D \end{bmatrix}$, where $d \in \mathbb{C}^{n-1}$ and $D \in M_{n-1}(\mathbb{C})$. Lemma 15 guarantees that $d = 0$, so that D is $\Lambda_{L_{k-1}}$-orthogonal. If $k = 2$, then D can be written as a product of $2(n-1) + 2$ $\Lambda_{L_{k-1}}$-Householder matrices. Thus, A can be written as a product of $2n + 2$ Λ_L-Householder matrices. If $k > 2$, repeat the reduction $k - 2$ more times. At this time, we have used $2(k-1)$ Λ_L-Householder matrices, and we need $2(n - k + 1) + 2$ more. Hence, A can be written as a product of $2n + 2$ Λ_L-Householder matrices.

If v is L-nonisotropic, then a similar calculation shows that A can be written as a product of $2n + 2$ Λ_L-Householder matrices.

Theorem 18. *Let $n \geqslant 2$ and $1 \leqslant k \leqslant n$ be integers. Let $L = I_k \oplus -I_{n-k}$. Let $A \in M_n(\mathbb{C})$ be Λ_L-orthogonal with* $\det(A) = \pm 1$. *Then A can be written as a product of at most $2n + 2$ Λ_L-Householder matrices.*

The following is part of Theorem 3 in [4]. We provide a different proof.

Corollary 19. *Let $n \geqslant 2$ and $k \geqslant 1$ be integers such that $n \geqslant k$. Let $L = I_k \oplus -I_{n-k}$. Let $A \in M_n(\mathbb{R})$ be Λ_L-orthogonal. Then A can be written as a product of at most $2n - 1$ Λ_L-Householder matrices.*

Proof. Let $A \in M_n(\mathbb{R})$ be Λ_L-orthogonal. Because A is real, we have $\det(A) = \pm 1$. Suppose $L_p A = B_1 C_1(\theta, 0, n)$, where $p = w$ or $p = v$ as in the proof of Theorem 18. Notice that we may take $\theta = 0$ so that $C_1 = I$ and $B_1 = \begin{bmatrix} \pm 1 & 0 \\ 0 & A_1 \end{bmatrix}$. So, far, we have only used 1 Λ_L-Householder matrix. We apply induction to show that we can use $n-2$ more Λ_L-Householder matrices to reduce A_1 to a diagonal matrix with diagonal entries ± 1. We only need $n-2$ Λ_L-Householder matrices because only 1 Λ_L-Householder matrix is needed to reduce a 2-by-2 matrix to a diagonal. Now, for each diagonal entry that is -1, multiply by L_{e_i}. Hence, every Λ_L-orthogonal A can be written as a product of at most $2n-1$ Λ_L-Householder matrices. □

3.3. Product of two Λ_L-Householder matrices

Let $n \geqslant 2$ and $k \geqslant 1$ be given integers with $k \leqslant n$. Let $L_k = I_k \oplus -I_{n-k}$. Let $Q = [q_i] \in M_n(\mathbb{C})$ be Λ_{L_k}-orthogonal. Then $q_i^* L_k q_i = 1$ for $i = 1, \ldots, k$, $q_i^* L_k q_i = -1$ for $i = k+1, \ldots, n$, and $q_i^* L_k q_j = 0$ for $i \neq j$.

Definition 20. Let $p \leqslant n$ be a given positive integer. Then $\{x_1, \ldots, x_p\} \subset \mathbb{C}^n$ is a Λ_{L_k}-*orthogonal set* if $x_i^* L_k x_j = 0$ for $i \neq j$ and $x_i^* L_k x_i = \pm 1$ for $i = 1, \ldots, p$.

Let $A = \{x_1, \ldots, x_p\} \subset \mathbb{C}^n$ be a Λ_{L_k}-orthogonal set. Let $y = \alpha_1 x_1 + \cdots + \alpha_p x_p = 0$. Then, for each $i = 1, \ldots, p$, we have $0 = x_i^* L_k y = \pm \alpha_i$, so that $\alpha_i = 0$. Hence, A is linearly independent. Let $Q \in M_n(\mathbb{C})$ be Λ_{L_k}-orthogonal. One checks that $QA = \{Qx_1, \ldots, Qx_p\}$ is also a Λ_{L_k}-orthogonal set. Suppose that $x_i^* L_k x_i = 1$ for $i = 1, \ldots, q$ and that $x_i^* L_k x_i = -1$ for $i = q+1, \ldots, p$. Set $B = [x_1 \ \cdots \ x_p]$. Lemma 12 guarantees that there exists a Λ_{L_k}-orthogonal P such that $Px_1 = e_1^{(n)}$. Because PA is a Λ_{L_k}-orthogonal set, we must have $PB = \begin{bmatrix} 1 & 0 \\ 0 & B_1 \end{bmatrix}$, where $B_1 = \left[b_i^{(1)}\right] \in M_{(n-1),(p-1)}(\mathbb{C})$ and $\left\{b_1^{(1)}, \ldots, b_{p-1}^{(1)}\right\}$ is a $\Lambda_{L_{k-1}}$-orthogonal set.

If $k = 1$, then B_1 has orthonormal columns. Extend $\left\{b_1^{(1)}, \ldots, b_{p-1}^{(1)}\right\}$ to an orthonormal basis of $\mathbb{C}^{n-1}$, say $\{c_1, \ldots, c_{n-p}\} \cup \left\{b_1^{(1)}, \ldots, b_{p-1}^{(1)}\right\}$. Set $C_1 = [c_i]$ and set $C = [B_1 \ C_1]$. Then $C \in M_{n-1}(\mathbb{C})$ is unitary. Moreover, $D \equiv [1] \oplus C$ is Λ_{L_k}-orthogonal. Let $P^{-1}D = [y_i]$. Notice that $y_i = x_i$ for $i = 1, \ldots, p$. Moreover, we have extended A to a Λ_{L_k}-orthogonal basis of $\mathbb{C}^n$.

If $k > 1$, then there exists a $\Lambda_{L_{k-1}}$-orthogonal $Q_1 \in M_{n-1}(\mathbb{C})$ such that $Q_1 B_1 = \begin{bmatrix} 1 & 0 \\ 0 & B_2 \end{bmatrix}$, where $B_2 = \left[b_i^{(2)}\right] \in M_{(n-2),(p-2)}(\mathbb{C})$ and $\left\{b_1^{(2)}, \ldots, b_{p-2}^{(2)}\right\}$ is a $\Lambda_{L_{k-2}}$-orthogonal set. Set $P_2 = [1] \oplus Q_1$ and notice that P_2 is Λ_{L_k}-orthogonal and that $P_2 PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B_2 \end{bmatrix}$. Continue the reduction until $i = q$, and let $T = P_q \cdots P_2 P$. Then $TB = \begin{bmatrix} I_q & 0 \\ 0 & C \end{bmatrix}$, where $C = [c_i] \in M_{(n-q),(p-q)}(\mathbb{C})$ and $\{c_1, \ldots, c_{p-q}\}$ is a $\Lambda_{L_{k-q}}$-orthogonal set. Notice that necessarily, $q \leqslant k$. Otherwise, we have $e_q^* L_k e_q = -1$, but $(TB)^* L_k (TB) = B^* L_k B = I_q \oplus -I_{p-q}$ implies $e_q^* L_k e_q = 1$. Now, $c_i^* L_{k-q} c_j = 0$ for $i \neq j$ and $c_i^* L_{k-q} c_i = -1$ for $i = 1, \ldots, p-q$. There exists a $\Lambda_{L_{k-q}}$-orthogonal $S \in M_{n-q}(\mathbb{C})$ such that $Sc_1 = e_{n-q}^{(n-q)}$. Then $SC = \begin{bmatrix} C_1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $N_1 = I_q \oplus S$. Then N_1 is Λ_{L_k}-orthogonal. Moreover, $C_1 = [f_i] \in M_{(n-q-1),(p-q-1)}(\mathbb{C})$ and $\{f_1, \ldots, f_{p-q-1}\}$ is a $\Lambda_{L_{k-q}}$-orthogonal set. Here, $L_{k-q} = I_{k-q} \oplus -I_{n-k-1}$. Now, there exists $\Lambda_{L_{k-q}}$-orthogonal R_1 such that $R_1 C_1 = \begin{bmatrix} C_2 & 0 \\ 0 & 1 \end{bmatrix}$. Set $S_2 = R_1 \oplus [1]$ and set $N_2 = I_q \oplus S_2$. Continue the reduction until $i = p - q$, and let $W = N_{p-q} \cdots N_1$. Necessarily, $p - q \leqslant n - k$ and

$$WTB = \begin{bmatrix} I_q & 0 \\ 0 & 0 \\ 0 & I_{p-q} \end{bmatrix}.$$

Let $M = WT$, and let $M^{-1} = [n_i]$. Set $F = [n_{q+1} \cdots n_{n-p+q}]$, set $E_1 = [x_1 \cdots x_q]$, set $E_2 = [x_{q+1} \cdots x_p]$, and set $D = [E_1\, F\, E_2]$. Then $MD = I$, so that $D = M^{-1}$. Now, notice that M is Λ_{L_k}-orthogonal, so that M^{-1} is also Λ_{L_k}-orthogonal. Hence, we have extended A to a Λ_{L_k}-orthogonal basis of $\mathbb{C}^n$.

Theorem 21. *Let $A = \{x_1, \ldots, x_p\} \subset \mathbb{C}^n$ be a Λ_{L_k}-orthogonal set. Then A is linearly independent. Suppose that $x_i^* L_k x_i = 1$ for $i = 1, \ldots, q$ and $x_i^* L_k x_i = -1$ for $i = q+1, \ldots, p$. Then $q \leqslant k$ and $p - q \leqslant n - k$. Moreover, A can be extended to a Λ_{L_k}-orthogonal basis of $\mathbb{C}^n$.*

Let $A \in M_n(\mathbb{C})$ be a product of two Λ_{L_k}-Householder matrices, say $A = L_u L_v$, where $u, v \in \mathbb{C}^n$. Then $\operatorname{rank}(A - I) = \operatorname{rank}(L_u (L_v - L_u)) = \operatorname{rank}(L_v - L_u) \leqslant 2$. If $\operatorname{rank}(L_u - L_v) = 0$, then $L_v = L_u$ and $A = I$. Suppose that $\operatorname{rank}(A - I) \neq 0$. Theorem 45 of [2] guarantees that the Jordan Canonical Form of A contains only blocks of the form (1) $J_k(\lambda) \oplus J_k\left(\frac{1}{\lambda}\right)$, where $|\lambda| > 1$ and any k, and (2) $J_k\left(e^{i\theta}\right)$, where $\theta \in \mathbb{R}$ and any k. If the Jordan Canonical Form of A contains blocks of the form (1), then λ must be real. Since $\operatorname{rank}(A - I) \leqslant 2$, we must have $k = 1$, that is, A is similar to $\operatorname{diag}\left(\lambda, \frac{1}{\lambda}\right) \oplus I_{n-2}$. If the Jordan Canonical Form of A contains blocks of the form (2) and if $\theta \neq k\pi$, where k is an integer, then the Jordan Canonical Form of A must also contain $J_k\left(e^{-i\theta}\right)$. In this case, we must have $k = 1$. If -1 is an eigenvalue of A, then A is similar to $-I_2 \oplus I_{n-2}$ or A is similar to $J_2(-1) \oplus I_{n-2}$. If 1 is the only eigenvalue of A, then A is similar to $J_2(1) \oplus I_{n-2}$ or A is similar to $J_3(1) \oplus I_{n-3}$.

It is without loss of generality to assume that $u^* L_k u = \pm 1$ and that $v^* L_k v = \pm 1$. We look at these cases.

Case 1. $u^* L_k u = v^* L_k v = 1$. There exists a Λ_{L_k}-orthogonal P such that $Pu = e_1$. Then $PAP^{-1} = L_{e_1} L_{Pv}$. Let $Pv = [a_i]_{i=1}^n$, let $z = [a_i]_{i=2}^n$.

Suppose that $k = 1$. If $z = 0$, then $Pv = a_1 e_1$ and $|a_1| = 1$, so that $L_{Pv} = L_{e_1}$ and $A = I$, a contradiction. Hence, $z \neq 0$. Let $||z||_2 = b$. Then, there exists a unitary $Q \in M_{n-1}(\mathbb{C})$ such that $Qz = be_1^{(n-1)}$. Set $P_1 = [1] \oplus Q$, so that P_1 is Λ_{L_k}-orthogonal. Moreover, $P_1 e_1 = e_1$ and $P_1 P v = a_1 e_1 + b e_2$. A direct computation shows that

$$P_1 P A P^{-1} P_1^{-1} = L_{e_1} L_{a_1 e_1 + b e_2} = \begin{bmatrix} 2|a_1|^2 - 1 & -2a_1 b \\ -2\overline{a_1} b & 1 + 2b^2 \end{bmatrix} \oplus I_{n-2}.$$

Here, we have $|a_1|^2 - b^2 = 1$ since $v^*L_kv = 1$ and P_1P is Λ_{L_k}-orthogonal. Let $\alpha = 1 + 2b^2$. Then the eigenvalues of A are the two positive numbers $\alpha \pm \sqrt{\alpha^2 - 1}$ and 1.

Suppose that $k \geqslant 2$. Notice that if $\{u, v\}$ is a Λ_{L_k}-orthogonal set, then $a_1 = 0$. Moreover, $z^*L_{k-1}z = 1$, so that there exists a $\Lambda_{L_{k-1}}$-orthogonal Q such that $Qz = e_1^{(n-1)}$. Set $P_1 = [1] \oplus Q$, so that P_1 is Λ_{L_k}-orthogonal. Moreover, $P_1e_1 = e_1$ and $P_1Pv = e_2$. In this case, $P_1PAPP_1^{-1} = L_{e_1}L_{e_2} = -I_2 \oplus I_{n-2}$.

Suppose that $\{u, v\}$ is not a Λ_{L_k}-orthogonal set. We have two subcases: z is L_{k-1}-isotropic or z is L_{k-1}-nonisotropic.

Suppose that z is L_{k-1}-isotropic. Notice that $n \geqslant 3$, otherwise, $z = 0$ and $A = I$. Now, $|a_1| = 1$, say, $a_1 = e^{i\theta}$, where $\theta \in \mathbb{R}$. Lemma 14 guarantees that there exists a $\Lambda_{L_{k-1}}$-orthogonal Q such that $Qz = e_1^{(n-1)} + e_k^{(n-1)}$. Set $P_1 = [1] \oplus Q$, so that P_1 is Λ_{L_k}-orthogonal. Moreover, $P_1e_1 = e_1$ and $P_1Pv = e^{i\theta}e_1 + e_2 + e_{k+1}$. A direct calculation shows that

$$P_1PAP^{-1}P_1^{-1} = \begin{bmatrix} 1 & 2e^{i\theta} & 0 & -2e^{i\theta} & 0 \\ -2e^{-i\theta} & -1 & 0 & 2 & 0 \\ 0 & 0 & I_{k-2} & 0 & 0 \\ -2e^{-i\theta} & -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & I_{n-k-1} \end{bmatrix}.$$

Let $A_1 = \begin{bmatrix} 1 & 2e^{i\theta} & -2e^{i\theta} \\ -2e^{-i\theta} & -1 & 2 \\ -2e^{-i\theta} & -2 & 3 \end{bmatrix}$. Then, notice that $(A_1 - I)^2$ has rank 1 and that $(A_1 - I)^3 = 0$. Hence, in this case, A is similar to $J_3(1) \oplus I_{n-3}$.

Suppose that z is L_{k-1}-nonisotropic. We have two subcases: $z^*L_{k-1}z > 0$ and $z^*L_{k-1}z < 0$.

Suppose that $b^2 = z^*L_{k-1}z > 0$, with $b > 0$. There exists a $\Lambda_{L_{k-1}}$-orthogonal Q such that $Qz = be_1^{(n-1)}$. Set $P_1 = [1] \oplus Q$, and notice that $P_1e_1 = e_1$ and that $P_1Pv = a_1e_1 + be_2$. Here, we have $|a_1|^2 + b^2 = 1$. In this case, we have

$$P_1PAP^{-1}P_1^{-1} = \begin{bmatrix} 2|a_1|^2 - 1 & -2a_1b \\ -2\overline{a_1}b & 1 - 2b^2 \end{bmatrix} \oplus I_{n-2}.$$

Let $\alpha = 1 - 2b^2$. Because $|\alpha| < 1$, the eigenvalues of A are $\alpha \pm i\sqrt{1 - \alpha^2}$ and 1.

Suppose that $-b^2 = z^*L_{k-1}z < 0$, with $b > 0$. There exists a $\Lambda_{L_{k-1}}$-orthogonal Q such that $Qz = be_k^{(n-1)}$. Set $P_1 = [1] \oplus Q$, and notice that $P_1e_1 = e_1$ and that $P_1Pv = a_1e_1 + be_{k+1}$. Here, we have $|a_1|^2 - b^2 = 1$. In this case, we have

$$P_1PAP^{-1}P_1^{-1} = \begin{bmatrix} 2|a_1|^2 - 1 & 0 & -2a_1b & 0 \\ 0 & I_{k-1} & 0 & 0 \\ -2\overline{a_1}b & 0 & 1 + 2b^2 & 0 \\ 0 & 0 & 0 & I_{n-k-1} \end{bmatrix}.$$

Let $\alpha = 1 + 2b^2$. Then the eigenvalues of A are $\alpha \pm \sqrt{\alpha^2 - 1}$ and 1.

Case 2. $u^*L_ku = v^*L_kv = -1$. Then $u^*(-L_k)u = 1$. Set $P = \begin{bmatrix} 0 & I_{n-k} \\ I_k & 0 \end{bmatrix}$. Then $P(-L_k)P^T = L_{n-k}$. Set $x = Pu$ and set $y = Pv$. Then $x^*L_{n-k}x = y^*L_{n-k}y = 1$.

Case 3. $u^*L_ku = 1$ and $v^*L_kv = -1$. There exists a Λ_{L_k}-orthogonal P such that $Pu = e_1$. Then $PAP^{-1} = L_{e_1}L_{Pv}$. Let $Pv = [a_i]_{i=1}^n$, let $z = [a_i]_{i=2}^n$.

Suppose that $k = 1$. Suppose further that $\{u, v\}$ is a Λ_{L_k}-orthogonal set. Then $a_1 = 0$ and $||z||_2 = 1$, so that there exists a unitary $Q \in M_{n-1}(\mathbb{C})$ such that $Qz = e_1^{(n-1)}$. Set $P_1 = [1] \oplus Q$ and notice that $P_1PAP^{-1}P_1^{-1} = -I_2 \oplus I_{n-2}$.

Suppose that $\{u, v\}$ is not a Λ_{L_k}-orthogonal set. Let $b = ||z||_2$. Then $|a_1|^2 - b^2 = 1$. Notice that $b \neq 0$, otherwise, $v^*L_kv = 1$. Now, there exists a unitary $Q \in M_{n-1}(\mathbb{C})$ such that $Qz = be_1^{(n-1)}$. Set $P_1 = [1] \oplus Q$, and notice that $P_1e_1 = e_1$ and $P_1Pv = a_1e_1 + be_2$. One checks that

$$P_1PAP^{-1}P_1^{-1} = \begin{bmatrix} -1-2|a_1|^2 & 2a_1b \\ 2\overline{a_1}b & 1-2b^2 \end{bmatrix} \oplus I_{n-2}.$$

Set $\alpha = 1 + 2b^2$. Then, the eigenvalues of A are $-\alpha \pm \sqrt{\alpha^2 - 1}$ and 1.

Suppose that $k \geqslant 2$. Suppose further that $\{u, v\}$ is a Λ_{L_k}-orthogonal set. Then $a_1 = 0$ and $z^*L_{k-1}z = -1$, so that there exists a $\Lambda_{L_{k-1}}$-orthogonal $Q \in M_{n-1}(\mathbb{C})$ such that $Qz = e_k^{(n-1)}$. Set $P_1 = [1] \oplus Q$ and notice that $P_1e_1 = e_1$, and that $P_1Pv = e_{k+1}$. In this case, $P_1PAP^{-1}P_1^{-1} = [-1] \oplus I_{k-1} \oplus [-1] \oplus I_{n-k-1}$, so that A is similar to $-I_2 \oplus I_{n-2}$.

Suppose that $\{u, v\}$ is not a Λ_{L_k}-orthogonal set. Notice that z is not L_{k-1}-isotropic, otherwise, we have $-1 = |a_1|^2 + z^*L_{k-1}z = |a_1|^2$. Moreover, $z^*L_{k-1}z = -1 - |a_1|^2 < 0$. Let $b = \sqrt{1 + |a_1|^2}$. There exists a $\Lambda_{L_{k-1}}$-orthogonal $Q \in M_{n-1}(\mathbb{C})$ such that $Qz = be_k^{(n-1)}$. Set $P_1 = [1] \oplus Q$, and notice that $P_1e_1 = e_1$ and that $P_1Pv = a_1e_1 + be_{k+1}$. Then,

$$P_1PAP^{-1}P_1^{-1} = \begin{bmatrix} -1-2|a_1|^2 & 0 & 2a_1b & 0 \\ 0 & I_{k-1} & 0 & 0 \\ 2\overline{a_1}b & 0 & 1-2b^2 & 0 \\ 0 & 0 & 0 & I_{n-k-1} \end{bmatrix}.$$

Set $\alpha = 2b^2 - 1 = 1 + 2|a_1|^2$. The eigenvalues of A are the two real numbers $-\alpha \pm \sqrt{\alpha^2 - 1}$ and 1.

Case 4. $u^*L_ku = -1$ and $v^*L_kv = 1$. Consider instead $-L_k$.

We summarize our results. Notice that neither $J_2(1)$ nor $J_2(-1)$ is a possible Jordan block of a product of two Λ_{L_k}-Householder matrices.

Theorem 22. *Let $n \geqslant 2$ and $k \geqslant 1$ be given integers. Let $A \in M_n(\mathbb{C})$ be given. Suppose that A is a product of two Λ_{L_k}-Householder matrices. Then A is similar to only one of the following:*

1. *$\mathrm{diag}\left(\lambda, \frac{1}{\lambda}\right) \oplus I_{n-2}$, where $\lambda \in \mathbb{R}$ and $|\lambda| \geqslant 1$,*
2. *$\mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right) \oplus I_{n-2}$, where $\theta \in \mathbb{R}$, or*
3. *$J_3(1) \oplus I_{n-3}$.*

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