On equitable $\Delta$-coloring of graphs with low average degree

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Abstract

An equitable coloring of a graph is a proper vertex coloring such that the sizes of any two color classes differ by at most 1. Hajnal and Szemerédi proved that every graph with maximum degree $\Delta$ is equitably $k$-colorable for every $k \geq \Delta + 1$. Chen, Lih, and Wu conjectured that every connected graph with maximum degree $\Delta \geq 3$ distinct from $K_{\Delta+1}$ and $K_{\Delta,\Delta}$ is equitably $\Delta$-colorable. This conjecture has been proved for graphs in some classes such as bipartite graphs, outerplanar graphs, graphs with maximum degree 3, interval graphs. We prove that this conjecture holds for graphs with average degree at most $\Delta/5$.

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1. Introduction

In several applications of coloring as a partition problem there is an additional requirement that color classes be not so large or be of approximately the same size. Examples are the mutual exclusion scheduling problem [1,17], scheduling in communication systems [7], construction timetables [9], and round-a-clock scheduling [18]. For other applications in scheduling, partitioning, and load balancing problems, one can look into [2,12,17]. A model imposing such a requirement is equitable coloring—a proper coloring such that color classes differ in size by at most one. A good survey on equitable colorings of graphs is given in [13]. Recently, Pemmaraju [16] and Janson and Ruciński [8] used equitable colorings to derive deviation bounds for sums of dependent random variables that exhibit limited dependence.

Unlike in the case of ordinary coloring, a graph may have an equitable $k$-coloring (i.e., an equitable coloring with $k$ colors) but have no equitable $(k + 1)$-coloring. For example, the complete bipartite graph $K_{2n+1,2n+1}$ has the obvious equitable 2-coloring, but has no equitable $(2n + 1)$-coloring. Thus, it is natural to look for the minimum number, eq($G$), such that for every $k \geq$ eq($G$), $G$ has an equitable $k$-coloring.

The difficulty of finding eq($G$) is not less than that of finding the chromatic number. Thus, already in the class of planar graphs, finding eq($G$) is an NP-hard problem. This situation prompted studying extremal problems on relations of eq($G$) with other graph parameters. Hajnal and Szemerédi [6] settled a conjecture of Erdős by proving that eq($G$) $\leq \Delta + 1$ for every graph $G$ with maximum degree at most $\Delta$. This bound is sharp, as shows the example of $K_{2n+1,2n+1}$ above. Other natural examples showing sharpness of Hajnal–Szemerédi Theorem are graphs with chromatic number greater

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than maximum degree, i.e. complete graphs and odd cycles. Chen et al. [3] proposed the following analogue of Brooks’ Theorem for equitable coloring:

**Conjecture 1** (Chen et al. [3]). Let \( G \) be a connected graph with maximum degree \( \Delta \). If \( G \) is distinct from \( K_{\Delta+1}, K_{\Delta,\Delta} \), and is not an odd cycle, then \( G \) has an equitable coloring with \( \Delta \) colors.

They proved the conjecture for graphs with maximum degree at most three. Later, Yap and Zhang [20,21] proved that the conjecture holds for outerplanar graphs and planar graphs with maximum degree at least 13. Nakprasit (unpublished) extended the result of Yap and Zhang [21] to planar graphs with maximum degree at least 9. Lih and Wu [14] verified the conjecture for bipartite graphs, and Chen et al. [4] verified it for interval graphs. Kostochka et al. [11] studied a list analogue of equitable coloring and proved the validity of Conjecture 1 for equitable list coloring in classes of interval graphs and 2-degenerate graphs. It follows from [10] that the conjecture holds for \( d \)-degenerate graphs with maximum degree \( \Delta \) if \( d \leq (\Delta - 1)/14 \).

Recall that a graph \( G \) is \( d \)-degenerate, if each subgraph \( G' \) of \( G \) has a vertex of degree (in \( G' \)) at most \( d \) (see, e.g., [19, p. 269]). In other words, one can destroy any \( d \)-degenerate graph by successively deleting vertices of degree at most \( d \). Forests are exactly 1-degenerate graphs. It is also well known that every outerplanar graph is 2-degenerate (see, e.g., [19, p. 240]), and every planar graph is 5-degenerate. To say that a graph has ‘low degeneracy’ is about the same as to say that every subgraph of \( G \) has a ‘small average degree’. In this paper, we prove Conjecture 1 for graphs that have ‘low average degree’ themselves without restrictions on average degrees of subgraphs.

**Theorem 1.** Let \( \Delta, n \geq 46 \). Suppose that an \( n \)-vertex graph \( G = (V, E) \) has maximum degree at most \( \Delta \) and \( |E| \leq \Delta n/10 \). If \( K_{\Delta+1} \) is not a subgraph of \( G \), then \( G \) has an equitable coloring with \( \Delta \) colors.

An immediate consequence of Theorem 1 is that Conjecture 1 holds for \( d \)-degenerate graphs with maximum degree \( \Delta \) if \( d \leq \Delta/10 \).

In order to prove Theorem 1, we will need the following statement.

**Theorem 2.** Let \( \Delta \geq 3 \) and \( G \) be a \( K_{\Delta+1} \)-free graph with \( \Delta(G) \leq \Delta \). Suppose that \( G - v \) has a \( \Delta \)-coloring with color classes \( M_1, M_2, \ldots, M_\Delta \). Then \( G \) has a \( \Delta \)-coloring with color classes \( M'_1, M'_2, \ldots, M'_\Delta \) such that \( |M_i| = |M'_i| \) for all \( i \) apart from one.

We call this statement a theorem, since it seems that it has its own merit. It can be considered as a slight refinement of the Brooks’ Theorem. For example, Theorem 2 has the following easy consequence.

**Corollary 1.** Let \( \Delta \geq 3 \) and \( G \) be a \( K_{\Delta+1} \)-free graph with \( \Delta(G) \leq \Delta \). Let \( 1 \leq k \leq \Delta \) and \( m_k \) be the order of a maximum \( k \)-colorable subgraph of \( G \). Then there is a proper coloring of \( G \) with at most \( \Delta \) colors in which the total number of vertices in some \( k \) color classes is \( m_k \). In particular, \( G \) has a proper coloring with at most \( \Delta \) colors in which one of the color classes has \( \chi(G) \) vertices.

The first step of the proof of Theorem 1 uses the Hajnal–Szemerédi Theorem whose complexity we have not analyzed. The rest of the proof can be rewritten as a polynomial time algorithm for equitable coloring of a graph.

The structure of the paper is as follows. In the next section, we prove Theorem 2. In Section 3 we introduce some useful notions and verify Theorem 1 for graphs with few vertices. In Section 4, we prove the bounded-size version of Theorem 1: we demand each color class to be of size at most \( [n/\Delta] \) but allow ‘small’ color classes. We finalize the proof of Theorem 1 in the last section.

Throughout the paper, we use standard graph-theoretic definitions and notation (see, e.g., [5,19]).

2. Proof of Theorem 2

The proof follows the steps of a proof of Brooks’ Theorem by Mel’nikov and Vizing [15] (see also [5, p. 99]).
Let $A \geq 3$ and $G$ be a $K_{A+1}$-free graph with $\Delta(G) \leq A$. Let $f$ be a proper coloring of $G - v$ with at most $A$ colors. Suppose that $M_1, M_2, \ldots, M_A$ are color classes (maybe empty) of $f$ and that $G$ has no $A$-coloring with color classes $M'_1, M'_2, \ldots, M'_A$ such that $|M'_i| = |M'_j|$ for all $i$ apart from one. Let $G_{ij}$ denote the subgraph of $G$ induced by $M_i \cup M_j \cup \{v\}$, and $G'_{ij}$ denote the component of $G_{ij}$ containing $v$. We will deliver the proof in a series of claims.

**Claim 2.1.** The vertex $v$ has exactly one neighbor in each $M_i$.

**Proof.** Otherwise, there is a color class $M_i$ containing no neighbors of $v$. We simply add $v$ to $M_i$. Since other color classes do not change, the conclusion of the theorem holds, a contradiction. \(\Box\)

In view of this claim, let $w_i$ denote the only neighbor of $v$ in $M_i$.

**Claim 2.2.** For each $1 \leq i < j \leq A$, the graph $G'_{ij}$ is an odd cycle.

**Proof.** Suppose that some $G'_{ij}$ is not an odd cycle. By Claim 2.1, $G'_{ij}$ cannot be an even cycle. Thus $G'_{ij}$ has a vertex of degree (in $G'_{ij}$) distinct from 2. Let $u$ be a vertex closest to $v$ in $G'_{ij}$ with degree in $G'_{ij}$ not equal to 2. Let $P = (v, u_1, \ldots, u_k), w_k = u,$ be the shortest $v, u$-path in $G'_{ij}$ (since $G'_{ij} - v$ is bipartite, this path is unique). Let $B_i = M_i \cup A_i - A_i$, and $B_j = M_j \cup A_j - A_j$. We may assume that $v_1 = w_j$. Since $P - v$ is a component in $G_{ij} - v$. Therefore, the sets $B_i$ and $B_j$ are independent. Note that $B_i$ does not contain neighbors of $v$, since the only neighbor, $v_1$, of $v$ in $M_i$ belongs to $B_j$ and $A_j$ does not contain any neighbor of $v$. Thus $M'_i = B_i \cup \{v\}, M'_j = B_j$ and $M'_m = M_m$, for $m \neq i, j$ are color classes of a proper coloring of $G$. If $k$ is odd, then $|M'_i| = |M'_i|$, and if $k$ is even, then $|M'_i| = |M'_i|$. This contradicts the choice of $G$ and $f$.

**Case 2:** $d_{G'_{ij}}(u) \geq 3$. Since $d_{G_{ij}}(u) \leq |d_{G'_{ij}}(u)| \geq 3$, there is a color class $M_i$, $l \neq i, j$, with no neighbors of $u$. Hence $M_i \cup \{u\}$ is independent. Similarly to Case 1, the sets $B_i \cup v_k$ and $B_j \cup v_k$ are independent. Thus, $M'_i = B_i \cup \{v\}, M'_j = B_j, M'_i = M_i \cup \{u\}$ and $M'_m = M_m$, for $m \neq i, j, l$, are color classes of a proper coloring of $G$. Note that independently of the parity of $k, |M'_i| = |M_i|$, and $|M'_j| = |M_j|$. This contradicts the choice of $G$ and $f$. \(\Box\)

**Claim 2.3.** For any distinct $i, j$ and $s$, the components $G'_{ij}$ and $G'_{js}$ share exactly two vertices, namely, the vertex $v$ and the neighbor, $w_j$, of $v$ in $M_j$.

**Proof.** By the definition, $\{v, w_j\} \subseteq G'_{ij} \cap G'_{js}$. Suppose $u \in G'_{ij} \cap G'_{js} - \{v, w_j\}$. By Claim 2.2, $u$ has four neighbors in $G'_{ij} \cup G'_{js}$. Hence there is a color class $M_l$, $l \neq j$, with no neighbors of $u$. Let $v, u_1, \ldots, u_k, u$ be the $v, u$-path in $G'_{ij}$ with $v_1 = w_j$. Note that $k$ is odd, since $u \in M_j$ and $v_1 = w_j \in M_i$. Denote $A_j = \{v_1, \ldots, v_k\} - M_i$, and $A_i = \{v_1, \ldots, v_k\} - M_j$. Let $B_i = M_i \cup A_j - A_i$, and $B_j = M_j \cup A_i - A_j - \{u\}$. Similarly to the proof in Claim 2.2, the sets $B_i$ and $B_j$ are independent. Also similarly to the proof of Claim 2.2, the sets $M'_i = B_i \cup \{v\}, M'_j = B_j, M'_i = M_i \cup \{u\}$ and $M'_m = M_m$, for $m \neq i, j, l$, are color classes of a proper coloring of $G$. Moreover, $|M'_m| = |M'_m|$ for all $m$ apart from $m = l$. This proves the claim. \(\Box\)

Now, we are ready to finish the proof of Theorem 2. Among all colorings of $G - v$ with color class sizes $|M_1|, \ldots, |M_A|$, choose a coloring $f = (M_1, \ldots, M_A)$ and indices $i$ and $j$ so that $G'_{ij} - v = G'_{ij}(f) - v$ has the largest order. Recall that by the claims above, $G'_{ij} - v$ is a path with end-vertices $v_i$ and $w_j$. We may assume that $i = 1, j = 2$, and $G'_{12} - v$ is a path $P_1 = (v_1, v_q)$, where $v_1 = w_1$ and $v_q = w_2$. If $q = 2$, then by the maximality of $G'_{ij}$, all $w_i$s are adjacent to each other and together with $v$ form a $K_{A+1}$, a contradiction. Thus, $q \geq 4$ and hence $v_{q-1} \neq w_1$.

Similarly, $G'_{23} - v$ is a path, $P_2$, with end-vertices $w_2$ and $w_3$. Let coloring $f_1$ be obtained from $f$ by swapping the colors 2 and 3 on vertices in $G'_{23} - v$. Since $G'_{23} - v$ is a $w_2, w_3$-path, the color class sizes in $f_1$ are the same as in $f$. By Claim 2.2, the new bicoloring subgraph $H = G'_{12}(f_1) - v$ has to be a $w_1, w_3$-path (because $w_3$ is the only neighbor of $v$ in the second color class of $f_1$). Note that $H$ contains $P_1 - v_q$. By the maximality of $q$, the vertices $v_{q-1}$ and $w_3$ must be adjacent. But then $v_{q-1}$ belongs to $G'_{13}(f)$, a contradiction to Claim 2.3. This proves the theorem. \(\Box\)
3. Background for the proof of Theorem 1

Proof of Corollary 1. Suppose that \( H \) is a \( k \)-colorable subgraph of \( G \) of the maximum order. Let \( V(G) - V(H) = \{v_1, \ldots, v_t\} \). Let \( G_0 = H \) and \( G_i = G[V(H) \cup \{v_1, \ldots, v_i\}] \) for \( i = 1, \ldots, t \). Let \( f_0 = (M_1, \ldots, M_A) \) be a coloring of \( H \) with \( M_{i+1} = M_{i+2} = \cdots = M_A = \emptyset \). Now, for \( i = 1, \ldots, t \), apply Theorem 2 to \( G_i \) and \( f_{i-1} \) to produce a coloring \( f_i \) of \( G_i \). Then by this theorem, the total number of vertices of colors 1, \ldots, \( k \) in \( G_t = G \) is at least \( |V(H)| \), but it cannot be greater by the maximality of \( H \). This proves the corollary. As an additional feature, we have that none of the sizes of the first \( k \) color classes changed. \( \square \)

Corollary 2. Let \( \Delta \geq 3 \) and \( G = (V, E) \) be a \( \Delta \)-free K\( \Delta \)+1-free graph with \( \Delta(G) \leq \Delta \). Let \( |V| = n = k(\Delta + 1) + r \), where \( 0 \leq r \leq \Delta \). Then \( G \) has a \( \Delta \)-coloring \( f \) with color classes \( M_1, M_2, \ldots, M_A \) such that

1. \( |M_i| \geq k \) for every \( i \);
2. for every set \( Z \) of color classes, \( |\bigcup_{M \in Z} M| \leq k + |Z||n/(\Delta + 1)| \); in particular, \( |M_i| \leq k + |n/(\Delta + 1)| \) for every \( i \);
3. if \( |M_i| = k + 1 + p \) for some \( p \geq 1 \), then the degree of every \( v \in M_i \) in \( G \) is at least \( \Delta - (k + r - 1)/p \).

Proof. By the Hajnal–Szemerédi Theorem, \( G \) has an equitable \( (\Delta + 1) \)-coloring \( f' \). Under the conditions of the corollary, exactly \( \Delta \) color classes of \( f' \) have size \( k + 1 \). Let \( M' \) be a color class of \( f' \) with \( |M'| = k \). Adding the vertices of \( M' \) one by one to \( G - M' \) and applying Theorem 2 on every step, we get a \( \Delta \)-coloring \( f'' \) of \( G \) satisfying (i) and (ii).

Now, consider the following procedure: If a vertex \( v \) in a color class \( M_z \) of size \( z \) has no neighbors in a color class \( M_j \) of size at most \( z - 2 \), then move \( v \) from \( M_z \) to \( M_j \). Clearly, we will stop after a finite number of steps. We claim that the final \( \Delta \)-coloring \( f \) is what we need. Indeed, once a coloring satisfies (i) and (ii), such moves do not destroy these properties. Since we have stopped our procedure, if for some \( i \), we have \( |M_i| = k + 1 + p \) and \( v \in M_i \), then \( v \) has neighbors in every color class of size at most \( k + p - 1 \). By (i), the number of color classes of size at least \( k + p \) (including \( M_z \)) is at most \( (n - k - k - 1)/p = (k + r - 1)/p \) (here \( -1 \) arises, because \( |M_i| = k + 1 + p \)). It follows that \( v \) is adjacent to vertices in at least \( \Delta - (k + r - 1)/p \) color classes. \( \square \)

3. Background for the proof of Theorem 1

In this section, we do preparatory work for the proof of Theorem 1: introduce some notions and prove Theorem 1 for \( n \leq 8.8\Delta \).

Let \( G = (V, E) \) be a graph with maximum degree \( \Delta \) and \( f \) be a vertex coloring of \( G \) with color classes \( M_1, \ldots, M_A \) (some color classes can be empty). For a set \( Y_0 \) of color classes of \( f \) and a subset \( V' \) of \( V \), we define the \((V', f)\)-expansion of \( Y_0 \) in \( G \) as follows.

Say that a vertex \( w \in V' - \bigcup_{M \in Y_0} M \) is a \( Y_0 \)-candidate if \( w \) has no neighbors in some color class \( M(w) \in Y_0 \). Let \( Y_1 \) be the set of color classes of \( f \) containing a \( Y_0 \)-candidate. Similarly, for \( h \geq 1 \), a vertex \( w \in V' - \bigcup_{M \in Y_0 \cup \cdots \cup Y_h} M \) is a \( Y_h \)-candidate if \( w \) has no neighbors in some color class \( M(w) \in Y_h \). Let \( Y_{h+1} \) be the set of color classes containing a \( Y_h \)-candidate. Finally, the set \( Y = \bigcup_{j=0}^{\infty} Y_j \) will be called the \((V', f)\)-expansion of \( Y_0 \) in \( G \).

By the construction, we have the following.

Claim 3.1. If \( Y \) is the \((V', f)\)-expansion of \( Y_0 \) in \( G \), then every vertex \( u \in V' - \bigcup_{M \in Y} M \) has a neighbor in every \( M \in Y \).

For each \( M \in Y - Y_0 \), we define an \((M, Y)\)-recoloring as follows. Suppose that \( M \in Y_{h+1} \) for some \( h \geq 0 \). By the definition of \( Y_{h+1} \), \( M \) contains a \( Y_h \)-candidate \( x_{h+1} \in V' \). Furthermore, for \( j = h, h - 1, \ldots, 1 \), the color class \( M_j = M(x_{j+1}) \) contains a \( Y_{j-1} \)-candidate \( x_j \in V' \). Then an \((M, Y)\)-recoloring of \( f \) is the coloring \( f' \) that differs from \( f \) only at \( x_1, \ldots, x_{h+1} \): for every \( x_j, j = h + 1, h, \ldots, 1 \), we let \( f'(x_j) = M_{j-1} = M(x_j) \). For different choices of \( x_1, \ldots, x_{h+1} \), we get different recolorings, but the following claim holds by the definition.

Claim 3.2. If \( h \geq 0 \) and \( M \in Y_{h+1} \) for some \((V', f)\)-expansion \( Y \) of \( Y_0 \) in \( G \) and \( f' \) is an \((M, Y)\)-recoloring of \( f \), then

(a) the sizes of almost all color classes in \( f \) and \( f' \) are the same, only the size of \( M \) decreases by one and the size of one color class in \( Y_0 \) increases by one;
(b) the colors of vertices in \( V(G) - V' \) are the same in \( f \) and \( f' \). \( \square \)
Lemma 1. Let $\Delta \geq 3$ and $G$ be a $K_{\Delta+1}$-free $n$-vertex graph with maximum degree $\Delta$ and
\[
|E(G)| \leq n\Delta/10. \tag{1}
\]
If $n/\Delta \leq 8.8$, then $G$ has an equitable $\Delta$-coloring.

Proof. Let $G$ be an inclusion minimal counterexample to the lemma, and $e$ be an edge adjacent to a vertex $v$ of the lowest positive degree $x$. If $x > 0.4\Delta$, then by (1) and the choice of $v$, the cardinality of the set $V'$ of non-isolated vertices in $G$ is at most $2|E(G)|/x < 2 \cdot 0.1n\Delta/0.4\Delta = n/2$. Applying Corollary 2 to $G[V']$, we obtain a $\Delta$-coloring of $G[V']$ in which every color class has at most $\lceil n/\Delta \rceil$ vertices and each but one color class has less than $\lceil n/\Delta \rceil$ vertices. Now we can add the remaining isolated vertices to these color classes in order to get the size of every class equal to $\lceil n/\Delta \rceil$ or $\lfloor n/\Delta \rfloor$. This contradicts the choice of $G$. Therefore,
\[
\deg(v) \leq 0.4\Delta. \tag{2}
\]

By the minimality of $G$, $G - e$ has an equitable $\Delta$-coloring $f$. We may assume that the color classes of $f$ are $M_1, \ldots, M_{\Delta}$ and that $v \in M_\Delta$. By the definition, the size of every $M_i$ is either $t$ or $t - 1$, where $t = \lceil n/\Delta \rceil$.

Let $M_0 = M_\Delta - v$. Then $f_0 = (M_0, \ldots, M_{\Delta-1})$ is a proper coloring of $G_0 = G - v$. Let $Y_0$ denote the set of color classes in $f_0$ of size $|M_0|$. If some $M_j \in Y_0$ contains no neighbors of $v$, then $|Y_0| > 1$ and hence $|M_0| = t - 1$. In this case, we color $v$ with $M_j$ and get an equitable $\Delta$-coloring of $G$, a contradiction. Thus, every $M_j \in Y_0$ contains a neighbor of $v$. Let $Y = \bigcup_{j=0}^\infty Y_j$ be the $(V(G_0), f_0)$-expansion of $Y_0$ in $G_0$ (defined at the beginning of the section) and $y = |Y|$.

Suppose that for some $h \geq 0$, a color class $M_{h+1}$ in $Y_{h+1}$ does not contain a neighbor of $v$. Let $f'$ be an $(M, Y)$-recoloring of $f_0$. By Claim 3.2, if we additionally color $v$ with $M_{h+1}$, then we obtain an equitable $\Delta$-coloring of $G$, a contradiction. Thus, every color class in $Y$ contains a neighbor of $v$ and therefore $y \leq \deg_G(v)$.

Let $V^+ = V(G) - \bigcup_{M \in \mathcal{Y}} M$.

Case 1: $|M_0| = t - 1$. By the definition of $Y_0$, every color class outside of $Y$ has size $t$. Therefore, $|V^+| = t(\Delta - y) \geq n(\Delta - y)/\Delta$. By (1) and Claim 3.1, we have
\[
\frac{n\Delta}{10} \geq |E(G)| \geq y(\Delta - y) \frac{n}{\Delta}. \tag{3}
\]
For $\lambda = y/\Delta$, (3) gives $\lambda^2 - \lambda + 0.1 \geq 0$. It follows that either $\lambda > 0.88$ or $\lambda < 0.12$. The former contradicts (2), so $y < 0.12\Delta$. Since every vertex in $V^+$ is adjacent to $M_0$, we get $\Delta(t - 1) \geq t(\Delta - y)$, i.e., $ty \geq \Delta$. This yields $t > \frac{1}{0.12} > 8$, which means $n > 8\Delta$ and $t \geq 9$. Furthermore, since every color class outside of $Y$ has size $t$ and $y < 0.12\Delta$, we get
\[
n \geq t(\Delta - y) + (t - 1)y \geq 9\Delta - y \geq 8.88\Delta.
\]

Case 2: $|M_0| = t - 2$. Recall that no other color class has size less than $t - 1$. Suppose that some $M \in Y$, say, $M \in Y_h$ has $t$ vertices. Then any $(M, Y)$-recoloring $f'$ of $f_0$ satisfies the conditions of Case 1 with $M_h - x_h$ in place of $M_0$. Since Case 1 is proved, we can assume that every $M \in Y$ has at most $t - 1$ vertices. It follows that the average size of color classes outside of $Y$ is higher than in $Y$, and therefore higher than $n/\Delta$. Thus, (3) holds, and we get $y < 0.12\Delta$ exactly as in Case 1. Similarly to Case 1, we obtain $\Delta(t - 2) \geq (t - 1)(\Delta - y)$, i.e., $(t - 1)y \geq \Delta$. It follows that $t - 1 > \frac{1}{0.12} > 8$. Since $t - 1$ is an integer, we have $n/\Delta > t - 1 \geq 9$. \hfill \Box

4. The bounded-size version of Theorem 1

An $l$-bounded coloring of a graph $G$ is a proper vertex coloring of $G$ in which the size of each color class is at most $l$. Clearly, every equitable $k$-coloring of an $n$-vertex graph is $\lceil n/k \rceil$-bounded, but not every $\lceil n/k \rceil$-bounded $k$-coloring of an $n$-vertex graph is equitable. Thus, the theorem below is a weaker statement than Theorem 1.

Theorem 3. Let $\Delta, n \geq 46$. Suppose that an $n$-vertex graph $G = (V, E)$ has maximum degree at most $\Delta$ and $|E| \leq \Delta n/10$. If $G$ does not contain $K_{\Delta+1}$, then $G$ has an $\lceil n/\Delta \rceil$-bounded coloring with $\Delta$ colors.
**Proof.** Let $G$ be a $K_{A+1}$-free graph with maximum degree $\Delta \geq 46$ and average degree at most $\Delta/5$. Let $t = \lceil n/\Delta \rceil$. Let $V_i = \{v_1, \ldots, v_l\}$ be the set of vertices of degree at least $4\Delta/5$. For $i = l + 1, \ldots, n$, consider the following procedure:

1. Let $d_i$ be the maximum degree in $G - V_{i-1}$.
2. If $d_i \geq 2\Delta/5$, then let $v_i$ be a vertex of degree $d_i$ in $G - V_{i-1}$.
3. If $d_i < 2\Delta/5$, then let $v_i$ be a vertex of maximum degree in $G$ among vertices in $G - V_{i-1}$.
4. Let $V_i = V_{i-1} \cup \{v_i\}$ and $G_i = G[V_i]$. Note that, in general, the resulting ordering is not unique.

**Claim 4.1.** Let $m$ be the largest index $i$ such that $d_i \geq 2\Delta/5$. Then $m \leq \lceil n/4 \rceil$.

**Proof.** By the definition, every $v_i$ for $l + 1 \leq i \leq m$ has at least $2\Delta/5$ adjacent vertices $v_j$ with $j > i$. Therefore, $G$ contains at least $(m - l)2\Delta/5$ edges not incident with $v_1, \ldots, v_l$. Thus, if $\Delta/4 < m$, then

$$|E(G)| \geq l \cdot \frac{4\Delta}{5} \cdot \frac{2\Delta}{5} + (m - l) \cdot \frac{2\Delta}{5} \geq \frac{n2\Delta}{4} = \frac{\Delta n}{10},$$

a contradiction. \[\square\]

Suppose that $m = k(\Delta + 1) + r$ with $0 \leq r \leq \Delta$. Then $G_m$ has a proper $\Delta$-coloring $f_m = (M_1^m, \ldots, M_d^m)$ satisfying Corollary 2. Note that by Claim 4.1, $k + \lceil m/\Delta + 1 \rceil = \lceil m/\Delta + 1 \rceil + \lceil m/\Delta + 1 \rceil < t$. Therefore, $f_m$ is a $t$-bounded coloring of $G_m$.

We will now complete $f_m$ to a $t$-bounded $\Delta$-coloring of $G$ by constructing consecutively colorings $f_i$ of $G_i$ for $i = 1 + m, 2 + m, \ldots, n$, in such a way that

$$f_i(v) = f_m(v) \quad \text{for every } v \in V_m.$$  \hspace{1cm} (4)

Observe that $f_m$ satisfies (4). Now, suppose that $m + 1 \leq i \leq n$ and $G_{i-1}$ has a $t$-bounded coloring $f_{i-1}$ satisfying (4). We will construct $f_i$ for $G_i$.

Let $M_1, \ldots, M_d$ be the color classes of $f_{i-1}$. Let $Y_0$ denote the set of color classes of cardinality less than $t$. If some $M_j \in Y_0$ contains no neighbors of $v_i$, then we color $v_i$ with $M_j$ and have a $t$-bounded coloring $f_i$ satisfying (4). Otherwise, let $Y = \bigcup_{j=0}^\infty Y_j$ be the $(V_{i-1} - V_m, f_{i-1})$-expansion of $Y_0$ in $G_{i-1}$ (defined in Section 3) and $y = |Y|$.

If for some $h \geq 0$, a color class $M_{h+1}$ in $Y_{h+1}$ does not contain a neighbor of $v$, then consider an $(M, Y)$-recoloring $f'$ of $f_{i-1}$. By Claim 3.2(a), if we additionally color $v_i$ with $M_{h+1}$, then we obtain a $t$-bounded $\Delta$-coloring of $G_i$. Moreover, by Claim 3.2(b), this new coloring also satisfies (4), as required. Thus, we may assume that every color class in $Y$ contains a neighbor of $v$. This together with the definition of $Y_0$ and Claim 3.1 yields the following.

**Claim 4.2.** If $G_i$ has no $t$-bounded coloring $f_i$ satisfying (4), then the $(V_{i-1} - V_m, f_{i-1})$-expansion $Y$ of $Y_0$ in $G_{i-1}$ possesses the following properties:

(a) every color class in $Y$ contains a neighbor of $v_i$ and thus $y \leq \deg_G(v_i)$,
(b) every vertex $u \in V_{i-1} - V_m - \bigcup_{M \in Y} M$ has a neighbor in every $M \in Y$,
(c) every color class outside of $Y$ has $t$ vertices.

Let

$$V^- = V_m - \bigcup_{M \in Y} M$$

and

$$V^+ = V_{i-1} - V_m - \bigcup_{M \in Y} M = V_{i-1} - \bigcup_{M \in Y} M - V^-.$$

**Claim 4.3.** Using the notation above, $|V^-| \leq 3n(\Delta - y)/(8\Delta)$. 

**Proof.** Recall that \( m = k(A + 1) + r \) with \( 0 \leq r \leq A \). By Corollary 2(ii),

\[
|V^−| \leq k + \left\lceil \frac{m}{A + 1} \right\rceil (A - y) \leq \frac{n}{4A} + \left\lceil \frac{n}{4A} \right\rceil (A - y).
\]

Since \( A \geq 46 \) and \( i > l \), we have \( A - y \geq A/5 > 9 \). By Lemma 1, \( n/(4A) \geq 2.2 \) and therefore, \( |V^−| \leq \frac{1}{2.2} \cdot n/(4A) \).

It follows that

\[
|V^−| \leq (A - y) \left( \frac{n}{4A(A - y)} + \frac{3}{2.2} \cdot \frac{n}{4A} \right) \leq \frac{n}{4A} (A - y) \left( \frac{1}{10} + \frac{3}{2.2} \right) < \frac{3n}{8A} (A - y).
\]

This proves the claim. □

**Claim 4.4.** The size \( y \) of \( Y \) is less than \( 0.15A \).

**Proof.** By Claim 4.2(b), at least \( y|V^+| \) edges connect \( V^+ \) with \( \bigcup_{M \in Y} M \). Recall that every \( v_q \) for \( l + 1 \leq q \leq m \) has at least \( 2A/5 \) adjacent vertices \( v_j \) with \( j > q \). Thus, at least \( 0.4A/|V^−| \) edges of \( G \) are incident with \( V^− \) and hence

\[
|E(G)| \geq y|V^+| + \frac{y}{2} |V^−|.
\]

Since \( |V^−| - \bigcup_{M \in Y} M| \geq t(A - y) \geq n/(A - y) \), we have

\[
|E(G)| \geq y \left( (A - y) \frac{n}{A} - |V^−| \right) + \frac{y}{2} |V^−| \geq y \left( (A - y) \frac{n}{A} - \frac{y}{2} |V^−| \right).
\]

This and Claim 4.3 yield

\[
\frac{nA}{10} \geq y \left( \frac{n}{A} (A - y) - (A - y) \frac{3n}{16A} \right) = \frac{13}{16} yn \left( 1 - \frac{y}{A} \right).
\]

Denoting \( \lambda = y/A \) and dividing both parts by \( nA \), we obtain \( \frac{1}{10} \geq \frac{13}{16} \lambda (1 - \lambda) \). Solving this inequality, we get \( \lambda > 0.85 \) or \( \lambda < 0.15 \). Since \( i > m \geq l \), we have \( y \leq \deg(v_i) < 0.8A \). We conclude that \( \lambda = y/A < 0.15 \). □

**Claim 4.5.** The size \( y \) of \( V^+ \) is greater than \( \frac{2n}{3} \).

**Proof.** Assume that \( |V^+| \leq \frac{2n}{3} \). As in the proof of Claim 4.4, at least \( y|V^+| \) edges connect \( V^+ \) with \( \bigcup_{M \in Y} M \) and at least \( 0.4A/|V^−| \) edges are incident with \( V^− \). Hence

\[
|E(G)| \geq y|V^+| + 0.4A \left( (A - y) \frac{n}{A} - |V^+| \right) \geq |V^+|(y - 0.4A) + 0.4 \frac{n}{A} (A - y).
\]

Recall that \( y < 0.15A \) by Claim 4.4 and \( |V^+| \leq \frac{2n}{3} \). Thus the last inequality yields

\[
\frac{nA}{10} > \frac{2n}{3} (y - 0.4A) + 0.4n(A - y).
\]

Dividing both parts by \( nA \), we obtain

\[
\frac{1}{10} > \frac{2}{3} \frac{y}{A} - \frac{8}{30} + \frac{2}{5} \frac{2y}{5A},
\]

which is false for \( y/A \geq 0 \). This contradiction proves the claim. □

**Claim 4.6.** Let \( M_1 \) be a color class of the smallest size in \( Y \). Then

\[
|M_1| \leq \frac{n + y}{A} - 1. \tag{5}
\]

**Proof.** Since \( v_i \) is not colored, \( |M_1| < n/A \). If \( |M_1| < t - 2 \), the conclusion is obvious. Suppose that \( |M_1| = t - 1 \). Since every color class not in \( Y \) has size \( t \), by the minimality of \( |M_1|, n + y \geq tA \). This proves the claim. □
Now we are ready to finish the proof. Define \( z = 4n/(9A) + \frac{17}{30} \).

Case 1: \(|M_1 \cap V_m| \leq z \). Then the number of neighbors of \( M_1 \) is at most \( zA + (|M_1| - z)(2A/5) = z(3A/5) + |M_1|2A/5 \). By Claims 4.6 and 4.4, this is less than

\[
\left( \frac{4n}{9A} + \frac{17}{30} \right) \frac{3A}{5} + \left( \frac{n}{A} - 0.85 \right) \cdot \frac{2A}{5} = \frac{2n}{3}.
\]

Since every vertex in \( V^+ \) is a neighbor of \( M_1 \), (6) contradicts Claim 4.5.

Case 2: \(|M_1 \cap V_m| = z + x \), where \( x > 0 \). We will prove that \( M_1 \) has at most \( z(3A/5) + |M_1|2A/5 \) neighbors in \( V^+ \), which by (6) would give the same contradiction as in Case 1.

By Corollary 2(iii), every vertex of \( M_1 \cap V_m \) has at most \((k + r - 1)/(z + x - k - 1)\) neighbors in \( V^+ \). Thus, it is enough to prove that

\[
\frac{k + r - 1}{z + x - k - 1} (z + x) \leq zA + x \cdot \frac{2A}{5},
\]

since the RHS of (7) is the maximum amount contributed by \( z + x \) vertices to \( zA + (|M_1| - z)(2A/5) \) in Case 1. Note that (7) is equivalent to

\[
\left( zA + x \cdot \frac{2A}{5} \right) (z + x - k - 1) - (k + r - 1)(z + x) \geq 0.
\]

For \( x = 0 \), (8) becomes

\[
zA(z + k - 1) - z(k + r - 1) \geq 0,
\]

which reduces to

\[
z \geq \frac{k + r - 1}{A} + k + 1.
\]

Recall that \( m = k(A + 1) + r \leq [n/4] \), i.e., \( n \geq 4(kA + k + r) \), and that \( z = 4n/9A + \frac{17}{30} \). It follows that

\[
z \geq \frac{16k}{9} + \frac{16k + r}{9} - \frac{17}{30}.
\]

which yields (9) for \( k \geq 1 \). This proves (8) for \( x = 0 \).

Now consider the LHS of (8) as a function \( g(x) \). Then

\[
g'(x) = \frac{2A}{5} (z + x - k - 1) + \left( zA + x \cdot \frac{2A}{5} \right) - (k + r - 1) = \frac{4A}{5} x + \frac{7A}{5} z - \frac{2A}{5} (k + 1) - (k + r - 1).
\]

We want to show that \( g'(x) \geq 0 \) for every \( x > 0 \). This would prove (8) and thus the theorem. By the last equality, the condition \( g'(x) \geq 0 \) is equivalent to

\[
\frac{7}{5} z \geq \frac{2}{5} (k + 1) - \frac{k + r - 1}{A}.
\]

This inequality is implied by (10) for \( k \geq 1 \).

5. Proof of Theorem 1

The algorithm in the previous section produces a \( t \)-bounded \( A \)-coloring of \( G \), but this coloring might have ‘small’ color classes. In order to ‘correct’ the coloring, we use a slight variation of the technique used above.

Consider \( t \)-bounded colorings of \( G \) obtained in the course of proof of Theorem 3. In particular, each vertex \( v \in V(G) - V_m \) has at most \( 0.8A \) neighbors in \( G \) and at most \( 0.4A \) neighbors in \( G - V_m \). Among such colorings with a fixed coloring \( f_m \) of \( G[V_m] \) satisfying Corollary 2, choose a coloring \( f_0 \) with fewest color classes of size \( t \). We will prove that \( f_0 \) has no color classes of size \( t - 2 \) or less.
Let $Y_0$ be the set of color classes of size at most $t - 2$ and assume that $Y_0$ is non-empty. Let $Y = \bigcup_{j=0}^{\infty} Y_j$ be the $(V(G) - V_m, f_0)$-expansion of $Y_0$ in $G$ and $y = |Y|$.

Claim 5.1. $Y$ possesses the following properties:
(a) every vertex $u \in V(G) - V_m - \bigcup_{M \in Y} M$ has a neighbor in every $M \in Y$,
(b) every color class in $Y$ has at most $t - 1$ vertices,
(c) every color class outside of $Y$ has at least $t - 1$ vertices.

Proof. Claim 3.1 implies (a), and the definition of $Y_0$ yields (c). To prove (b), assume by contradiction that for some $h \geq 0$, a color class $M_{h+1}$ in $Y_{h+1}$ has cardinality $t$. Consider an $(M, Y)$-recoloring $f'$ of $f_0$. By Claim 3.2(a), $f'$ is a $t$-bounded $A$-coloring of $G$ with fewer color classes of size $t$. Moreover, by Claim 3.2(b), $f'$ satisfies (4). This contradicts the choice of $f_0$. □

Since there is a color class $M'$ of size $t$, $y < A$. Since every vertex in $M' - V_m$ has neighbors in each color class of $Y$, $y \leq 0.8A$. (11)

Let

$$V^- = V_m - \bigcup_{M \in Y} M$$

and

$$V^+ = V - V_m - \bigcup_{M \in Y} M = V - \bigcup_{M \in Y} M - V^-.$$

Now, Claim 4.3 holds: the proof simply repeats that in the previous section. By Claim 5.1(b) and (c),

$$\left| V(G) - \bigcup_{M \in Y} M \right| \geq (A - y) \frac{n}{A}. \quad (12)$$

Thus we can essentially repeat the proofs of Claims 4.4 and 4.5, and conclude that they hold for our $Y$.

Let $M_1 \in Y_0$. By the definition of $Y_0$, $|M_1| \leq t - 2 < n/A - 1$, which is stronger than Claim 4.6. Therefore, all the calculations in the previous section following Claim 4.6 go through and we get a contradiction to our assumption.

References