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Double Schubert polynomials for the classical groups

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Abstract

For each infinite series of the classical Lie groups of type B, C or D, we construct a family of polynomials parametrized by the elements of the corresponding Weyl group of infinite rank. These polynomials represent the Schubert classes in the equivariant cohomology of the appropriate flag variety. They satisfy a stability property, and are a natural extension of the (single) Schubert polynomials of Billey and Haiman, which represent non-equivariant Schubert classes. They are also positive in a certain sense, and when indexed by maximal Grassmannian elements, or by the longest element in a finite Weyl group, these polynomials can be expressed in terms of the factorial analogues of Schur's Q- or P-functions defined earlier by Ivanov.

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1. Introduction

A classical result of Borel states that the (rational) cohomology ring of the flag variety of any simple complex Lie group G is isomorphic, as a graded ring, to the *coinvariant* algebra of the corresponding Weyl group W, i.e. to the quotient of a polynomial ring modulo the ideal generated by the W-invariant polynomials of positive degree. There is a distinguished additive basis of the cohomology ring, given by Schubert classes, and indexed by the elements in the Weyl group. Bernstein, Gelfand and Gelfand [3] (see also Demazure [9]) showed that if one starts with a polynomial that represents the cohomology class of highest codimension (the Schubert class of a point), one obtains all the other Schubert classes by applying a succession of *divided difference* operators corresponding to simple roots. This construction depends on the choice of a polynomial representative for the "top" cohomology class. For $SL(n,\mathbb{C})$, Lascoux and Schützenberger [32] considered one particular choice, which yielded polynomials – the Schubert polynomials – with particularly good combinatorial and geometric properties.

It is a natural problem to extend the construction in [32] to the other classical Lie groups. To this end, Fomin and Kirillov [11] listed up five properties that characterize the Schubert polynomials in type A, but they showed that it is impossible to construct a theory of "Schubert polynomials" in type B satisfying the same properties. For type B_n, they constructed several families of polynomials which satisfy all but one of these properties.

There is another approach to this problem due to Billey and Haiman [6]. If G_n is a Lie group of classical type $\mathrm{B}_n, \mathrm{C}_n$, or D_n, there is an embedding $G_n \hookrightarrow G_{n+1}$ which induces an embedding of the corresponding flag varieties $\mathcal{F}_n = G_n/B_n \hookrightarrow \mathcal{F}_{n+1}$; here B_n denotes a Borel subgroup of G_n. This embedding determines reverse maps in cohomology $H^*(\mathcal{F}_{n+1},\mathbb{Q}) \to H^*(\mathcal{F}_n,\mathbb{Q})$, compatible with the Schubert classes $\sigma_w^{(n)}$, where w belongs to W_n, the Weyl group of G_n. Therefore there is a well-defined *stable* Schubert class $\sigma_w^{(\infty)} = \varprojlim \sigma_w^{(n)}$ in the inverse system $\varprojlim H^*(\mathcal{F}_n,\mathbb{Q})$; the stable classes are indexed by w in the infinite Weyl group $W_\infty = \bigcup_{n\geqslant 1} W_n$. A priori, the class $\sigma_w^{(\infty)}$ is represented by a homogeneous element in the ring of power series $\mathbb{Q}[[z_1,z_2,\ldots]]$, but Billey and Haiman showed in [6] that it is represented by a *unique* element $\mathfrak{S}_w$ in the polynomial subring[2]

$$\mathbb{Q}\big[z_1,z_2,\ldots;p_1(z),p_3(z),p_5(z),\ldots\big], \tag{1.1}$$

where $p_k(z)=\sum_{i=1}^{\infty} z_i^k$ denotes the power-sum symmetric function. It is known that the images of the even power-sums $p_{2i}(z)$ in the limit of the coinvariant rings vanish for each of the types B, C and D. The polynomials $\{\mathfrak{S}_w\}$ are *canonical*: they form the unique solution of a system of equations involving infinitely many BGG divided difference operators. These polynomials will satisfy the main combinatorial properties of the type A Schubert polynomials, if interpreted appropriately. In particular, the polynomial $\mathfrak{S}_w$ is stable, i.e. it represents the Schubert classes $\sigma_w^{(n)}$ in $H^*(\mathcal{F}_n,\mathbb{Q})$ simultaneously for all positive integers n.

The flag varieties admit an action of the maximal torus T_n, and the inclusion $\mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}$ is equivariant with respect to $T_n \hookrightarrow T_{n+1}$. Therefore one can define an *equivariant* version of the stable Schubert classes, and one can ask whether we can 'lift' – in a canonical way – the polynomials of Billey and Haiman to the equivariant setting. The major advantage in the equivariant

[2] The elements $\{z_1,z_2,\ldots;p_1(z),p_3(z),p_5(z),\ldots\}$ are algebraically independent, so the ring (1.1) can also be regarded simply as the polynomial ring in z_i and p_k $(k=1,3,5,\ldots)$.

setting is that the equivariant classes are much more rigid, as they are specified by localization conditions. However, the naive generalization of the Billey–Haiman results does not work, as the BGG divided difference operators do not suffice. Fortunately – in equivariant cohomology only – there exists another set of operators, defined by Knutson in [25] (see also [41]). The analogous system formed by the two sets of operators has the *double Schubert polynomials* as unique solution. The terminology comes from type A, since these polynomials coincide with those defined by Lascoux in [31] (see also [33]).

As shown by Fulton in [12], the type A double Schubert polynomials can also be constructed as polynomials which represent the cohomology classes of some degeneracy loci. More recently, two related constructions connecting double Schubert polynomials to equivariant cohomology of flag manifolds, using either Thom polynomials or Gröbner degenerations were obtained independently by Fehér and Rimányi [10] and by Knutson and Miller [26]. The degeneracy locus construction was extended to other types by Fulton [13], Pragacz and Ratajski [39] and Kresch and Tamvakis [29]. The resulting polynomials are expressed in terms of Chern classes canonically associated to the geometric situation at hand. Their construction depends again on the choice of a polynomial to represent the "top class" – the diagonal class in the cohomology of a flag bundle. Unfortunately different choices lead to polynomials having some desirable combinatorial properties – but not all. In particular, the polynomials in [29,13] do not satisfy the stability property, and the precise combinatorial relationship to the double Schubert polynomials in this paper is unclear.

Rather than pursuing the degeneracy loci approach – which uses cohomology classes, defined only up to the ideal of relations – we will use localization techniques, which yield (canonically defined) *polynomials*. We study the basic properties of the double Schubert polynomials, and we find a formula for the "top class" class of the point, surprisingly compact. A brief, and more precise, description of the results, is given in the next subsection.

1.1. Infinite hyperoctahedral groups

To fix notations, let W_∞ be the infinite hyperoctahedral group, i.e. the Weyl group of type C_∞ (or B_∞). It is generated by elements $s_0, s_1, \ldots$ subject to the braid relations described in (3.1) below. For each non-negative integer n, the subgroup W_n of W_∞ generated by $s_0, \ldots, s_{n-1}$ is the Weyl group of type C_n. W_∞ contains a distinguished subgroup W'_∞ of index 2 – the Weyl group of D_∞ – which is generated by $s_{\hat{1}}, s_1, s_2, \ldots$, where $s_{\hat{1}} = s_0 s_1 s_0$. The corresponding finite subgroup $W'_n = W'_\infty \cap W_n$ is the type D_n Weyl group. To be able to make statements which are uniform across all classical types, we use $\boldsymbol{W}_\infty$ to denote W_∞ when we consider types C or B and W'_∞ for type D; similar notation is used for $\boldsymbol{W}_n \subset \boldsymbol{W}_\infty$. Finally, set $\boldsymbol{I}_\infty$ to be the indexing set $\{0, 1, 2, \ldots\}$ for types B, C and $\{\hat{1}, 1, 2, \ldots\}$ for type D.

1.2. Schubert polynomials via localization

A crucial part in the theory of Schubert polynomials of classical types is played by the Schur P- and Q-functions [40]. These are symmetric functions $P_\lambda(x)$ and $Q_\lambda(x)$ in a new set of variables $x = (x_1, x_2, \ldots)$, and are indexed by strict partitions λ (see Section 4 below for details). The Schur P- or Q-function corresponding to λ with one part of length i is denoted respectively by $P_i(x)$ and $Q_i(x)$. Define $\Gamma = \mathbb{Z}[Q_1(x), Q_2(x), \ldots]$ and $\Gamma' = \mathbb{Z}[P_1(x), P_2(x), \ldots]$. Note that Γ and Γ' are not polynomial rings, since $Q_i(x)$ respectively $P_i(x)$ are not algebraically independent (see Section 4 for the relations among them), but they have canonical $\mathbb{Z}$-bases consisting of the Schur Q- functions $Q_\lambda(x)$ (respectively Schur P-functions $P_\lambda(x)$).[3] We define next the $\mathbb{Z}[t]$-algebras of *Schubert polynomials*

$$R_\infty = \mathbb{Z}[t] \otimes_{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[z], \qquad R'_\infty = \mathbb{Z}[t] \otimes_{\mathbb{Z}} \Gamma' \otimes_{\mathbb{Z}} \mathbb{Z}[z],$$

where $\mathbb{Z}[z] = \mathbb{Z}[z_1, z_2, \ldots]$ is the polynomial ring in $z = (z_1, z_2, \ldots)$. We will justify the terminology in the next paragraph. Again, in order to state results uniformly in all types, we use the bold letter $\boldsymbol{R}_\infty$ to denote R_∞ for type C and R'_∞ for types B and D.

There exists a homomorphism

$$\Phi = (\Phi_v)_{v \in \boldsymbol{W}_\infty} : \boldsymbol{R}_\infty \to \prod_{v \in \boldsymbol{W}_\infty} \mathbb{Z}[t] \subset \varprojlim H^*_{T_n}(\mathcal{F}_n)$$

of graded $\mathbb{Z}[t]$-algebras, where the last inclusion means that $\prod_{v \in \boldsymbol{W}_\infty} \mathbb{Z}[t]$ is identified with a $\mathbb{Z}[t]$-subalgebra of $\varprojlim H^*_{T_n}(\mathcal{F}_n)$ using localization maps; we call this homomorphism the *universal localization map*. Its precise (algebraic) definition is given in Section 6 and it has a natural geometrical interpretation explained in Section 10. One of the main results in this paper is that Φ is an isomorphism from $\boldsymbol{R}_\infty$ onto the $\mathbb{Z}[t]$-span of the stable Schubert classes $\sigma_w^{(\infty)}$ (cf. Theorem 6.3 below). Then the stable Schubert polynomial $\mathfrak{S}_w = \mathfrak{S}_w(z, t; x)$ is defined to be $\Phi^{-1}(\sigma_w^{(\infty)})$.

1.3. Divided difference operators

Alternatively, $\{\mathfrak{S}_w(z, t; x)\}$ can be characterized purely algebraically by using the divided difference operators. There are two families of operators ∂_i, δ_i $(i \in \boldsymbol{I}_\infty)$ on $\boldsymbol{R}_\infty$, such that operators from one family commute with those from the other (see Section 2.5 for the definition). Then:

Theorem 1.1. *There exists a unique family of elements $\mathfrak{S}_w = \mathfrak{S}_w(z, t; x)$ in $\boldsymbol{R}_\infty$, where $w \in \boldsymbol{W}_\infty$, satisfying the equations*

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise}, \end{cases} \qquad \delta_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise}, \end{cases} \tag{1.2}$$

for all $i \in \boldsymbol{I}_\infty$, and such that $\mathfrak{S}_w$ has no constant term except for $\mathfrak{S}_e = 1$.

The operators ∂_i, δ_i are the limits of the same operators on the equivariant cohomology $H^*_{T_n}(\mathcal{F}_n)$, since the latter are compatible with the projections $H^*_{T_{n+1}}(\mathcal{F}_{n+1}) \to H^*_{T_n}(\mathcal{F}_n)$. In this context, the operator ∂_i is an equivariant generalization of the operator defined in [3,9], and it can be shown that it is induced by the right action of the Weyl group on the equivariant cohomology (cf. [28,25]). The operator δ_i exists only in equivariant cohomology, and it was used in [25,41] to study equivariant Schubert classes. It turns out that it corresponds to a left Weyl group action on $H^*_{T_n}(\mathcal{F}_n)$. A different approach to define both these operators can also be found in the unpublished notes [37].

[3] However, Γ and Γ' are polynomial rings when tensored with rational numbers: $\Gamma \otimes \mathbb{Q} = \mathbb{Q}[Q_1(x), Q_3(x), \ldots]$ and $\Gamma' \otimes \mathbb{Q} = \mathbb{Q}[P_1(x), P_3(x), \ldots]$ – see e.g. [34, III.8]. Since we will be interested in integral cohomology classes, we will not use these facts later on in this paper.

1.4. Combinatorial properties of the double Schubert polynomials

We state next the combinatorial properties of the double Schubert polynomials $\mathfrak{S}_w(z,t;x)$:

- (Basis) $\mathfrak{S}_w(z,t;x)$ form a $\mathbb{Z}[t]$-basis of $\boldsymbol{R}_\infty$.
- (Symmetry) $\mathfrak{S}_w(z,t;x) = \mathfrak{S}_{w^{-1}}(-t,-z;x)$.
- (Positivity) The double Schubert polynomial $\mathfrak{S}_w(z,t;x)$ can be uniquely written as

$$\mathfrak{S}_w(z,t;x) = \sum_\lambda f_\lambda(z,t) F_\lambda(x),$$

  where the sum is over strict partitions $\lambda = (\lambda_1,\dots,\lambda_r)$ such that $\lambda_1 + \cdots + \lambda_r \leqslant \ell(w)$, $f_\lambda(z,t)$ is a homogeneous polynomial in $\mathbb{N}[z,-t]$, and $F_\lambda(x)$ is the Schur Q-function $Q_\lambda(x)$ in type C, respectively the Schur P function $P_\lambda(x)$ in types B, D. For a precise combinatorial formula for the coefficients $f_\lambda(z,t)$ see Corollary 8.10 and Lemma 8.11 below.
- (Restriction) When all $t_i = 0$, the polynomial $\mathfrak{S}_w(z,0;x)$ equals the Billey–Haiman polynomial $\mathfrak{S}_w(z;x)$.

The basis property implies that we can define the structure constants $c_{uv}^w(t) \in \mathbb{Z}[t]$ by the expansion

$$\mathfrak{S}_u(z,t;x)\mathfrak{S}_v(z,t;x) = \sum_{w \in \boldsymbol{W}_\infty} c_{uv}^w(t) \mathfrak{S}_w(z,t;x).$$

These coincide with the structure constants in equivariant cohomology of $\mathcal{F}_n$, written in a stable form. The same phenomenon happens in [32,6].

1.5. Grassmannian Schubert classes

To each strict partition λ one can associate a *Grassmannian element* $w_\lambda \in \boldsymbol{W}_\infty$. Geometrically these arise as the elements in $\boldsymbol{W}_\infty$ which index the pull-backs of the Schubert classes from the appropriate Lagrangian or Orthogonal Grassmannian, via the natural projection from the flag variety. For the Lagrangian Grassmannian, the first author [16] identified the equivariant Schubert classes with the factorial analogues of Schur Q-function defined by Ivanov [20]. This result was extended to the maximal isotropic Grassmannians of orthogonal types B and D by Ikeda and Naruse [18]. See Section 4 for the definition of Ivanov's functions $Q_\lambda(x|t)$, $P_\lambda(x|t)$. We only mention here that if all $t_i = 0$ they coincide with the ordinary Q or P functions; in that case, these results recover Pragacz's results from [38] (see also [21]). We will show in Theorem 6.6 that the polynomial $\mathfrak{S}_{w_\lambda}(z,t;x)$ coincides with $Q_\lambda(x|t)$ or $P_\lambda(x|t)$, depending on the type at hand. In particular, the double Schubert polynomials for the Grassmannian elements are Pfaffians – this is a *Giambelli formula* in this case.

1.6. Longest element formulas

Perhaps the most unexpected result of this paper is a very compact combinatorial formula for the double Schubert polynomial indexed by $w_0^{(n)}$, the longest element in $\boldsymbol{W}_n$ (regarded as a subgroup of $\boldsymbol{W}_\infty$). This formula has a particular significance since this is the top class mentioned in the first section, and all the other polynomials can be obtained from it, using the appropriate sequence of divided difference operators. We denote by $\mathfrak{B}_w, \mathfrak{C}_w, \mathfrak{D}_w$ the double Schubert polynomial $\mathfrak{S}_w$ for types B, C, and D respectively. Note that $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$, where $s(w)$ is the number of signs changed by w (cf. Section 3.3 below).

Theorem 1.2 *(Top classes). The double Schubert polynomial associated with the longest element $w_0^{(n)}$ in $\boldsymbol{W}_n$ is equal to*:

(1) $\mathfrak{C}_{w_0^{(n)}}(z,t;x) = Q_{\rho_n+\rho_{n-1}}(x|t_1,-z_1,t_2,-z_2,\ldots,t_{n-1},-z_{n-1})$,
(2) $\mathfrak{D}_{w_0^{(n)}}(z,t;x) = P_{2\rho_{n-1}}(x|t_1,-z_1,t_2,-z_2,\ldots,t_{n-1},-z_{n-1})$,

where $\rho_k = (k,k-1,\ldots,1)$.

1.7. Comparison with degeneracy loci formulas

One motivation for the present paper was to give a geometric interpretation to the factorial Schur Q-function by means of degeneracy loci formulas. In type A, this problem was treated by the second author in [35], where the Kempf–Laksov formula for degeneracy loci is identified with the Jacobi–Trudi type formula for the factorial (ordinary) Schur function. To this end, we will reprove a multi-Pfaffian expression for σ_{w_λ} (see Section 11 below) obtained by Kazarian [23] while studying Lagrangian degeneracy loci.

1.8. Organization

Section 2 is devoted to some general facts about the equivariant cohomology of the flag variety. In Section 3 we fix notation concerning root systems and Weyl groups, while in Section 4 we give the definitions and some properties of Schur Q- and P-functions, and of their factorial analogues. The stable (equivariant) Schubert classes $\{\sigma_w^{(\infty)}\}$ and the ring H_∞ spanned by these classes are introduced in Section 5. In Section 6 we define the ring of Schubert polynomials $\boldsymbol{R}_\infty$ and establish the isomorphism $\Phi : \boldsymbol{R}_\infty \to H_\infty$. In the course of the proof, we recall the previous results on isotropic Grassmannians (Theorem 6.6). In Section 7 we define the left and right action of the infinite Weyl group on ring $\boldsymbol{R}_\infty$, and then use them to define the divided difference operators. We also discuss the compatibility of the actions on both $\boldsymbol{R}_\infty$ and H_∞ under the isomorphism Φ. We will prove the existence and uniqueness theorem for the double Schubert polynomials in Section 8, along with some basic combinatorial properties of them. The formula for the Schubert polynomials indexed by the longest Weyl group element is proved in Section 9. Finally, in Section 10 we give an alternative geometric construction of our universal localization map Φ, and in Section 11, we prove the formula for $Q_\lambda(x|t)$ in terms of a multi-Pfaffian.

1.9. Note

After the present work was completed we were informed that A. Kirillov [24] had introduced double Schubert polynomials of type B (and C) in 1994 by using Yang–Baxter operators (cf. [11]), independently to us, although no connection with (equivariant) cohomology had been established. His approach is quite different from ours, nevertheless the polynomials are the same, after a suitable identification of variables. Details will be given elsewhere.

This is the full paper version of 'extended abstract' [17] for the FPSAC 2008 conference held in Viña del Mar, Chile, June 2008. Some results in this paper were announced without proof in [19].

2. Equivariant Schubert classes of the flag variety

In this section we will recall some basic facts about the equivariant cohomology of the flag variety $\mathcal{F} = G/B$. The main references are [2] and [28] (see also [30]).

2.1. Schubert varieties and equivariant cohomology

Let G be a complex connected semisimple Lie group, T a maximal torus, $W = N_G(T)/T$ its Weyl group, and B a Borel subgroup such that $T \subset B$. The flag variety is the variety $\mathcal{F} = G/B$ of translates of the Borel subgroup B, and it admits a T-action, induced by the left G-action. Each Weyl group element determines a T-fixed point e_w in the flag variety (by taking a representative of w), and these are all the torus-fixed points. Let B^- denote the opposite Borel subgroup. The *Schubert variety* X_w is the closure of $B^- e_w$ in the flag variety; it has codimension $\ell(w)$ – the length of w in the Weyl group W.

In general, if X is a topological space with a left T-action, the equivariant cohomology of X is the ordinary cohomology of a "mixed space" $(X)_T$, whose definition (see e.g. [15] and references therein) we recall. Let $ET \to BT$ be the universal T-bundle. The T-action on X induces an action on the product $ET \times X$ by $t \cdot (e, x) = (et^{-1}, tx)$. The quotient space $(X)_T = (ET \times X)/T$ is the "homotopic quotient" of X and the (T-)equivariant cohomology of X is by definition

$$H_T^i(X) = H^i(X_T).$$

In particular, the equivariant cohomology of a point, denoted by $\mathcal{S}$, is equal to the ordinary cohomology of the classifying space BT. If χ is a character in $\hat{T} = Hom(T, \mathbb{C}^*)$ it determines a line bundle $L_\chi : ET \times_T \mathbb{C}_\chi \to BT$ where $\mathbb{C}_\chi$ is the 1-dimensional T-module determined by χ. It turns out that the morphism $\hat{T} \to H_T^2(pt)$ taking the character χ to the first Chern class $c_1(L_\chi)$ extends to an isomorphism from the symmetric algebra of $\hat{T}$ to $H_T^*(pt)$. Therefore, if one chooses a basis $t_1, \dots, t_n$ for $\hat{T}$, then $\mathcal{S}$ is the polynomial ring $\mathbb{Z}[t_1, \dots, t_n]$.

Returning to the situation when $X = \mathcal{F}$, note that X_w is a T-stable, therefore its fundamental class determines the (*equivariant*) *Schubert class* $\sigma_w = [X_w]_T$ in $H_T^{2\ell(w)}(\mathcal{F})$. It is well known that the Schubert classes form an $H_T^*(pt)$-basis of $H_T^*(\mathcal{F})$.

2.2. Localization map

Denote by $\mathcal{F}^T = \{e_v \mid v \in W\}$ the set of T-fixed points in $\mathcal{F}$; the inclusion $\iota : \mathcal{F}^T \hookrightarrow \mathcal{F}$ is T-equivariant and induces a homomorphism $\iota^* : H_T^*(\mathcal{F}) \to H_T^*(\mathcal{F}^T) = \prod_{v \in W} H_T^*(e_v)$. We identify each $H_T^*(e_v)$ with $\mathcal{S}$ and for $\eta \in H_T^*(\mathcal{F})$ we denote its localization in $H_T^*(e_v)$ by $\eta|_v$. Let R^+ denote the set of positive roots corresponding to B and set $R^- = -R^+$, $R = R^+ \cup R^-$. Each root α in R can be regarded as a linear form in $\mathcal{S}$. Let s_α denote the reflection corresponding to the root α. Remarkably, the localization map ι^* is injective, and the elements $\eta = (\eta|_v)_v$ in $\prod_{v \in W} \mathcal{S}$ in the image of ι^* are characterized by the *GKM conditions* (see e.g. [2]):

$$\eta|_v - \eta|_{s_\alpha v} \quad \text{is a multiple of } \alpha$$

for all v in W and $\alpha \in R^+$, where $s_\alpha \in W$ is the reflection associated to α.

2.3. Schubert classes

We recall a characterization of the Schubert class σ_w. Let $\leqslant$ denote the Bruhat–Chevalley ordering on W; then $e_v \in X_w$ if and only if $w \leqslant v$.

Proposition 2.1. *(See [2,28].) The Schubert class σ_w is characterized by the following conditions*:

(1) *$\sigma_w|_v$ vanishes unless $w \leqslant v$,*
(2) *if $w \leqslant v$ then $\sigma_w|_v$ is homogeneous of degree $\ell(w)$,*
(3) *$\sigma_w|_w = \prod_{\alpha \in R^+ \cap wR^-} \alpha$.*[4]

Proposition 2.2. *Any cohomology class η in $H_T^*(\mathcal{F})$ can be written uniquely as an $H_T^*(pt)$-linear combination of σ_w using only those w such that $w \geqslant u$ for some u with $\eta|_u \neq 0$.*

Proof. The corresponding fact for the Grassmann variety is proved in [27]. The same proof works for the general flag variety also. $\square$

2.4. Actions of Weyl group

There are two actions of the Weyl group on the equivariant cohomology ring $H_T^*(\mathcal{F})$, which are used to define corresponding divided-difference operators. In this subsection we will follow the approach presented in [25]. Identify $\eta \in H_T^*(\mathcal{F})$ with the sequence of polynomials $(\eta|_v)_{v \in W}$ arising from the localization map. For $w \in W$ define

$$\left(w^R \eta\right)\big|_v = \eta|_{vw}, \qquad \left(w^L \eta\right)\big|_v = w \cdot (\eta|_{w^{-1}v}).$$

It is proved in [25] that these are well-defined actions on $H_{T_n}^*(\mathcal{F}_n)$, and that w^R is $H_T^*(pt)$-linear, while w^L is not (precisely because it acts on the polynomials' coefficients).

2.5. Divided difference operators

For each simple root α_i, we define the *divided difference operators* ∂_i and δ_i on $H_T^*(\mathcal{F})$ by

$$(\partial_i \eta)|_v = \frac{\eta|_v - (s_i^R \eta)|_v}{-v(\alpha_i)}, \qquad (\delta_i \eta)_v = \frac{\eta|_v - (s_i^L \eta)|_v}{\alpha_i} \quad (v \in W).$$

These rational functions are proved to be actually polynomials. They satisfy the GKM conditions, and thus give elements in $H_T^*(\mathcal{F})$ (see [25]). We call ∂_i's (resp. δ_i's) *right* (resp. *left*) divided

[4] Explicit formulas for the more general localized class $\sigma_w|_v$ have been obtained independently in [1, App. D.3] and [5, Theorem 3]. We will not use these formulas in the present paper.

difference operators. The operator ∂_i was introduced in [28]. On the ordinary cohomology, analogous operators to ∂_i's are introduced independently by Bernstein et al. [3] and Demazure [9]. The left divided difference operators δ_i was studied by Knutson in [25] (see also [41]). Note that ∂_i is $H_T^*(pt)$-linear whereas δ_i is not. Both these operators appeared in the unpublished notes of D. Peterson [37], in the disguise of a left and a right action of the nil-Hecke ring from [28] on the equivariant cohomology. The next proposition was stated in [25, Proposition 2] (see also [28,41, 37]).

Proposition 2.3.

(1) *Operators ∂_i and δ_i are well defined on the ring $H_T^*(\mathcal{F})$.*
(2) *The left and right divided difference operators commute with each other.*
(3) *We have*

$$\partial_i\sigma_w = \begin{cases} \sigma_{ws_i} & \text{if } \ell(ws_i) = \ell(w)-1, \\ 0 & \text{if } \ell(ws_i) = \ell(w)+1, \end{cases} \qquad \delta_i\sigma_w = \begin{cases} \sigma_{s_iw} & \text{if } \ell(s_iw) = \ell(w)-1, \\ 0 & \text{if } \ell(s_iw) = \ell(w)+1. \end{cases} \tag{2.1}$$

Proof. We only prove (2.1) for δ_i here, as the rest is proved in [25]. By Proposition 2.1 $(\delta_i\sigma_w)|_v$ is non-zero only for $\{v \mid v \geqslant w \text{ or } s_iv \geqslant w\}$. This implies that the element $\delta_i\sigma_w$ is an $H_T^*(pt)$-linear combination of $\{\sigma_v \mid v \geqslant w \text{ or } s_iv \geqslant w\}$ by Proposition 2.2. Moreover σ_v appearing in the linear combination have degree at most $\ell(w)-1$. Thus if $\ell(s_iw) = \ell(w)+1$ then $\delta_i\sigma_w$ must vanish. If $\ell(s_iw) = \ell(w)-1$ the only possible term is a multiple of σ_{s_iw}. In this case we calculate

$$(\delta_i\sigma_w)|_{s_iw} = \frac{\sigma_w|_{s_iw} - s_i(\sigma_w|_w)}{\alpha_i} = -\frac{s_i(\sigma_w|_w)}{\alpha_i},$$

where we used $\sigma_w|_{s_iw} = 0$ since $s_iw < w$. Here we recall the following well-known fact that

$$w > s_iw \quad \Longrightarrow \quad s_i\bigl(R^+ \cap wR^-\bigr) = \bigl(R^+ \cap s_iwR^-\bigr) \sqcup \{-\alpha_i\}.$$

So we have

$$s_i(\sigma_w|_w) = \prod_{\beta\in s_i(R^+\cap wR^-)} \beta = (-\alpha_i) \prod_{\beta\in(R^+\cap s_iwR^-)} \beta = (-\alpha_i)\cdot \sigma_{s_iw}|_{s_iw}.$$

By the characterization (Proposition 2.1), we have $\delta_i\sigma_w = \sigma_{s_iw}$. $\square$

3. Classical groups

In this section, we fix the notations for the root systems, Weyl groups, for the classical groups used throughout the paper.

3.1. Root systems

Let G_n be the classical Lie group of one of the types B_n, C_n and D_n, i.e. the symplectic group $Sp(2n,\mathbb{C})$ in type C_n, the odd orthogonal group $SO(2n+1,\mathbb{C})$ in type B_n and $SO(2n,\mathbb{C})$ in type D_n. Correspondingly we have the set R_n of roots, and the set of simple roots. These are subsets of the character group $\hat{T}_n = \bigoplus_{i=1}^n \mathbb{Z}t_i$ of T_n, the maximal torus of G_n.

The positive roots R_n^+ (set $R_n^- := -R_n^+$ the negative roots) are given by

$$\text{Type B}_n\text{:}\quad R_n^+ = \{t_i \mid 1 \leqslant i \leqslant n\} \cup \{t_j \pm t_i \mid 1 \leqslant i < j \leqslant n\},$$
$$\text{Type C}_n\text{:}\quad R_n^+ = \{2t_i \mid 1 \leqslant i \leqslant n\} \cup \{t_j \pm t_i \mid 1 \leqslant i < j \leqslant n\},$$
$$\text{Type D}_n\text{:}\quad R_n^+ = \{t_j \pm t_i \mid 1 \leqslant i < j \leqslant n\}.$$

The following are the simple roots:

$$\text{Type B}_n\text{:}\quad \alpha_0 = t_1, \qquad \alpha_i = t_{i+1} - t_i \quad (1 \leqslant i \leqslant n-1),$$
$$\text{Type C}_n\text{:}\quad \alpha_0 = 2t_1, \qquad \alpha_i = t_{i+1} - t_i \quad (1 \leqslant i \leqslant n-1),$$
$$\text{Type D}_n\text{:}\quad \alpha_{\hat{1}} = t_1 + t_2, \qquad \alpha_i = t_{i+1} - t_i \quad (1 \leqslant i \leqslant n-1).$$

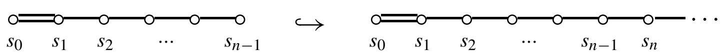
We introduce a symmetric bilinear form $(\cdot,\cdot)$ on $\hat{T}_n \otimes_{\mathbb{Z}} \mathbb{Q}$ by $(t_i, t_j) = \delta_{i,j}$. The simple *coroots* $\alpha_i^\vee$ are defined to be $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. Let ω_i denote the *fundamental weights*, i.e. those elements in $\hat{T}_n \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $(\omega_i, \alpha_i^\vee) = \delta_{i,j}$. They are explicitly given as follows:

$$\text{Type B}_n\text{:}\quad \omega_0 = \frac{1}{2}(t_1 + t_2 + \cdots + t_n), \qquad \omega_i = t_{i+1} + \cdots + t_n \quad (1 \leqslant i \leqslant n-1),$$
$$\text{Type C}_n\text{:}\quad \omega_i = t_{i+1} + \cdots + t_n \quad (0 \leqslant i \leqslant n-1),$$
$$\text{Type D}_n\text{:}\quad \omega_{\hat{1}} = \frac{1}{2}(t_1 + t_2 + \cdots + t_n), \qquad \omega_1 = \frac{1}{2}(-t_1 + t_2 + \cdots + t_n),$$
$$\omega_i = t_{i+1} + \cdots + t_n \quad (2 \leqslant i \leqslant n-1).$$

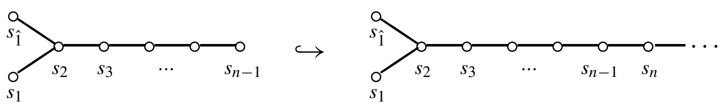
3.2. Weyl groups

Set $I_\infty = \{0, 1, 2, \ldots\}$ and $I'_\infty = \{\hat{1}, 1, 2, \ldots\}$. We define the Coxeter group (W_∞, I_∞) (resp. (W'_∞, I_∞)) of infinite rank, and its finite *parabolic* subgroup W_n (resp. W'_n) by the following Coxeter graphs:

$\mathrm{C}_n \subset \mathrm{C}_\infty$ ($\mathrm{B}_n \subset \mathrm{B}_\infty$)



$\mathrm{D}_n \subset \mathrm{D}_\infty$



More explicitly, the group W_∞ (resp. W'_∞) is generated by the simple reflections s_i $(i \in I_\infty)$ (resp. s_i $(i \in I'_\infty)$) subject to the relations:

$$\begin{cases} s_i^2 = e \quad (i \in I_\infty), \\ s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \bigl(i \in I_\infty \setminus \{0\}\bigr), \\ s_i s_j = s_j s_i \quad \bigl(|i-j| \geqslant 2\bigr), \end{cases}$$

$$\begin{cases} s_i^2 = e \quad \bigl(i \in I'_\infty\bigr), \\ s_{\hat{1}} s_2 s_{\hat{1}} = s_2 s_{\hat{1}} s_2, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \bigl(i \in I'_\infty \setminus \{\hat{1}\}\bigr), \\ s_{\hat{1}} s_i = s_i s_{\hat{1}} \quad \bigl(i \in I'_\infty,\ i \neq 2\bigr), \\ s_i s_j = s_j s_i \quad \bigl(i, j \in I'_\infty \setminus \{\hat{1}\},\ |i-j| \geqslant 2\bigr). \end{cases} \tag{3.1}$$

For general facts on Coxeter groups, we refer to [7]. Let $\leqslant$ denote the *Bruhat–Chevalley order* on W_∞ or W'_∞. The *length* $\ell(w)$ of $w \in W_\infty$ (resp. $w \in W'_\infty$) is defined to be the least number k of simple reflections in any reduced expression of $w \in W_\infty$.

The subgroups $W_n \subset W_\infty$, $W'_n \subset W'_\infty$ are the Weyl groups of the following types:

$$\text{Types } \mathrm{B}_n, \mathrm{C}_n\text{:} \quad W_n = \langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle, \qquad \text{Type } \mathrm{D}_n\text{:} \quad W'_n = \langle s_{\hat{1}}, s_1, s_2, \ldots, s_{n-1} \rangle.$$

It is known that the inclusion $W_n \subset W_\infty$ (resp. $W'_n \subset W'_\infty$) preserves the length and the Bruhat–Chevalley order, while $W'_\infty \subset W_\infty$ (resp. $W'_n \subset W_n$) is not (using terminology from [7] this says that W_n is a *parabolic subgroup* of W_∞, while W'_∞ is not). From now on, whenever possible, we will employ the notation explained in Section 1.1, and use bold fonts $\boldsymbol{W}_\infty$ respectively $\boldsymbol{W}_n$ to make uniform statements.

3.3. Signed permutations

The group W_∞ is identified with the set of all permutations w of the set $\{1, 2, \ldots\} \cup \{\bar{1}, \bar{2}, \ldots\}$ such that $w(i) \neq i$ for only finitely many i, and $\overline{w(i)} = w(\bar{i})$ for all i. These can also be considered as signed (or barred) permutation of $\{1, 2, \ldots\}$; we often use one-line notation $w = (w(1), w(2), \ldots)$ to denote an element $w \in W_\infty$. The simple reflections are identified with the transpositions $s_0 = (1, \bar{1})$ and $s_i = (i+1, i)(\bar{i}, \overline{i+1})$ for $i \geqslant 1$. The subgroup $W_n \subset W_\infty$ is described as

$$W_n = \bigl\{ w \in W_\infty \bigm| w(i) = i \text{ for } i > n \bigr\}.$$

In one-line notation, we often denote an element $w \in W_n \subset W_\infty$ by the finite sequence $(w(1), \ldots, w(n))$.

The group W'_∞, as a (signed) permutation group, can be realized as the subgroup of W_∞ consisting of elements in W_∞ with even number of sign changes. The simple reflection $s_{\hat{1}}$ is identified with $s_0 s_1 s_0 \in W_\infty$, so as a permutation $s_{\hat{1}} = (1, \bar{2})(2, \bar{1})$.

3.4. Grassmannian elements

An element $w \in W_\infty$ is a *Grassmannian element* if

$$w(1) < w(2) < \cdots < w(i) < \cdots$$

in the ordering $\cdots < \bar{3} < \bar{2} < \bar{1} < 1 < 2 < 3 < \cdots$. Let W_∞^0 denote the set of all Grassmannian elements in W_∞. For $w \in W_\infty^0$, let r be the number such that

$$w(1) < \cdots < w(r) < 1 \quad \text{and} \quad \bar{1} < w(r+1) < w(r+2) < \cdots. \tag{3.2}$$

Then we define the r-tuple of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ by $\lambda_i = \overline{w(i)}$ for $1 \leqslant i \leqslant r$. This is a *strict partition* i.e. a partition with distinct parts: $\lambda_1 > \cdots > \lambda_r > 0$. Let $\mathcal{SP}$ denote the set of all strict partitions. The correspondence gives a bijection $W_\infty^0 \to \mathcal{SP}$. We denote by $w_\lambda \in W_\infty^0$ the Grassmannian element corresponding to $\lambda \in \mathcal{SP}$; then $\ell(w_\lambda) = |\lambda| = \sum_i \lambda_i$. Note that this bijection preserves the partial order when $\mathcal{SP}$ is considered to be a partially ordered set given by the inclusion $\lambda \subset \mu$ of strict partitions.

We denote by $W_\infty^{\hat{1}}$ the set of all Grassmannian elements contained in W'_∞. For $w \in W_\infty^{\hat{1}}$, the number r in (3.2) is always even. Define the strict partition $\lambda' = (\lambda'_1 > \cdots > \lambda'_r \geqslant 0)$ by setting $\lambda'_i = \overline{w(i)} - 1$ for $1 \leqslant i \leqslant r$. Note that λ'_r can be zero this time. This correspondence gives also a bijection $W_\infty^{\hat{1}} \to \mathcal{SP}$. We denote by $w'_\lambda \in W_\infty^{\hat{1}}$ the element corresponding to $\lambda \in \mathcal{SP}$. As before, $\ell(w'_\lambda) = |\lambda|$ where $\ell(w)$ denotes the length of w in W'_∞.

Example. Let $\lambda = (4, 2, 1)$. Then the corresponding Grassmannian elements are given by $w_\lambda = \bar{4}\bar{2}\bar{1}3 = s_0 s_1 s_0 s_3 s_2 s_1 s_0$ and $w'_\lambda = \bar{5}\bar{3}\bar{2}\bar{1}4 = s_{\hat{1}} s_2 s_1 s_4 s_3 s_2 s_{\hat{1}}$.

The group W_∞ (resp. W'_∞) has a parabolic subgroup generated by s_i $(i \in I_\infty \setminus \{0\})$ (resp. s_i $(i \in I'_\infty \setminus \{\hat{1}\})$). We denote these subgroups by $S_\infty = \langle s_1, s_2, \ldots \rangle$ since it is isomorphic to the infinite Weyl group of type A. The product map

$$W_\infty^0 \times S_\infty \to W_\infty \quad \big(\text{resp. } W_\infty^{\hat{1}} \times S_\infty \to W'_\infty\big),$$

given by $(u, w) \mapsto uw$ is a bijection satisfying $\ell(uw) = \ell(u) + \ell(w)$ (cf. [7, Proposition 2.4.4]). As a consequence, w_λ (resp. w'_λ) is the unique element of minimal length in the left coset $w_\lambda S_\infty$ (resp. $w'_\lambda S_\infty$).

4. Schur's Q-functions and their factorial analogues

4.1. Schur's Q-functions

Our main reference for symmetric functions is [34]. Let $x = (x_1, x_2, \ldots)$ be infinitely many indeterminates. Define $Q_i(x)$ as the coefficient of u^i in the generating function

$$f(u) = \prod_{i=1}^{\infty} \frac{1 + x_i u}{1 - x_i u} = \sum_{k \geqslant 0} Q_k(x) u^k.$$

Note that $Q_0 = 1$. Define Γ to be $\mathbb{Z}[Q_1(x), Q_2(x), \ldots]$. The identity $f(u) f(-u) = 1$ yields

$$Q_i(x)^2 + 2 \sum_{j=1}^{i} (-1)^j Q_{i+j}(x) Q_{i-j}(x) = 0 \quad \text{for } i \geqslant 1. \tag{4.1}$$

It is known that the ideal of relations among the functions $Q_k(x)$ is generated by the previous relations. For $i \geqslant j \geqslant 0$, define elements

$$Q_{i,j}(x) := Q_i(x)Q_j(x) + 2\sum_{k=1}^{j}(-1)^k Q_{i+k}(x)Q_{j-k}(x).$$

Note that $Q_{i,0}(x) = Q_i(x)$ and $Q_{i,i}(x)$ $(i \geqslant 1)$ is identically zero. For λ a strict partition we write $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r \geqslant 0)$ with r even. Then the corresponding Schur's Q-function $Q_\lambda = Q_\lambda(x)$ is defined by

$$Q_\lambda(x) = \mathrm{Pf}\big(Q_{\lambda_i,\lambda_j}(x)\big)_{1\leqslant i<j\leqslant r},$$

where Pf denotes the Pfaffian. It is known then that the functions $Q_\lambda(x)$ for $\lambda \in \mathcal{SP}$ form a $\mathbb{Z}$-basis of Γ. The *Schur P-function* is defined to be $P_\lambda(x) = 2^{-\ell(\lambda)}Q_\lambda(x)$ where $\ell(\lambda)$ is the number of non-zero parts in λ. The next lemma shows that the Schur Q-function is *supersymmetric*.

Lemma 4.1. *Each element $\varphi(x)$ in Γ satisfies*

$$\varphi(t, -t, x_1, x_2, \ldots) = \varphi(x_1, x_2, \ldots)$$

where t is an indeterminate.

Proof. It suffices to show this for the ring generators $Q_i(x)$. This follows immediately from the generating function. □

4.2. *Factorial Schur Q- and P-functions*

In this subsection we recall the definition and some properties of the factorial Schur Q- and P-functions defined by V.N. Ivanov in [20]. Fix $n \geqslant 1$ an integer, λ a strict partition of length $r \leqslant n$ and $a = (a_i)_{i\geqslant 1}$ an infinite sequence. By $(x|a)^k$ we denote the product $(x - a_1)\cdots(x - a_k)$. According to [20, Definition 2.10] the factorial Schur P-function $P_\lambda^{(n)}(x_1, \ldots, x_n|a)$ is defined by:

$$P_\lambda^{(n)}(x_1, \ldots, x_n|a) = \frac{1}{(n-r)!}\sum_{w\in S_n} w \cdot \left(\prod_{i=1}^{r}(x_i|a)^{\lambda_i}\prod_{i\leqslant r, i<j\leqslant n}\frac{x_i + x_j}{x_i - x_j}\right),$$

where w acts on the variables x_i. If $a_1 = 0$ this function is stable, i.e. $P_\lambda^{(n+1)}(x_1, \ldots, x_n, 0|a) = P_\lambda^{(n)}(x_1, \ldots, x_n|a)$, therefore there is a well-defined limit denoted by $P_\lambda(x|a)$. It was proved in [18, Proposition 8] that if $a_1 \neq 0$, $P_\lambda^{(n)}(x_1, \ldots, x_n|a)$ is stable modulo 2, i.e. $P_\lambda^{(n+2)}(x_1, \ldots, x_n, 0, 0|a) = P_\lambda^{(n)}(x_1, \ldots, x_n|a)$; so in this case there is a well-defined even and odd limit. From now on we will denote by $P_\lambda(x|a)$ the *even* limit of these functions. For a sequence $a = (a_i)_{i\geqslant 1}$, define also the factorial Schur Q-function $Q_\lambda(x|a)$ to be

$$Q_\lambda(x|a) = 2^{\ell(\lambda)}P_\lambda(x|0, a),$$

where $\ell(\lambda)$ is the number of non-zero parts of λ. The sequence $\bar{a} = (0, a)$ is defined to be $\bar{a}_1 = 0$ and $\bar{a}_{i+1} = a_i$ for $i \geqslant 1$. As explained in [18], the situation $\bar{a}_1 \neq 0$ is needed to study type D; in types B, C, the case $\bar{a}_1 = 0$ will suffice. For simplicity, we will also denote $P_\lambda^{(n)}(x_1, \ldots, x_n|a)$ by $P_\lambda(x_1, \ldots, x_n|a)$. For each strict partition λ define two sequences t_λ and t'_λ as follows:

$$t_\lambda = (t_{\lambda_1}, \ldots, t_{\lambda_r}, 0, 0, \ldots), \qquad t'_\lambda = \begin{cases} (t_{\lambda_1+1}, \ldots, t_{\lambda_r+1}, 0, 0, \ldots) & \text{if } r \text{ is even,} \\ (t_{\lambda_1+1}, \ldots, t_{\lambda_r+1}, t_1, 0, \ldots) & \text{if } r \text{ is odd.} \end{cases}$$

Let also $w_\lambda \in W_\infty^0$ (resp. $w'_\lambda \in W_\infty^{\hat{1}}$) be the Grassmann element corresponding to $\lambda \in \mathcal{SP}$. We associate to λ its *shifted Young diagram* Y_λ as the set of *boxes* with coordinates (i, j) such that $1 \leqslant i \leqslant r$ and $i \leqslant j \leqslant i + \lambda_i - 1$. We set $\lambda_j = 0$ for $j > r$ by convention. Define

$$H_\lambda(t) = \prod_{(i,j)\in Y_\lambda} (t_{\overline{w_\lambda(i)}} + t_{\overline{w_\lambda(j)}}), \qquad H'_\lambda(t) = \prod_{(i,j)\in Y_\lambda} (t_{\overline{w'_\lambda(i)}} + t_{\overline{w'_\lambda(j+1)}}).$$

Example. Let $\lambda = (3, 1)$. Then $w_\lambda = \bar{3}\bar{1}2$, $w'_\lambda = \bar{4}\bar{2}13$, and

$$H_\lambda(t) = 4t_1t_3(t_3 + t_1)(t_3 - t_2), \qquad H'_\lambda(t) = (t_4 + t_2)(t_4 - t_1)(t_4 - t_3)(t_2 - t_1).$$

Type C
$$\begin{array}{c|c|c|c|} & 3 & 1 & \bar{2} \\ \cline{2-4} 3 & 2t_3 & t_3 + t_1 & t_3 - t_2 \\ \cline{2-4} 1 & \multicolumn{1}{c|}{} & 2t_1 & \\ \cline{3-3} \end{array}$$
$$w_\lambda = s_0s_2s_1s_0 = \bar{3}\bar{1}2$$

Type D
$$\begin{array}{c|c|c|c|} & 2 & \bar{1} & \bar{3} \\ \cline{2-4} 4 & t_4 + t_2 & t_4 - t_1 & t_4 - t_3 \\ \cline{2-4} 2 & \multicolumn{1}{c|}{} & t_2 - t_1 & \\ \cline{3-3} \end{array}$$
$$w'_\lambda = s_1s_3s_2s_{\hat{1}} = \bar{4}\bar{2}13$$

Proposition 4.2. *(See [20].) For any strict partition λ, the factorial Schur Q-function $Q_\lambda(x|t)$ (resp. $P_\lambda(x|t)$) satisfies the following properties*:

(1) *$Q_\lambda(x|t)$ (resp. $P_\lambda(x|t)$) is homogeneous of degree $|\lambda| = \sum_{i=1}^r \lambda_i$,*
(2) *$Q_\lambda(x|t) = Q_\lambda(x) +$ lower order terms in x (resp. $P_\lambda(x|t) = P_\lambda(x) +$ lower order terms in x),*
(3) *$Q_\lambda(t_\mu|t) = 0$ (resp. $P_\lambda(t'_\mu|t) = 0$) unless $\lambda \subset \mu$,*
(4) *$Q_\lambda(t_\lambda|t) = H_\lambda(t)$ (resp. $P_\lambda(t'_\lambda|t) = H'_\lambda(t)$).*

Moreover $Q_\lambda(x|t)$ (resp. $P_\lambda(x|t)$) ($\lambda \in \mathcal{SP}$) form a $\mathbb{Z}[t]$-basis of $\mathbb{Z}[t] \otimes_{\mathbb{Z}} \Gamma$ (resp. $\mathbb{Z}[t] \otimes_{\mathbb{Z}} \Gamma'$).

Proof. In the case $t_1 = 0$ this was proved in [20, Theorem 5.6]. If $t_1 \neq 0$, the identity (3) follows from the definition (cf. [18, Proposition 9]), while (4) follows from a standard computation. $\square$

Remark 4.3. The statement in the previous proposition can be strengthened by showing that the properties (1)–(4) characterize the factorial Schur Q- (respectively P-) functions. For $t_1 = 0$ this was shown in [20, Theorem 5.6]. A similar proof can be given for $t_1 \neq 0$, but it also follows from Theorem 6.3 below. The characterization statement will not be used in this paper.

Remark 4.4. The function $Q_\lambda(x|t)$ belongs actually to $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[t_1, t_2, \ldots, t_{\lambda_1 - 1}]$ and $P_\lambda(x|t)$ to $\Gamma' \otimes_{\mathbb{Z}} \mathbb{Z}[t_1, t_2, \ldots, t_{\lambda_1}]$. For example we have

$$Q_i(x|t) = \sum_{j=0}^{i-1} (-1)^j e_j(t_1, \ldots, t_{i-1}) Q_{i-j}(x), \qquad P_i(x|t) = \sum_{j=0}^{i-1} (-1)^j e_j(t_1, \ldots, t_i) P_{i-j}(x).$$

Remark 4.5. An alternative formula for $Q_\lambda(x|t)$, in terms of a *multi-Pfaffian*, will be given below in Section 11.

The following proposition will only be used within the proof of the formula for the Schubert polynomial for the longest element in each type, presented in Section 9 below.

Proposition 4.6. *(See [20].) Let $\lambda = (\lambda_1 > \cdots > \lambda_r \geqslant 0)$ be a strict partition with r even. Then*

$$Q_\lambda(x|t) = \mathrm{Pf}\big(Q_{\lambda_i, \lambda_j}(x|t)\big)_{1 \leqslant i < j \leqslant r}, \qquad P_\lambda(x|t) = \mathrm{Pf}\big(P_{\lambda_i, \lambda_j}(x|t)\big)_{1 \leqslant i < j \leqslant r}.$$

Proof. Again, for $t_1 = 0$, this was proved in [20, Theorem 3.2], using the approach described in [34, III.8, Example 13]. The same approach works in general, but for completeness we briefly sketch an argument. Lemma 6.5 below shows that there is an injective universal localization map $\Phi : \mathbb{Z}[z] \otimes \mathbb{Z}[t] \otimes \Gamma' \to \prod_{w \in W'_\infty} \mathbb{Z}[t]$. The image of $P_\lambda(x|t)$ is completely determined by the images at Grassmannian Weyl group elements w'_μ and it is given by $P_\lambda(t'_\mu|t)$. But by the results from [18, Section 10] we have that $P_\lambda(t'_\mu|t) = \mathrm{Pf}(P_{\lambda_i, \lambda_j}(t'_\mu|t))_{1 \leqslant i < j \leqslant r}$. The result follows by injectivity of Φ. $\square$

We record here the following formula used later. The proof is by a standard computation (see e.g. the proof of [20, Theorem 8.4]).

Lemma 4.7. *We have*

$$P_{k,1}(x|t) = P_k(x|t) P_1(x|t) - P_{k+1}(x|t) - (t_{k+1} + t_1) P_k(x|t) \quad \textit{for } k \geqslant 1. \tag{4.2}$$

4.3. Factorization formulas

In this subsection we present several *factorization* formulas for the factorial Schur P- and Q-functions, which will be used later in Section 9. To this end, we first consider the case of ordinary factorial Schur functions.

4.3.1. Factorial Schur polynomials

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. Define the *factorial Schur polynomial* by

$$s_\lambda(x_1, \ldots, x_n|t) = \frac{\det((x_j|t)^{\lambda_i + n - i})_{1 \leqslant i, j \leqslant n}}{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)},$$

where $(x|t)^k$ denotes $\prod_{i=1}^k (x - t_i)$. It turns out that $s_\lambda(x|t)$ is an element in $\mathbb{Z}[x_1, \ldots, x_n]^{S_n} \otimes \mathbb{Z}[t_1, \ldots, t_{\lambda_1 + n - 1}]$. For some basic properties of these polynomials, the reader can consult [36]. The following formula will be used to prove Lemma 9.5 in Section 9.

Lemma 4.8. *We have* $s_{\rho_{n-1}}(t_1,\ldots,t_n|t_1,-z_1,t_2,-z_2,\ldots,t_{n-1},-z_{n-1})=\prod_{1\leqslant i<j\leqslant n}(t_j+z_i)$.

Proof. When variables z_i, t_i are specialized as in this lemma, the numerator is an anti-diagonal lower triangular matrix. The entry on the i-th row on the anti-diagonal is given by $\prod_{j=1}^{i-1}(t_i-t_j)(t_i+z_j)$. The lemma follows immediately from this. □

Next formula is a version of Lemma 4.8 which will be used in the proof of Lemma 9.7.

Lemma 4.9. *If n is odd then we have*

$$s_{\rho_{n-1}+1^{n-1}}(t_2,\ldots,t_n|t_1,-z_1,\ldots,t_{n-1},-z_{n-1})=\prod_{j=2}^{n}(t_j-t_1)\prod_{1\leqslant i<j\leqslant n}(t_j+z_i).$$

Proof. Similar to the proof of Lemma 4.8. □

4.3.2. Factorial Schur P- and Q-functions

We need the following factorization formula.

Lemma 4.10. *Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be a partition. Then we have*

$$Q_{\rho_n+\lambda}(x_1,\ldots,x_n|t)=\prod_{i=1}^{n}2x_i\prod_{1\leqslant i<j\leqslant n}(x_i+x_j)\times s_\lambda(x_1,\ldots,x_n|t),$$

Proof. By their very definition

$$Q_{\rho_n+\lambda}(x_1,\ldots,x_n|t)=2^n\sum_{w\in S_n}w\left[\prod_{i=1}^{n}x_i(x_i|t)^{\lambda_i+n-i}\prod_{1\leqslant i<j\leqslant n}\frac{x_i+x_j}{x_i-x_j}\right],$$

where w acts as permutation of the variables $x_1,\ldots,x_n$. Since the polynomial $\prod_{i=1}^{n}x_i\prod_{1\leqslant i<j\leqslant n}(x_i+x_j)$ in the parenthesis is symmetric in x, the last expression factorizes into

$$2^n\prod_{i=1}^{n}x_i\prod_{1\leqslant i<j\leqslant n}(x_i+x_j)\times\sum_{w\in S_n}w\left[\prod_{i=1}^{n}(x_i|t)^{\lambda_i+n-i}\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)^{-1}\right].$$

Then by the definition of $s_\lambda(x_1,\ldots,x_n|t)$ we have the lemma. □

The following two lemmas are proved in the same way:

Lemma 4.11. *Assume n is even. Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be a partition. Then we have*

$$P_{\rho_{n-1}+\lambda}(x_1,\ldots,x_n|t)=\prod_{1\leqslant i<j\leqslant n}(x_i+x_j)\cdot s_\lambda(x_1,\ldots,x_n|t).$$

Lemma 4.12. *Assume n is odd. Let $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ be a partition. Then we have*

$$P_{\rho_{n-1}+\lambda}(x_1, \dots, x_{n-1}|t) = \prod_{1 \leqslant i<j \leqslant n-1} (x_i + x_j) \times s_{\lambda+1^{n-1}}(x_1, \dots, x_{n-1}|t).$$

5. Stable Schubert classes

The aim of this section is to introduce *stable* Schubert classes indexed by the Weyl group of infinite rank $\boldsymbol{W}_\infty$. Recall that the embeddings of Dynkin diagrams shown in Section 3.2 induce embeddings $i : \boldsymbol{W}_n \to \boldsymbol{W}_{n+1}$; then $\boldsymbol{W}_\infty = \bigcup_{n \geqslant 1} \boldsymbol{W}_n$.

5.1. Stable Schubert classes

Let us denote by $\sigma_w^{(n)}$ the equivariant Schubert class on $\mathcal{F}_n$ labeled by $w \in \boldsymbol{W}_n$.

Proposition 5.1. *The localization of Schubert classes is stable, i.e.*

$$\sigma_{i(w)}^{(n+1)}\big|_{i(v)} = \sigma_w^{(n)}\big|_v \quad \text{for all } w, v \in \boldsymbol{W}_n.$$

Proof. First we claim that $\sigma_{i(w)}^{(n+1)}|_{i(v)} \in \mathbb{Z}[t]^{(n)}$ for any $w, v \in \boldsymbol{W}_n$. Let $w_0^{(n)}$ be the longest element in $\boldsymbol{W}_n$. By Proposition 2.1, we have for $v \in \boldsymbol{W}_n$

$$\sigma_{i(w_0^{(n)})}^{(n+1)}\big|_{i(v)} = \begin{cases} \prod_{\beta \in R_n^+} \beta & \text{if } v = w_0^{(n)}, \\ 0 & \text{if } v \neq w_0^{(n)}. \end{cases}$$

In particular these polynomials belong to $\mathbb{Z}[t]^{(n)}$. For arbitrary $w \in \boldsymbol{W}_n$, any reduced expression of $i(w)$ contains only simple reflections $s_0, \dots, s_{n-1}$ (in type D, s_0 is replaced by $s_{\hat{1}}$). Hence we obtain the Schubert class $\sigma_{i(w)}^{(n+1)}$ by applying the divided difference operators $\partial_0, \dots, \partial_{n-1}$ (in type D, ∂_0 is replaced by $\partial_{\hat{1}}$) successively to the class $\sigma_{i(w_0^{(n)})}^{(n+1)}$. In this process only the variables $t_1, \dots, t_n$ are involved to compute $\sigma_{i(w)}^{(n+1)}|_{i(v)}$ $(v \in \boldsymbol{W}_n)$. Hence the claim is proved.

For $w \in \boldsymbol{W}_n$, we consider the element η_w in $\prod_{v \in \boldsymbol{W}_n} \mathbb{Z}[t]^{(n)}$ given by $\eta_w|_v = \sigma_{i(w)}^{(n+1)}|_{i(v)}$ $(v \in \boldsymbol{W}_n)$. We will show the element η_w satisfies the conditions in Proposition 2.1 that characterize $\sigma_w^{(n)}$. In fact, the vanishing condition holds since $i(w) \leqslant i(v)$ if and only if $w \leqslant v$. Homogeneity and the degree condition is satisfied because $\ell(i(w)) = \ell(w)$. The normalization follows from the fact $R_n^+ \cap w R_n^- = R_{n+1}^+ \cap i(w) R_{n+1}^-$. Thus we have $\eta_w = \sigma_w^{(n)}$ and the proposition is proved. □

Fix w to be in $\boldsymbol{W}_\infty$. Then, by the previous proposition, for any $v \in \boldsymbol{W}_\infty$, and for any sufficiently large n such that $w, v \in \boldsymbol{W}_n$, the polynomial $\sigma_w^{(n)}|_v$ does not depend on the choice of n. Thus we can introduce a unique element $\sigma_w^{(\infty)} = (\sigma_w^{(\infty)}|_v)_v$ in $\prod_{v \in \boldsymbol{W}_\infty} \mathbb{Z}[t]$ such that

$$\sigma_w^{(\infty)}\big|_v = \sigma_w^{(n)}\big|_v$$

for all sufficiently large n. We call this element the *stable* Schubert class.

Remark 5.2. As pointed out by a referee, the stability of the equivariant Schubert classes may have a geometric interpretation using the cohomology of Kashiwara's thick Kac–Moody flag variety defined in [22, Section 5]. We plan to investigate this elsewhere.

Definition 5.3. Let H_∞ be the $\mathbb{Z}[t]$-submodule of $\prod_{v\in \boldsymbol{W}_\infty}\mathbb{Z}[t]$ spanned by the stable Schubert classes $\sigma_w^{(\infty)}$, $w\in \boldsymbol{W}_\infty$, where the $\mathbb{Z}[t]$-module structure is given by the diagonal multiplication.

We will show later in Corollary 6.4 that H_∞ is actually a $\mathbb{Z}[t]$-subalgebra in the product ring $\prod_{v\in \boldsymbol{W}_\infty}\mathbb{Z}[t]$. The properties of the (finite-dimensional) Schubert classes extend immediately to the stable case. For example, the classes $\sigma_w^{(\infty)}$ ($w\in W_\infty$) are linearly independent over $\mathbb{Z}[t]$ (Proposition 2.2), and they satisfy the properties from Proposition 2.1. To state the latter, define $R^+=\bigcup_{n\geqslant 1}R_n^+$, regarded as a subset of $\mathbb{Z}[t]$. Then:

Proposition 5.4. *The stable Schubert class satisfies the following*:

(1) (*Homogeneity*) $\sigma_w^{(\infty)}|_v$ *is homogeneous of degree* $\ell(w)$ *for each* $v\geqslant w$.
(2) (*Normalization*) $\sigma_w^{(\infty)}|_w=\prod_{\beta\in R^+\cap w(R^-)}\beta$.
(3) (*Vanishing*) $\sigma_w^{(\infty)}|_v$ *vanishes unless* $v\geqslant w$.

It is natural to consider the following stable version of the GKM conditions in the ring $\prod_{v\in \boldsymbol{W}_\infty}\mathbb{Z}[t]$:

$$\eta|_v-\eta|_{s_\alpha v}\quad \text{is a multiple of } \alpha \text{ for all } \alpha\in R^+,\ v\in \boldsymbol{W}_\infty.$$

Then the stable Schubert class $\sigma_w^{(\infty)}$ is the unique element in $\prod_{v\in W_\infty}\mathbb{Z}[t]$ that satisfies the GKM conditions and the three conditions of Proposition 5.4. It follows that all the elements from H_∞ satisfy the GKM conditions. In particular, the proofs from [25] can be retraced, and one can define the left and right actions of $\boldsymbol{W}_\infty$ on H_∞ by the same formulas as in Section 2.4 but for $i\in \boldsymbol{I}_\infty$. Using these actions, we define also the divided difference operators ∂_i,δ_i on H_∞ (see Section 2.5). The next result follows again from the finite-dimensional case (Proposition 2.3).

Proposition 5.5. *We have*

$$\partial_i\sigma_w^{(\infty)}=\begin{cases}\sigma_{ws_i}^{(\infty)}, & \ell(ws_i)=\ell(w)-1,\\ 0, & \ell(ws_i)=\ell(w)+1,\end{cases}\qquad \delta_i\sigma_w^{(\infty)}=\begin{cases}\sigma_{s_iw}^{(\infty)}, & \ell(s_iw)=\ell(w)-1,\\ 0, & \ell(s_iw)=\ell(w)+1.\end{cases}$$

5.2. Inverse limit of cohomology groups

Let H_n denote the image of the localization map

$$\iota_n^*: H_{T_n}^*(\mathcal{F}_n)\to H_{T_n}^*\big(\mathcal{F}_n^{T_n}\big)=\prod_{v\in \boldsymbol{W}_n}\mathbb{Z}[t]^{(n)}.$$

By the stability property for the localization of Schubert classes, the natural projections $H_\infty\to H_n\simeq H_{T_n}^*(\mathcal{F}_n)$ are compatible with the homomorphisms $H_{T_{n+1}}^*(\mathcal{F}_{n+1})\to H_{T_n}^*(\mathcal{F}_n)$ induced by the equivariant embeddings $\mathcal{F}_n\to\mathcal{F}_{n+1}$. Therefore there is a $\mathbb{Z}[t]$-module homomorphism

$$j : H_\infty \hookrightarrow \varprojlim H^*_{T_n}(\mathcal{F}_n).$$

The injectivity of localization maps in the finite-dimensional setting implies that j is injective as well.

6. Universal localization map

In this section, we introduce a $\mathbb{Z}[t]$-algebra $\boldsymbol{R}_\infty$ and establish an explicit isomorphism from $\boldsymbol{R}_\infty$ onto H_∞, the $\mathbb{Z}[t]$-module spanned by the stable Schubert classes. This isomorphism will be used in the proof of the existence of the double Schubert polynomials from Section 8.

6.1. The ring $\boldsymbol{R}_\infty$ and the universal localization map

Set $\mathbb{Z}[z] = \mathbb{Z}[z_1, z_2, z_3, \ldots]$ and define the following rings:

$$R_\infty := \mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[z] \otimes_{\mathbb{Z}} \Gamma, \quad \text{and} \quad R'_\infty := \mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[z] \otimes_{\mathbb{Z}} \Gamma'.$$

As usual, we will use $\boldsymbol{R}_\infty$ to denote R_∞ for type C and R'_∞ for types B and D.

We introduce next the most important algebraic tool of the paper. Let v be in W_∞. Set $t_v = (t_{v,1}, t_{v,2}, \ldots)$ to be

$$t_{v,i} = \begin{cases} t_{\overline{v(i)}} & \text{if } v(i) \text{ is negative,} \\ 0 & \text{otherwise,} \end{cases}$$

where we set $t_{\bar{i}}$ to be $-t_i$. Define a homomorphism of $\mathbb{Z}[t]$-algebras

$$\Phi_v : R'_\infty \to \mathbb{Z}[t] \qquad (x \mapsto t_v, \quad z_i \mapsto t_{v(i)}).$$

Note that since $v(i) = i$ for all sufficiently large i, the substitution $x \mapsto t_v$ to $P_\lambda(x)$ gives a *polynomial* $P_\lambda(t_v)$ in $\mathbb{Z}[t]$ (rather than a formal power series). Since R_∞ is a subalgebra of R'_∞, the restriction map $\Phi_v : R_\infty \to \mathbb{Z}[t]$ sends $Q_\lambda(x)$ to $Q_\lambda(t_v)$.

Definition 6.1. Define the "universal localization map" to be the homomorphism of $\mathbb{Z}[t]$-algebras given by

$$\Phi : \boldsymbol{R}_\infty \to \prod_{v \in \boldsymbol{W}_\infty} \mathbb{Z}[t], \quad f \mapsto \big(\Phi_v(f)\big)_{v \in \boldsymbol{W}_\infty}.$$

Remark 6.2. A geometric interpretation of the map Φ, in terms of the usual localization map, will be given later in Section 10.

The main result of this section is

Theorem 6.3. *The map Φ is an isomorphism of graded $\mathbb{Z}[t]$ algebras from $\boldsymbol{R}_\infty$ onto its image. Moreover, the image of Φ is equal to H_∞.*

Corollary 6.4. *H_∞ is a $\mathbb{Z}[t]$-subalgebra in $\prod_{v \in \boldsymbol{W}_\infty} \mathbb{Z}[t]$.*

The proof of the theorem will be given in several lemmata and propositions, and it occupies the remaining part of Section 6. The more involved part is to show surjectivity, which relies on the analysis of the "transition equations" implied by the equivariant Chevalley rule, and on study of factorial Schur P- and Q-functions. The proof of injectivity is rather short, and we present it next.

Lemma 6.5. *The map Φ is injective.*

Proof. We first consider type B case. Write $f \in R'_\infty$ as $f = \sum_\lambda c_\lambda(t,z)P_\lambda(x)$. Suppose $\Phi(f) = 0$. There are m, n such that

$$c_\lambda \in \mathbb{Z}[t_1, \ldots, t_m, z_1, \ldots, z_n]$$

for all λ such that $c_\lambda \neq 0$. Define $v \in W_\infty$ by $v(i) = m + i$ $(1 \leqslant i \leqslant n)$, $v(n+i) = i$ $(1 \leqslant i \leqslant m)$, $v(m+n+i) = \overline{m+n+i}$ $(1 \leqslant i \leqslant N)$, and $v(i) = i$ $(i > N)$, where $N \geqslant m+n+1$. Then we have $\Phi_v(f) = \sum_\lambda c_\lambda(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n})P_\lambda(t_{m+n+1}, t_{m+n+2}, \ldots, t_{m+n+N}) = 0$. Since this holds for all sufficiently large N, we have

$$\sum_\lambda c_\lambda(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n})P_\lambda(t_{m+n+1}, t_{m+n+2}, \ldots) = 0.$$

Since $P_\lambda(t_{m+n+1}, t_{m+n+2}, \ldots)$ are linearly independent over $\mathbb{Z}$ (see [34, III, (8.9)]), we have

$$c_\lambda(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) = 0$$

for all λ. This implies $c_\lambda(t_1, \ldots, t_m; z_1, \ldots, z_n) = 0$ for all λ. Since $R_\infty \subset R'_\infty$, type C case follows immediately. Type D case is proved by a minor modification. Take N to be always even, and consider the sufficiently large *even* N. □

6.2. Factorial Q- (and P-)functions and Grassmannian Schubert classes

Recall that there is a natural bijection between W_∞^0, $W_\infty^{\hat{1}}$ and the set of strict partitions $\mathcal{SP}$. The next result was proved by Ikeda in [16] for type C, and Ikeda and Naruse in [18] for types B, D.

Theorem 6.6. *(See [16,18].) Let $\lambda \in \mathcal{SP}$ and $w_\lambda \in W_\infty^0$ and $w'_\lambda \in W_\infty^{\hat{1}}$ be the corresponding Grassmannian elements. Then we have*

(1) $\Phi(Q_\lambda(x|t)) = \sigma_{w_\lambda}^{(\infty)}$ *for type* C,
(2) $\Phi(P_\lambda(x|0,t)) = \sigma_{w_\lambda}^{(\infty)}$ *for type* B,
(3) $\Phi(P_\lambda(x|t)) = \sigma_{w'_\lambda}^{(\infty)}$ *for type* D.

Proof. We consider first the type C case. The map on W_∞ given by $v \mapsto \sigma_{w_\lambda}^{(\infty)}|_v$ is constant on each left coset of $W_{\infty,0} \cong S_\infty$ and it is determined by the values at the Grassmannian elements. Let $v \in W_\infty$ and w_μ be the minimal length representative of the coset vS_∞ corresponding to a strict partition μ. Then t_v defined in Section 6.1 is a permutation of t_μ. Since $Q_\lambda(x|t)$ is symmetric with respect to x we have $\Phi_v(Q_\lambda(x|t)) = Q_\lambda(t_v|t) = Q_\lambda(t_\mu|t)$. In [16], it was shown that

$Q_\lambda(t_\mu|t) = \sigma^{(\infty)}_{w_\mu}|_{w_\mu}$, which is equal to $\sigma^{(\infty)}_{w_\mu}|_v$. This completes the proof of this case. Proofs of the other cases are the same with appropriate identification of the functions and strict partitions. $\square$

6.3. Equivariant Chevalley formula

The *Chevalley formula* is a rule to multiply a Schubert class with a divisor class. To state it we need some notation. For a positive root $\alpha \in R^+$ and a simple reflection s_i, set

$$c_{\alpha,s_i} = \left(\omega_i, \alpha^\vee\right), \quad \alpha^\vee = 2\alpha/(\alpha,\alpha),$$

where ω_i is the i-th fundamental weight of one of the classical types A_n–D_n for sufficiently large n. The number c_{α,s_i} – called *Chevalley multiplicity* – does not depend on the choice of n.

Proposition 6.7. *(Cf. [28].) For any $w \in \boldsymbol{W}_\infty$, the Chevalley multiplicity $\sigma^{(\infty)}_{s_i}|_w$ is given by $\omega_i - w(\omega_i)$, where ω_i is the fundamental weight for a classical type A_n–D_n such that $n \geqslant i$.*

Lemma 6.8 *(Equivariant Chevalley formula).*

$$\sigma^{(\infty)}_{s_i}\sigma^{(\infty)}_{w} = \sum_{\alpha\in R^+,\, \ell(ws_\alpha)=\ell(w)+1} c_{\alpha,s_i}\sigma^{(\infty)}_{ws_\alpha} + \sigma^{(\infty)}_{s_i}\big|_w \cdot \sigma^{(\infty)}_{w}.$$

Proof. The non-equivariant case is due to Chevalley [8], but the stable version of this formula was given in [4]. An easy argument using localization shows that the only difference in the equivariant case is the appearance of the equivariant term $\sigma^{(\infty)}_{s_i}|_w \cdot \sigma^{(\infty)}_{w}$. $\square$

Remark 6.9. There are only finitely many non-zero terms in the sum in the right-hand side.

Lemma 6.10. *The elements $\Phi(z_i) \in H_\infty$ are expressed in terms of Schubert classes as follows*:

Type B: $\Phi(z_1) = \sigma^{(\infty)}_{s_1} - 2\sigma^{(\infty)}_{s_0} + t_1$, $\quad \Phi(z_i) = \sigma^{(\infty)}_{s_i} - \sigma^{(\infty)}_{s_{i-1}} + t_i \quad (i \geqslant 2)$,

Type C: $\Phi(z_i) = \sigma^{(\infty)}_{s_i} - \sigma^{(\infty)}_{s_{i-1}} + t_i \quad (i \geqslant 1)$,

Type D: $\Phi(z_1) = \sigma^{(\infty)}_{s_1} - \sigma^{(\infty)}_{s_{\hat{1}}} + t_1$, $\quad \Phi(z_2) = \sigma^{(\infty)}_{s_2} - \sigma^{(\infty)}_{s_1} - \sigma^{(\infty)}_{s_{\hat{1}}} + t_2$, *and*

$\Phi(z_i) = \sigma^{(\infty)}_{s_i} - \sigma^{(\infty)}_{s_{i-1}} + t_i \quad (i \geqslant 3)$.

Proof. This follows by localizing both sides of the formulas, and then using Proposition 6.7. $\square$

Lemma 6.11. *We have* $\mathrm{Im}(\Phi) \subset H_\infty$.

Proof. The ring R_∞ has a $\mathbb{Z}[t]$-basis $z^\alpha Q_\lambda(x|t)$ where z^α are monomials in $\mathbb{Z}[z]$ and λ are strict partitions. Since Φ is $\mathbb{Z}[t]$-linear, it is enough to show that $\Phi(z^\alpha Q_\lambda(x|t))$ belongs to H_∞. We use induction on degree d of the monomial z^α. The case $d = 0$ holds by Theorem 6.6. Let $d \geqslant 1$ and assume that $\Phi(z^\alpha Q_\lambda(x|t))$ lies in H_∞ for any monomial z^α of degree less than d. Note that, by Lemma 6.10, we have $\Phi(z_i) \in H_\infty$. Choose any index i such that $z^\alpha = z_i \cdot z^\beta$. By induction hypothesis $\Phi(z^\beta Q_\lambda(x|t))$ is an element in H_∞, i.e., a linear combination of $\sigma_w^{(\infty)}$'s with coefficients in $\mathbb{Z}[t]$. Lemma 6.10 together with equivariant Chevalley formula imply that $\Phi(z_i)\sigma_w^{(\infty)}$ belongs to H_∞. It follows that $z^\alpha Q_\lambda(x|t)$ belongs to H_∞. $\square$

6.4. *Transition equations*

To finish the proof of surjectivity of Φ, we need certain recursive relations for the Schubert classes – the *transition equations* – implied by the (equivariant) Chevalley formula. The arguments in this subsection are very similar to those given by S. Billey in [4]. Let t_{ij} denote the reflection with respect to the root $t_j - t_i$, s_{ij} the reflection with respect to $t_i + t_j$ and s_{ii} the reflection with respect to t_i or $2t_i$ (depending on type). From now on we regard $\mathbb{Z}[z]$ as subalgebra of H_∞ via Φ and we identify z_i with its image $\Phi(z_i)$ in H_∞ (cf. Lemma 6.11).

Proposition 6.12 *(Transition equations). The Schubert classes σ_w of types* B, C *and* D *satisfy the following recursion formula*:

$$\sigma_w^{(\infty)} = \big(z_r - v(t_r)\big)\sigma_v^{(\infty)} + \sum_{1\leqslant i<r} \sigma_{vt_{ir}}^* + \sum_{i\neq r} \sigma_{vs_{ir}}^* + \chi\sigma_{vs_{rr}}^*, \tag{6.1}$$

where r is the last descent of w, s is the largest index such that $w(s) < w(r)$, $v = wt_{rs}$, $\chi = 2, 1, 0$ according to the types B, C, D, *and for each $\sigma_{vt}^* = 0$ unless $\ell(vt) = \ell(v) + 1 = \ell(w)$ for $v, t \in \boldsymbol{W}_\infty$ in which case $\sigma_{vt}^* = \sigma_{vt}^{(\infty)}$.*

Proof. The same as in [4, Theorem 4] using the equivariant Chevalley formula (Lemma 6.8). $\square$

Remark 6.13. The precise recursive nature of Eq. (6.1) will be explained in the proof of the next proposition below.

Proposition 6.14. *If $w \in \boldsymbol{W}_n$ then the Schubert class $\sigma_w^{(\infty)}$ is expressed as a $\mathbb{Z}[z, t]$-linear combination of the Schubert classes of maximal Grassmannian type. More precisely we have*

$$\sigma_w^{(\infty)} = \sum_\lambda g_{w,\lambda}(z, t)\sigma_\lambda^{(\infty)}, \tag{6.2}$$

for some polynomials $g_{w,\lambda}(z, t)$ in variables t_i and z_i, and the sum is over strict partitions λ such that $|\lambda| \leqslant n$.

Proof. We will show that the recursion (6.1) terminates in a finite number of steps to get the desired expression. Following [4], we define a partial ordering on the elements of $\boldsymbol{W}_\infty$. Given w in $\boldsymbol{W}_\infty$, let $LD(w)$ be the position of the last descent. Define a partial ordering on the elements of $\boldsymbol{W}_\infty$ by $w <_{LD} u$ if $LD(u) < LD(w)$ or if $LD(u) = LD(w)$ and $u(LD(u)) < w(LD(w))$. In [4] it was shown that each element appearing on the right-hand side of (6.1) is less than w under this ordering. Moreover it was proved in [4, Theorem 4] that recursive applications of (6.1) give only terms which correspond to the elements in $\boldsymbol{W}_{n+r}$ where r is the last descent of w. Therefore we obtain the expansion (6.2). $\square$

6.5. Proof of Theorem 6.3

Proof. By Lemma 6.11 we know $\mathrm{Im}(\Phi) \subset H_\infty$. Clearly Φ preserves the degree. So it remains to show $H_\infty \subset \mathrm{Im}(\Phi)$. In order to show this, it suffices to $\sigma_w^{(\infty)} \in \mathrm{Im}(\Phi)$. In fact we have

$$\Phi\left(\sum_\lambda g_{w,\lambda}(z,t) Q_\lambda(x|t)\right) = \sigma_w^{(\infty)}, \tag{6.3}$$

since Φ is $\mathbb{Z}[z,t]$-linear. $\square$

7. Weyl group actions and divided difference operators on $\boldsymbol{R}_\infty$

We define two commuting actions of $\boldsymbol{W}_\infty$ on the ring $\boldsymbol{R}_\infty$. It is shown that the Weyl group actions are compatible with the action on H_∞ via Φ.

7.1. Weyl group actions on R_∞

We start from type C. We make W_∞ act as ring automorphisms on R_∞ by letting s_i^z interchange z_i and z_{i+1}, for $i > 0$, and letting s_0^z replace z_1 and $-z_1$, and also

$$s_0^z Q_i(x) = Q_i(x) + 2\sum_{j=1}^{i} z_1^j Q_{i-j}(x).$$

The operator s_0^z was introduced in [6]. Let $\omega : R_\infty \to R_\infty$ be an involutive ring automorphism defined by

$$\omega(z_i) = -t_i, \qquad \omega(t_i) = -z_i, \qquad \omega\big(Q_k(x)\big) = Q_k(x).$$

Define the operators s_i^t on R_∞ by $s_i^t = \omega s_i^z \omega$ for $i \in I_\infty$. More explicitly, s_i^t interchange t_i and t_{i+1}, for $i > 0$, and s_0^t replace t_1 and $-t_1$ and also

$$s_0^t Q_i(x) = Q_i(x) + 2\sum_{j=1}^{i} (-t_1)^j Q_{i-j}(x).$$

Lemma 7.1. *The action of operators s_0^z, s_0^t on any $\varphi(x) \in \Gamma$ are written as*

$$s_0^z \varphi(x_1, x_2, \ldots) = \varphi(z_1, x_1, x_2, \ldots), \qquad s_0^t \varphi(x_1, x_2, \ldots) = \varphi(-t_1, x_1, x_2, \ldots).$$

Note that the right-hand side of both the formulas above belong to R_∞.

Proof. We show this for the generators $\varphi(x) = Q_k(x)$ of Γ. By the definition of s_0^z we have

$$\sum_{k=0}^\infty s_0^z Q_k(x) \cdot u^k = \left(\prod_{i=1}^\infty \frac{1+x_i u}{1-x_i u}\right)\frac{1+z_1 u}{1-z_1 u} = \sum_{k=0}^\infty Q_k(z_1, x_1, x_2, \ldots) u^k.$$

Thus we have the result for $s_0^z Q_k(x)$ for $k \geqslant 1$. The proof for s_0^t is similar. $\square$

Proposition 7.2.

(1) *The operators* s_i^z $(i \geqslant 0)$ *give an action of* W_∞ *on* R_∞.
(2) *The operators* s_i^t $(i \geqslant 0)$ *give an action of* W_∞ *on* R_∞.
(3) *The two actions of* W_∞ *commute with each other.*

Proof. We show that s_i^z satisfy the Coxeter relations for W_∞. The calculation for s_i^t is the same. We first show that $(s_0^z)^2 = 1$. For $f(z) \in \mathbb{Z}[z]$, $(s_0^z)^2 f(z) = f(z)$ is obvious. We have for $\varphi(x) \in \Gamma$,

$$\left(s_0^z\right)^2\left(\varphi(x)\right) = s_0^z\varphi(z_1, x_1, x_2, \ldots) = \varphi(z_1, -z_1, x_1, x_2, \ldots) = \varphi(x_1, x_2, \ldots),$$

where we used the super-symmetry (Lemma 4.1) at the last equality. The verification of the remaining relations and the commutativity are left for the reader. □

In type B, the action of W_∞ on R'_∞ is obtained by extending in the canonical way the action from R_∞. Finally, we consider the type D case. In this case, the action is given by restriction the action of W_∞ on R'_∞ to the subgroup W'_∞. Namely, if we set $s_{\hat{1}}^z = s_0^z s_1^z s_0^z$ and $s_{\hat{1}}^t = s_0^t s_1^t s_0^t$, then we have the corresponding formulas for $s_{\hat{1}}^t$ and $s_{\hat{1}}^t$ (in type D):

$$s_{\hat{1}}^z\varphi(x_1, x_2, \ldots) = \varphi(z_1, z_2, x_1, x_2, \ldots), \qquad s_{\hat{1}}^t\varphi(x_1, x_2, \ldots) = \varphi(-t_1, -t_2, x_1, x_2, \ldots).$$

7.2. Divided difference operators

The divided difference operators on $\boldsymbol{R}_\infty$ are defined by

$$\partial_i f = \frac{f - s_i^z f}{\omega(\alpha_i)}, \qquad \delta_i f = \frac{f - s_i^t f}{\alpha_i},$$

where s_i and α_i $(i \in \boldsymbol{I}_\infty)$ are the simple reflections and the corresponding simple roots. Clearly we have $\delta_i = \omega\partial_i\omega$ $(i \in \boldsymbol{I}_\infty)$.

7.3. Weyl group action and commutativity with divided difference operators

Proposition 7.3. *We have* (1) $s_i^L\Phi = \Phi s_i^t$, (2) $s_i^R\Phi = \Phi s_i^z$.

Proof. We will only prove this for type C; the other types can be treated similarly. We first show (1). This is equivalent to $s_i(\Phi_{s_i v}(f)) = \Phi_v(s_i^t f)$ for all $f \in R_\infty$. If $f \in \mathbb{Z}[z, t]$ the computation is straightforward and we omit the proof. Suppose $f = \varphi(x) \in \Gamma$. We will only show $s_0(\Phi_{s_0 v}(f)) = \Phi_v(s_0^t f)$ since the case $i \geqslant 1$ is straightforward.

By Lemma 7.1, the right-hand side of this equation is written as

$$\varphi(-t_1, x_1, x_2, \ldots)|_{x_j = t_{v,j}}. \tag{7.1}$$

Let k be the (unique) index such that $v(k) = 1$ or $\bar{1}$. Then the string $t_{s_0 v}$ differs from t_v only in k-th position. If $v(k) = \bar{1}$, then $t_{v,k} = t_1$, $t_{s_0 v,k} = 0$ and $t_{v,j} = t_{s_0 v,j}$ for $j \neq k$. In this case (7.1) is

$$\varphi(-t_1, t_{v,1}, \ldots, t_{v,k-1}, t_1, t_{v,k+1}, \ldots).$$

This polynomial is equal to $\varphi(t_{v,1}, \ldots, t_{v,k-1}, t_{v,k+1}, \ldots)$ because $\varphi(x)$ is supersymmetric. It is straightforward to see that $s_0\Phi_{s_0 v}(\varphi(x))$ is equal to $\varphi(t_{v,1}, \ldots, t_{v,k-1}, t_{v,k+1}, \ldots)$. The case for $v(k) = 1$ is easier, so we left it to the reader.

Next we show (2), i.e. $\Phi_{vs_i}(f) = \Phi_v(s_i^z f)$ for all $f \in R_\infty$. Again, the case $f \in \mathbb{Z}[z, t]$ is straightforward, so we omit the proof of it. We show $\Phi_{vs_0}(\varphi(x)) = \Phi_v(s_0^z \varphi(x))$ for $\varphi(x) \in \Gamma$. The right-hand side is

$$\varphi(z_1, x_1, x_2, \ldots)|_{z_1 = v(t_1),\, x_j = t_{v,j}}, \tag{7.2}$$

where $t_{v,j} = t_{\overline{v(j)}}$ if $v(j)$ is negative and otherwise $t_{v,j}$ is zero. If $v(1) = -k$ is negative, the above function (7.2) is

$$\varphi(-t_k, t_k, t_{v,2}, t_{v,3}, \ldots).$$

This is equal to $\varphi(0, 0, t_{v,2}, t_{v,3}, \ldots)$ because φ is supersymmetric. Then also this is equal to $\varphi(0, 0, t_{v,2}, t_{v,3}, \ldots) = \varphi(0, t_{v,2}, t_{v,3}, \ldots)$ by stability property. Now since $\overline{v(1)}$ is positive we have $t_{vs_0} = (0, t_{v,2}, t_{v,3}, \ldots)$. Therefore the polynomial (7.2) coincides with $\Phi_{vs_0}(\varphi(x))$. If $v(1)$ is positive, then $t_v = (0, t_{v,2}, t_{v,3}, \ldots)$ and $t_{vs_0} = (t_{v(1)}, t_{v,2}, t_{v,3}, \ldots)$. Hence the substitution $x \mapsto t_{vs_0}$ to the function $\varphi(x_1, x_2, \ldots)$ gives rise to the polynomial (7.2). Next we show $\Phi_{vs_i}(\varphi(x)) = \Phi_v(s_i^z \varphi(x))$ for $i \geqslant 1$. First recall that $s_i^z \varphi(x) = \varphi(x)$. In this case t_{vs_i} is obtained from t_v by exchanging $t_{v,i}$ and $t_{v,i+1}$. So $\varphi(t_{vs_i}) = \varphi(t_v)$. This completes the proof. □

Using the above proposition, the next result follows:

Proposition 7.4. *The localization map $\Phi : \boldsymbol{R}_\infty \to H_\infty$ commutes with the divided difference operators both on $\boldsymbol{R}_\infty$ and H_∞, i.e.,*

$$\Phi \partial_i = \partial_i \Phi, \qquad \Phi \delta_i = \delta_i \Phi.$$

Proof. Let $f \in R_\infty$. Applying Φ on the both hand sides of equation $\omega(\alpha_i) \cdot \partial_i f = f - s_i^z f$ we have $\Phi(-\omega(\alpha_i)) \cdot \Phi(\partial_i f) = \Phi(f) - s_i^R \Phi(f)$, where we used Proposition 7.3 and linearity. Localizing at v we obtain $v(\alpha_i) \cdot \Phi_v(\partial_i f) = \Phi_v(f) - \Phi_{vs_i}(f)$. Note that we used the definition of s_i^R and $\Phi_v(\omega(\alpha_i)) = -v(\alpha_i)$. The proof for the statement regarding δ_i is similar, using $\Phi(\alpha_i) = \alpha_i$. □

7.4. *Proof of the existence and uniqueness Theorem 1.1*

Proof. (Uniqueness) Let $\{\mathfrak{S}_w\}$ and $\{\mathfrak{S}'_w\}$ be two families both satisfying the defining conditions of the double Schubert polynomials. By induction on the length of w, we see $\partial_i(\mathfrak{S}_w - \mathfrak{S}'_w) = \delta_i(\mathfrak{S}_w - \mathfrak{S}'_w) = 0$ for all $i \in \boldsymbol{I}_\infty$. This implies that the difference $\mathfrak{S}_w - \mathfrak{S}'_w$ is invariant for both left and right actions of $\boldsymbol{W}_\infty$. It is easy to see that the only such invariants in $\boldsymbol{R}_\infty$ are the constants. So $\mathfrak{S}_w - \mathfrak{S}'_w = 0$ by the constant term condition.

(Existence) Define $\mathfrak{S}_w(z, t; x) = \Phi^{-1}(\sigma_w^{(\infty)})$. By Proposition 7.4 and Proposition 5.5, $\mathfrak{S}_w(z, t; x)$ satisfies the defining equations for the double Schubert polynomials. The conditions on the constant term are satisfied since $\sigma_w^{(\infty)}$ is homogeneous of degree $\ell(w)$ (Proposition 5.4) and we have $\mathfrak{S}_e = 1$. $\square$

Remark 7.5. By construction, $\mathfrak{S}_w(z,t;x)$ satisfies the transition equation (6.1) with $\sigma_w^{(\infty)}$ replaced by $\mathfrak{S}_w(z,t;x)$. This equation provides an effective way to calculate the double Schubert polynomials.

Remark 7.6. The ring $\mathbb{Z}[z] \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ is stable under the actions of the divided difference operators ∂_i, δ_i $(i \geqslant 1)$ of type A, and the type A double Schubert polynomials $\mathfrak{S}_w^A(z,t)$, $w \in S_\infty$ form the unique family of solutions of the system of equations involving only ∂_i, δ_i for $i \geqslant 1$, and which satisfy the constant term conditions.

7.5. Projection to the cohomology of flag manifolds

We close this section with a brief discussion of the projection from $\boldsymbol{R}_\infty$ onto $H^*_{T_n}(\mathcal{F}_n)$. For $f \in \mathbb{Z}[t]$, we denote by $f^{(n)} \in \mathbb{Z}[t]^{(n)}$ the polynomial given by setting $t_i = 0$ for $i > n$ in f. Let $\mathrm{pr}_n : H_\infty \to H_n$ be the projection given by $(f_v)_{v \in \boldsymbol{W}_\infty} \mapsto (f_v^{(n)})_{v \in \boldsymbol{W}_n}$. Consider the following composition of maps

$$\pi_n : \boldsymbol{R}_\infty \xrightarrow{\Phi} H_\infty \xrightarrow{\mathrm{pr}_n} H_n \cong H^*_{T_n}(\mathcal{F}_n).$$

Explicitly, we have $\pi_n(f)|_v = \Phi_v(f)^{(n)}$ $(f \in \boldsymbol{R}_\infty,\ v \in \boldsymbol{W}_n)$. We will give an alternative geometric description for π_n in Section 10.

Proposition 7.7. *We have $\pi_n(\mathfrak{S}_w) = \sigma_w^{(n)}$ for $w \in \boldsymbol{W}_n$ and $\pi_n(\mathfrak{S}_w) = 0$ for $w \notin \boldsymbol{W}_n$. Moreover π_n commutes with divided difference operators*

$$\partial_i^{(n)} \circ \pi_n = \pi_n \circ \partial_i, \qquad \delta_i^{(n)} \circ \pi_n = \pi_n \circ \delta_i \quad (i \in \boldsymbol{I}_\infty),$$

where $\partial_i^{(n)}, \delta_i^{(n)}$ are divided difference operators on $H^*_{T_n}(\mathcal{F}_n)$.

Proof. The first statement follows from the construction of $\sigma_w^{(\infty)}$ and the vanishing property (Proposition 5.4). The second statement follows from Proposition 7.4 and the commutativity $\partial_i^{(n)} \circ \mathrm{pr}_n = \mathrm{pr}_n \circ \partial_i$, $\delta_i^{(n)} \circ \mathrm{pr}_n = \mathrm{pr}_n \circ \delta_i$ which is obvious from the construction of ∂_i, δ_i. $\square$

Corollary 7.8. *There exists an injective homomorphism of $\mathbb{Z}[t]$-algebras $\pi_\infty : \boldsymbol{R}_\infty \to \varprojlim H^*_{T_n}(\mathcal{F}_n)$.*

Proof. The proof follows from the above construction and Section 5.2. $\square$

8. Double Schubert polynomials

8.1. Basic properties

Recall that the *double Schubert polynomial* $\mathfrak{S}_w(z,t;x)$ is equal to the inverse image of the stable Schubert class $\sigma_w^{(\infty)}$ under the algebra isomorphism $\Phi : \boldsymbol{R}_\infty \to H_\infty$. In the next two sections we will study the algebraic properties of these polynomials.

Theorem 8.1. *The double Schubert polynomials satisfy the following*:

(1) (*Basis*) *The double Schubert polynomials* $\{\mathfrak{S}_w\}_{w\in\boldsymbol{W}_\infty}$ *form a* $\mathbb{Z}[t]$*-basis of* $\boldsymbol{R}_\infty$.
(2) (*Relation to Billey–Haiman's polynomials*) *For all* $w \in \boldsymbol{W}_\infty$ *we have*

$$\mathfrak{S}_w(z, 0; x) = \mathfrak{S}_w(z; x), \tag{8.1}$$

where $\mathfrak{S}_w(z; x)$ *denotes Billey–Haiman's polynomial.*
(3) (*Symmetry*) *We have* $\mathfrak{S}_w(-t, -z; x) = \mathfrak{S}_{w^{-1}}(z, t; x)$.

Proof. Property (1) holds because the stable Schubert classes $\sigma_w^{(\infty)}$ form a $\mathbb{Z}[t]$-basis for H_∞ (cf. Section 1.2). Property (2) holds because $\mathfrak{S}_w(z, 0; x) \in \mathbb{Z}[z] \otimes \Gamma'$ satisfies the defining conditions for Billey–Haiman's polynomials involving the right divided difference operators ∂_i. Then by the uniqueness of Billey–Haiman's polynomials, we have the results. For (3), set $\mathfrak{X}_w = \omega(\mathfrak{S}_{w^{-1}})$. Then by the relation $\delta_i = \omega\partial_i\omega$ we can show that $\{\mathfrak{X}_w\}$ satisfies the defining conditions of the double Schubert polynomials. So the uniqueness of the double Schubert polynomials implies $\mathfrak{X}_w = \mathfrak{S}_w$. Then we have $\omega(\mathfrak{S}_w) = \omega(\mathfrak{X}_w) = \omega(\omega\mathfrak{S}_{w^{-1}}) = \mathfrak{S}_{w^{-1}}$. $\square$

Remark 8.2. For type D we have $s_0^z s_0^t \mathfrak{D}_w = \mathfrak{D}_{\hat{w}}$ where $\hat{w}$ is the image of w under the involution of W'_∞ given by interchanging s_1 and $s_{\hat{1}}$. This is shown by the uniqueness of solution as in the proof of the symmetry property. See [6, Corollary 4.10] for the corresponding fact for the Billey–Haiman polynomials.

8.2. Relation to type A *double Schubert polynomials*

Let $\mathfrak{S}_w^A(z, t)$ denote the type A double Schubert polynomials. Recall that $\boldsymbol{W}_\infty$ has a parabolic subgroup generated by s_i $(i \geqslant 1)$ which is isomorphic to S_∞.

Lemma 8.3. *Let* $w \in \boldsymbol{W}_\infty$. *If* $w \in S_\infty$ *then* $\mathfrak{S}_w(z, t; 0) = \mathfrak{S}_w^A(z, t)$ *otherwise we have* $\mathfrak{S}_w(z, t; 0) = 0$.

Proof. The polynomials $\{\mathfrak{S}_w(z, t; 0)\}$, $w \in S_\infty$, in $\mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[z] \subset \boldsymbol{R}_\infty$ satisfy the defining divided difference equations for the double Schubert polynomials of type A (see Remark 7.6). This proves the first statement. Suppose $w \notin S_\infty$. In order to show $\mathfrak{S}_w(z, t; 0) = 0$, we use the universal localization map $\Phi^A : \mathbb{Z}[z] \otimes \mathbb{Z}[t] \to \prod_{v\in S_\infty} \mathbb{Z}[t]$ of type A, which is defined in the obvious manner. A similar proof to Lemma 6.5 shows that the map Φ^A is injective. For any $v \in S_\infty$ we have $\Phi_v(\mathfrak{S}_w(z, t; 0)) = \Phi_v(\mathfrak{S}_w(z, t; x))$, which is equal to $\sigma_w^{(\infty)}|_v$ by construction of $\mathfrak{S}_w(z, t; x)$. Since $v \ngeq w$, we have $\sigma_w^{(\infty)}|_v = 0$. This implies that the image of $\mathfrak{S}_w$ under the universal localization map Φ^A is zero, thus $\mathfrak{S}_w(z, t; 0) = 0$. $\square$

8.3. Divided difference operators and the double Schubert polynomials

We collect here some properties concerning actions of the divided difference operators on the double Schubert polynomials. These will be used in the next subsection. The next proposition seems well known (see e.g. unpublished notes [37]), but we could not find a proof in the literature.

Proposition 8.4. *Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of $w \in \boldsymbol{W}_\infty$. Then the operators*

$$\partial_w = \partial_{i_1} \cdots \partial_{i_r}, \qquad \delta_w = \delta_{i_1} \cdots \delta_{i_r}$$

do not depend on the reduced expressions and are well defined for $w \in \boldsymbol{W}_\infty$. Moreover we have

$$\partial_w \mathfrak{S}_u = \begin{cases} \mathfrak{S}_{uw^{-1}} & \text{if } \ell(uw^{-1}) = \ell(u) - \ell(w), \\ 0 & \text{otherwise}, \end{cases} \tag{8.2}$$

$$\delta_w \mathfrak{S}_u = \begin{cases} \mathfrak{S}_{wu} & \text{if } \ell(wu) = \ell(u) - \ell(w), \\ 0 & \text{otherwise}. \end{cases} \tag{8.3}$$

Proof. Since $\{\mathfrak{S}_u\}$ is a $\mathbb{Z}[t]$-basis of $\boldsymbol{R}_\infty$, Eq. (8.2) uniquely determines a $\mathbb{Z}[t]$-linear operator, which we denote by φ_w. One can prove $\partial_{i_1} \cdots \partial_{i_r} = \varphi_w$ by induction on the length of w. The proof for δ_i is similar. $\square$

Remark 8.5. The argument here is based on the existence of $\{\mathfrak{S}_w\}$, but one can also prove it in the classical way – using braid relations (cf. e.g. [3]) – by a direct calculation.

The next result will be used in the next subsection (Proposition 8.7).

Lemma 8.6. *We have $\Phi_e(\partial_u \mathfrak{S}_w) = \delta_{u,w}$.*

Proof. First note that $\Phi_e(\mathfrak{S}_w) = \sigma_w^{(\infty)}|_e = \delta_{w,e}$. If $\ell(wu^{-1}) = \ell(w) - \ell(u)$ is satisfied then by Proposition 8.4, we have $\Phi_e(\partial_u \mathfrak{S}_w) = \Phi_e(\mathfrak{S}_{wu^{-1}}) = \delta_{w,u}$. Otherwise we have $\partial_u \mathfrak{S}_w = 0$ again by Proposition 8.4. $\square$

8.4. Interpolation formulas and their applications

In this subsection we obtain an explicit combinatorial formula for the double Schubert polynomials, based on the explicit formulas for the single Schubert polynomials from [6]. The main tool for doing this is the interpolation formula, presented next.

Proposition 8.7 *(Interpolation formula). For any $f \in \boldsymbol{R}_\infty$, we have*

$$f = \sum_{w \in \boldsymbol{W}_\infty} \Phi_e\big(\partial_w(f)\big) \mathfrak{S}_w(z, t; x).$$

Proof. Since the double Schubert polynomials $\{\mathfrak{S}_w\}$ form a $\mathbb{Z}[t]$-basis of the ring $\boldsymbol{R}_\infty$, we write $f = \sum_{w \in \boldsymbol{W}_\infty} c_w \mathfrak{S}_w$, $c_w(t) \in \mathbb{Z}[t]$. As ∂_w is $\mathbb{Z}[t]$-linear we obtain by using Lemma 8.6

$$\Phi_e(\partial_w f) = \sum_{u \in \boldsymbol{W}_\infty} c_u(t) \Phi_e(\partial_w \mathfrak{S}_u) = \sum_{u \in \boldsymbol{W}_\infty} c_u(t) \delta_{w,u} = c_w(t). \qquad \square$$

Remark 8.8. Let $y = (y_1, y_2, \ldots)$ be formal parameters. On the extended ring $\mathbb{Z}[y] \otimes \boldsymbol{R}_\infty$, we can introduce the Weyl group actions, divided difference operators, and the localization map in the trivial way such that they are $\mathbb{Z}[y]$-linear. Since the elements $\mathfrak{S}_w$ ($w \in \boldsymbol{W}_\infty$) clearly form a $\mathbb{Z}[y] \otimes \mathbb{Z}[t]$-basis of $\mathbb{Z}[y] \otimes \boldsymbol{R}_\infty$, the interpolation formula holds also for any $f \in \mathbb{Z}[y] \otimes \boldsymbol{R}_\infty$.

Proposition 8.9. *Let $y = (y_1, y_2, \dots)$ be formal parameters. Then*

$$\mathfrak{S}_w(z, t; x) = \sum_{u,v} \mathfrak{S}^A_u(y, t)\mathfrak{S}_v(z, y; x)$$

summed over all $u \in S_\infty$, $v \in \boldsymbol{W}_\infty$ such that $w = uv$, $\ell(u) + \ell(v) = \ell(w)$.

Proof. By the interpolation formula (see Remark 8.8), we have

$$\mathfrak{S}_w(z, y; x) = \sum_v \Phi_e\big(\partial_v \mathfrak{S}_w(z, y; x)\big)\mathfrak{S}_v(z, t; x).$$

By Proposition 8.4, we see $\partial_v \mathfrak{S}_w(z, y; x)$ is equal to $\mathfrak{S}_{wv^{-1}}(z, y; x)$ if $\ell(wv^{-1}) = \ell(w) - \ell(v)$, and zero otherwise. Suppose $\ell(wv^{-1}) = \ell(w) - \ell(v)$, then $\Phi_e(\mathfrak{S}_{wv^{-1}}(z, y; x)) = \mathfrak{S}_{wv^{-1}}(t, y; 0)$ by the definition of Φ_e. By Lemma 8.3 this is $\mathfrak{S}^A_{wv^{-1}}(t, y)$ if $wv^{-1} = u \in S_\infty$ and zero otherwise. Then interchanging t and y we have the proposition. □

Making $y = 0$ in the previous proposition, and using that $\mathfrak{S}^A_u(y, t) = \mathfrak{S}^A_{u^{-1}}(-t, -y)$ (cf. Theorem 8.1(3) above, for type A double Schubert polynomials) we obtain

Corollary 8.10. *Let $\mathfrak{S}^A_w(z)$ denote the (single) Schubert polynomial of type* A. *We have*

$$\mathfrak{S}_w(z, t; x) = \sum_{u,v} \mathfrak{S}^A_{u^{-1}}(-t)\mathfrak{S}_v(z; x)$$

summed over all $u \in S_\infty$, $v \in \boldsymbol{W}_\infty$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$.

There is an explicit combinatorial expression for the Billey–Haiman polynomials $\mathfrak{S}_w(z; x)$ in terms of Schur Q-functions and type A (single) Schubert polynomials (cf. Theorems 3 and 4 in [6]). This, together with the above corollary implies also an explicit formula in our case. Moreover, the formula for $\mathfrak{S}_w(z; x)$ is *positive*, and therefore this yields a positivity property for the double Schubert polynomials (see Theorem 8.13 below). We will give an alternative proof for this positivity result, independent of the results from [6].

8.5. Positivity property

To prove the positivity of the double Schubert polynomials, we begin with the following lemma (compare with Theorems 3 and 4 in [6]):

Lemma 8.11. *We have $\mathfrak{S}_w(z; x) = \sum_{u,v} \mathfrak{S}^A_u(z)\mathfrak{S}_v(0, 0; x)$ summed over all $u \in \boldsymbol{W}_\infty$, $v \in S_\infty$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$.*

Remark 8.12. The function $\mathfrak{S}_v(0, 0; x)$ is the Stanley's symmetric function involved in the combinatorial expression for $\mathfrak{S}_w(z; x)$ from [6]. This follows from comparing the present lemma and the Billey–Haiman's formulas (4.6) and (4.8).

Proof. By (8.1) and symmetry property we have $\mathfrak{S}_w(z;x) = \mathfrak{S}_w(z,0;x) = \mathfrak{S}_{w^{-1}}(0,-z;x)$. Applying Proposition 8.9 with $y=0$ we can rewrite this as follows:

$$\sum_{w^{-1}=u^{-1}v^{-1}} \mathfrak{S}^A_{u^{-1}}(0,-z)\mathfrak{S}_{v^{-1}}(0,0;x) = \sum_{w=vu} \mathfrak{S}^A_u(z)\mathfrak{S}_v(0,0;x),$$

where the sum is over $v \in \boldsymbol{W}_\infty, u \in S_\infty$ such that $w^{-1} = u^{-1}v^{-1}$, and $\ell(w^{-1}) = \ell(u^{-1}) + \ell(v^{-1})$. The last equality follows from symmetry property. □

We are finally ready to prove the positivity property of $\mathfrak{S}_w(z,t;x)$. Expand $\mathfrak{S}_w(z,t;x)$ as

$$\mathfrak{S}_w(z,t;x) = \sum_{\lambda \in \mathcal{SP}} f_{w,\lambda}(z,t)F_\lambda(x),$$

where $F_\lambda(x) = Q_\lambda(x)$ for type C and $P_\lambda(x)$ for type D.

Theorem 8.13 *(Positivity of double Schubert polynomials). For any $w \in W_n$, the coefficient $f_{w,\lambda}(z,t)$ is a polynomial in $\mathbb{N}[-t_1,\ldots,-t_{n-1},z_1,\ldots,z_{n-1}]$.*

Proof. The proof follows from the expression on Corollary 8.10, Lemma 8.14 below, combined with Lemma 8.11 and the fact that $\mathfrak{S}^A_u(z) \in \mathbb{N}[z]$. □

Lemma 8.14. $\mathfrak{S}_v(0,0;x)$ *is a linear combination of Schur's Q- (respectively P-) functions with non-negative integral coefficients.*

Proof. This follows from the transition equations in Section 6.4 (see also Remark 7.5). In fact, the functions $\mathfrak{S}_w(0,0;x)$ satisfy the transition equations specialized at $z=t=0$ with the Grassmannian Schubert classes identified with the Schur's Q- or P-functions. In fact, the recursive formula for $F_w(x) = \mathfrak{S}_w(0,0;x)$ is positive, in the sense that the right-hand side of the equation is a certain non-negative integral linear combination of the functions $\{F_w(x)\}$. This implies that $F_w(x) = \mathfrak{S}_w(0,0;x)$ can be expressed as a linear combination of Schur's Q- (or P-) functions with coefficients in non-negative integers. □

9. Formula for the longest element

In this section, we give explicit formula for the double Schubert polynomials associated with the longest element $w_0^{(n)}$ in W_n (and W_n'). We note that our proof of Theorem 1.1 is independent of this section.

9.1. Removable boxes

We start this subsection with some combinatorial properties of factorial Schur Q- and P-functions. The goal is to prove Proposition 9.3, which shows how the divided difference operators act on the aforementioned functions. See Section 4.2 to recall the convention for the shifted Young diagram Y_λ.

Definition 9.1. A box $x \in Y_\lambda$ is removable if $Y_\lambda - \{x\}$ is again a shifted Young diagram of a strict partition. Explicitly, $x = (i, j)$ is removable if $j = \lambda_i + i - 1$ and $\lambda_{i+1} \leqslant \lambda_i - 2$.

To each box $x = (i, j)$ in Y_λ we define its *content* $c(x) \in I_\infty$, $c'(x) \in I'_\infty$ by $c(x) = j - i$, and $c'(x) = j - i + 1$ if $i \neq j$, $c'(i,i) = \hat{1}$ if i is odd, and $c'(i,i) = 1$ if i is even. Let $i \in I_\infty$ (resp. $i \in I'_\infty$). We call λ i*-removable* if there is a removable box x in Y_λ such that $c(x) = i$ (resp. $c'(x) = i$). Note that there is at most one such x for each $i \in I_\infty$ (resp. $i \in I'_\infty$). We say λ is i*-unremovable* if it is not i*-removable*.

Type C

0	1	2	3	4
	0	1	2	3
		0	1	
			0	

0 or 3-removable

Type D

$\hat{1}$	2	3	4	5
	1	2	3	4
		$\hat{1}$	2	
			1	

1 or 4-removable

The following facts are well known (see e.g. Section 7 in [18]).

Lemma 9.2. *Let $w_\lambda \in W_\infty^0$ (resp. $w'_\lambda \in W_\infty^{\hat{1}}$) denote the Grassmannian element corresponding to $\lambda \in \mathcal{SP}$. For $i \in I_\infty$ (resp. $i \in I'_\infty$), a strict partition λ is i-removable if and only if $\ell(s_i w_\lambda) = \ell(w_\lambda) - 1$ (resp. $\ell(s_i w'_\lambda) = \ell(w'_\lambda) - 1$). If λ is i-removable then $s_i w_\lambda$ (resp. $s_i w'_\lambda$) is also a Grassmannian element and the corresponding strict partition is the one obtained from λ by removing a (unique) box of content i.*

Proposition 9.3. *Let λ be a strict partition and $i \in I_\infty$ (resp. $i \in I'_\infty$).*

(1) *If λ is i-removable, then $\delta_i Q_\lambda(x|t) = Q_{\lambda'}(x|t)$ (resp. $\delta_i P_\lambda(x|t) = P_{\lambda'}(x|t)$), where λ' is the strict partition obtained by removing the (unique) box of content i from λ.*
(2) *If λ is i-unremovable, then $\delta_i Q_\lambda(x|t) = 0$ (resp. $\delta_i P_\lambda(x|t) = 0$), that is to say $s_i^t Q_\lambda(x|t) = Q_\lambda(x|t)$ (resp. $s_i^t P_\lambda(x|t) = P_\lambda(x|t)$).*

Proof. This follows from Lemma 9.2 and from the fact that $\mathfrak{C}_{w_\lambda} = Q_\lambda(x|t)$ and $\mathfrak{D}_{w'_\lambda} = P_\lambda(x|t)$, hence we can apply the divided difference equations from Theorem 1.1. $\square$

9.2. Type C_n *case*

For $\lambda \in \mathcal{SP}$ we define

$$K_\lambda = K_\lambda(z, t; x) = Q_\lambda(x | t_1, -z_1, t_2, -z_2, \ldots, t_n, -z_n, \ldots).$$

We need the following two lemmata to prove Theorem 1.2.

Lemma 9.4. *Set $\Lambda_n = \rho_n + \rho_{n-1}$. We have $\delta_{n-1} \cdots \delta_1 \delta_0 \delta_1 \cdots \delta_{n-1} K_{\Lambda_n} = K_{\Lambda_{n-1}}$.*

Lemma 9.5. *We have $\pi_n(K_{\Lambda_n}) = \sigma^{(n)}_{w_0^{(n)}}$, where $\pi_n : R_\infty \to H^*_{T_n}(\mathcal{F}_n)$ is the projection defined in Section* 7.5.

9.2.1. Proof of Theorem 1.2 for type C

Proof. Let $w_0^{(n)}$ in W_n be the longest element in W_n. We need to show that

$$\mathfrak{C}_{w_0^{(n)}}(z, t; x) = K_{\Lambda_n}(z, t; x). \tag{9.1}$$

Let $w \in W_\infty$. Choose any n such that $w \in W_n$ and set

$$F_w := \delta_{ww_0^{(n)}} K_{\Lambda_n}.$$

Since $\ell(ww_0^{(n)}) + 2n - 1 = \ell(ww_0^{(n+1)})$ and $ww_0^{(n)} s_n \cdots s_1 s_0 s_1 \cdots s_n = ww_0^{(n+1)}$, it follows that $\delta_{ww_0^{(n)}} \cdot \delta_n \cdots \delta_1 \delta_0 \delta_1 \cdots \delta_n = \delta_{ww_0^{(n+1)}}$. Then Lemma 9.4 yields $\delta_{ww_0^{(n+1)}} K_{\Lambda_{n+1}} = \delta_{ww_0^{(n)}} K_{\Lambda_n}$ for any $w \in W_n$, so F_w is independent of the choice of n. In order to prove the theorem it is enough to prove $F_w = \mathcal{C}_w$ for all $w \in W_\infty$.

By definition of F_w and basic properties of divided differences we can show that

$$\delta_i F_w = \begin{cases} F_{s_i w}, & \ell(s_i w) = \ell(w) - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{9.2}$$

Now we claim that $\pi_n(F_w) = \sigma_w^{(n)}$ (for any n such that $w \in W_n$). In fact, by commutativity of π_n and divided difference operators (Proposition 7.7), we have

$$\pi_n(F_w) = \delta_{ww_0^{(n)}} \pi_n(K_{\Lambda_n}) = \delta_{ww_0^{(n)}} \sigma^{(n)}_{w_0^{(n)}} = \sigma_w^{(n)}.$$

In the second equality we used Lemma 9.5, and the last equality is a consequence of (9.2). Thus the claim is proved. Since the claim holds for any sufficiently large n, we have $\Phi(F_w) = \sigma_w^{(\infty)}$ (cf. Proposition 5.1). □

9.2.2. Proof of Lemma 9.4

Proof. The lemma follows from the successive use of the following equations (see the example below):

(1) $\delta_i K_{\Lambda_n - 1^{n-i-1}} = K_{\Lambda_n - 1^{n-i}}$ $(0 \leqslant i \leqslant n-1)$,
(2) $\delta_i K_{\Lambda_n - 1^n - 0^{n-i} 1^{i-1}} = K_{\Lambda_n - 1^n - 0^{n-i-1} 1^i}$ $(1 \leqslant i \leqslant n-1)$.

We first prove (1). For the case $i = 0$, we can apply Proposition 9.3(1) directly to get the equation. Suppose $1 \leqslant i \leqslant n-1$. Before applying δ_i to $K_{\Lambda_n - 1^{n-i-1}}$ we switch the parameters at $(2i-1)$-th and $2i$-th positions to get

$$K_{\Lambda_n - 1^{n-i-1}} = Q_{\Lambda_n - 1^{n-i-1}}(x | t_1, -z_1, \ldots, -z_i, t_i, t_{i+1}, -z_{i+1}, \ldots, t_n, -z_n).$$

This is valid in view of Proposition 9.3(2) and the fact that $\Lambda_n - 1^{n-i-1}$ is $(2i-1)$-unremovable. In the right-hand side, the parameters t_i and t_{i+1} are on $2i$-th and $(2i+1)$-th positions. Thus the operator δ_i on this function is equal to the $2i$-th divided difference operator "δ_{2i}" with respect to the sequence of the rearranged parameters $(t_1, -z_1, \ldots, -z_i, t_i, t_{i+1}, -z_{i+1}, \ldots, t_n, -z_n)$ (see the example below). Thus we have by Proposition 9.3(1)

$$\delta_i K_{\Lambda_n - 1^{n-i-1}} = Q_{\Lambda_n - 1^{n-i}}(x|t_1, -z_1, \ldots, -z_i, t_i, t_{i+1}, -z_{i+1}, \ldots, t_n, -z_n),$$

namely we remove the box of content $2i$ from $\Lambda_n - 1^{n-i-1}$. Then again by Proposition 9.3(2), the last function is equal to $K_{\Lambda_n - 1^{n-i}}$; here we notice $\Lambda_n - 1^{n-i}$ is $(2i-1)$-unremovable.

Next we prove (2). In this case, by Proposition 9.3(2), we can switch $2i$-th and $(2i+1)$-th parameters to get

$$K_{\Lambda_n - 1^n - 0^{n-i}1^{i-1}} = Q_{\Lambda_n - 1^n - 0^{n-i}1^{i-1}}(x|t_1, -z_1, \ldots, -z_{i-1}, t_i, t_{i+1}, -z_i, \ldots, t_n, -z_n).$$

Here we used the fact that $\Lambda_n - 1^n - 0^{n-i}1^{i-1}$ is $2i$-unremovable. Now we apply δ_i to the function. The operator δ_i is now "δ_{2i-1}" with respect to the sequence of the rearranged parameters $(t_1, -z_1, \ldots, -z_{i-1}, t_i, t_{i+1}, -z_i, \ldots, t_n, -z_n)$. By applying Proposition 9.3(1), we have

$$\delta_i K_{\Lambda_n - 1^n - 0^{n-i}1^{i-1}} = Q_{\Lambda_n - 1^n - 0^{n-i-1}1^i}(x|t_1, -z_1, \ldots, -z_{i-1}, t_i, t_{i+1}, -z_i, \ldots, t_n, -z_n).$$

The last expression is equal to $K_{\Lambda_n - 1^n - 0^{n-i-1}1^i}$ since $\Lambda_n - 1^n - 0^{n-i-1}1^i$ is $2i$-unremovable. $\square$

Examples. Here we illustrate the process to show $\delta_2\delta_1\delta_0\delta_1\delta_2 K_{\Lambda_3} = K_{\Lambda_2}$ (case $n = 3$ in Lemma 9.4).

$$\underset{K_{5,3,1}}{\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline &0&1&2&\\ \hline &&0&&\\ \hline\end{array}} \xrightarrow{\delta_2} \underset{K_{4,3,1}}{\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline &0&1&2\\ \hline &&0&\\ \hline\end{array}} \xrightarrow{\delta_1} \underset{K_{4,2,1}}{\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline &0&1&\\ \hline &&0&\\ \hline\end{array}} \xrightarrow{\delta_0} \underset{K_{4,2}}{\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline &0&1&\\ \hline\end{array}} \xrightarrow{\delta_1} \underset{K_{4,1}}{\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline &0&&\\ \hline\end{array}} \xrightarrow{\delta_2} \underset{K_{3,1}}{\begin{array}{|c|c|c|}\hline 0&1&2\\\hline &0&\\ \hline\end{array}}$$

We pick up the first arrow: $\delta_2 K_{5,3,1} = K_{4,3,1}$ (equation in (1) in the proof of Lemma 9.4 for $n = 3$, $i = 2$). As is indicated in the proof, we divide this equality into the following four steps:

$$\begin{aligned} K_{5,3,1} &= Q_{5,3,1}(x|t_1, -z_1, \underline{t_2, -z_2}, t_3, -z_3) \underset{\text{(a)}}{=} Q_{5,3,1}(x|t_1, -z_1, -z_2, t_2, t_3, -z_3) \\ &\xrightarrow{\delta_2} Q_{4,3,1}(x|t_1, -z_1, \underline{-z_2, t_2}, t_3, -z_3) \underset{\text{(b)}}{=} Q_{4,3,1}(x|t_1, -z_1, t_2, -z_2, t_3, -z_3) = K_{4,3,1}. \end{aligned}$$

In the equality (a) we used the fact that $\Lambda_3 = (5, 3, 1)$ is 3-unremovable, so the underlined pair of variables can be exchanged (by Proposition 9.3(2)). Then we apply δ_2 to this function. Note that the variables t_2, t_3 are in the 4-th and 5-th positions in the parameters of the function. So if we rename the parameters as $f = Q_{5,3,1}(x|t_1, -z_1, -z_2, t_2, t_3, -z_3) = Q_{5,3,1}(x|u_1, u_2, u_3, u_4, u_5, u_6)$, then δ_2 is "δ_4" with respect to the parameter sequence $(u_i)_i$. Namely we have

$$\delta_2 f = \frac{f - s_2^t f}{t_3 - t_2} = \frac{f - s_4^u f}{u_5 - u_4},$$

where s_4^u exchanges u_4 and u_5. Since $\Lambda_3 = (5, 3, 1)$ is 4-removable, we see from Proposition 9.3(1) that $\delta_2 =$ “δ_4” removes the box of content 4 from $(5, 3, 1)$ to obtain the shape $(4, 3, 1)$. Then finally, in the equality (b), we exchange the variables $-z_2, t_2$ again using Proposition 9.3(2). This is valid since $(4, 3, 1)$ is 3-unremovable. Thus we obtained $K_{4,3,1}$.

9.2.3. Proof of Lemma 9.5

Proof. We calculate $\Phi_v(K_{\Lambda_n})$ for $v \in W_n$. Recall that the map $\Phi_v : R_\infty \to \mathbb{Z}[t]$ is the $\mathbb{Z}[t]$-algebra homomorphism given by $x_i \mapsto t_{v,i}$ and $z_i \mapsto t_{v(i)}$. So we have

$$\Phi_v(K_{\Lambda_n}) = Q_{\Lambda_n}(t_{v,1}, \dots, t_{v,n} | t_1, -t_{v(1)}, \dots, t_n, -t_{v(n)}).$$

Note that $t_{v,i} = 0$ for $i > n$ since v is an element in W_n. From the factorization formula (4.10), this is equal to

$$\prod_{1 \leqslant i \leqslant n} 2t_{v,i} \prod_{1 \leqslant i < j \leqslant n} (t_{v,i} + t_{v,j}) \times s_{\rho_{n-1}}(t_{v,1}, \dots, t_{v,n} | t_1, -t_{v(1)}, \dots, t_n, -t_{v(n)}).$$

The presence of the factor $\prod_i 2t_{v,i}$ implies that $\Phi_v(K_{\Lambda_n})$ vanishes unless $v(1), \dots, v(n)$ are all negative. So from now on we assume $v = (\overline{\sigma(1)}, \dots, \overline{\sigma(n)})$ for some permutation $\sigma \in S_n$. Then we have $t_{v,i} = t_{\sigma(i)}$ and $t_{v(i)} = -t_{\sigma(i)}$ so the last factor of factorial Schur polynomial becomes

$$s_{\rho_{n-1}}(t_{\sigma(1)}, \dots, t_{\sigma(n)} | t_1, t_{\sigma(1)}, \dots, t_n, t_{\sigma(n)}).$$

This is equal to $s_{\rho_{n-1}}(t_1, \dots, t_n | t_1, t_{\sigma(1)}, \dots, t_n, t_{\sigma(n)})$ because $s_{\rho_{n-1}}$ is symmetric in their first set of variables. From Lemma 4.8 we know that this polynomial factors into $\prod_{1 \leqslant i < j \leqslant n} (t_j - t_{\sigma(i)})$. This is zero except for the case $\sigma = \mathrm{id}$, namely $v = w_0^{(n)}$. If $\sigma = \mathrm{id}$ then $\Phi_v(K_{\Lambda_n})$ becomes $\prod_{1 \leqslant i \leqslant n} 2t_i \prod_{1 \leqslant i < j \leqslant n} (t_i + t_j) \prod_{1 \leqslant i < j \leqslant n} (t_j - t_i) = \sigma^{(n)}_{w_0^{(n)}}|_{w_0^{(n)}}$. □

9.3. Type D_n case

Set $K'_\lambda(z, -t; x) = P_\lambda(x | t_1, -z_1, \dots, t_{n-1}, -z_{n-1}, \dots)$. Our goal in this subsection is

$$\mathfrak{D}_{w_0^{(n)}} = K'_{2\rho_{n-1}}(z, t; x).$$

We use the same strategy as in Section 9.2 to prove this. Actually the proof in Section 9.2.1 works also in this case using the following two lemmata, which will be proved below.

Lemma 9.6. *We have* $\delta_{n-1} \cdots \delta_2 \delta_{\hat{1}} \delta_1 \delta_2 \cdots \delta_{n-1} K'_{2\rho_{n-1}} = K'_{2\rho_{n-2}}$.

Lemma 9.7. *We have* $\pi_n(K'_{2\rho_{n-1}}) = \sigma^{(n)}_{w_0^{(n)}}$.

9.3.1. A technical lemma

We need the following technical lemma which is used in the proof of Lemma 9.6. Throughout the section, $(u_1, u_2, u_3, \dots)$ denote any sequence of variables independent of t_1, t_2.

Lemma 9.8. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a strict partition such that r is odd and $\lambda_r \geqslant 3$. Set $\tilde{t} = (u_1, t_1, t_2, u_2, u_3, \dots)$. Then $\delta_{\hat{1}} P_{\lambda_1,\dots,\lambda_r,1}(x|\tilde{t}) = P_{\lambda_1,\dots,\lambda_r}(x|\tilde{t})$.*

Sublemma 9.9. *Suppose λ is $1, 2$ and $\hat{1}$-unremovable. Then we have $\delta_{\hat{1}} P_\lambda(x|\tilde{t}) = 0$.*

Proof. Since λ is $1, 2$-unremovable, we can rearrange the first three parameters by using Proposition 9.3, so we have $P_\lambda(x|\tilde{t}) = P_\lambda(x|t_1, t_2, u_1, u_2, u_3, \dots)$. Because λ is also $\hat{1}$-unremovable, it follows that $\delta_{\hat{1}} P_\lambda(t_1, t_2, u_1, u_4, \dots) = 0$ from Proposition 9.3. $\square$

Sublemma 9.10 *(Special case of Lemma 9.8 for $r = 1$). We have $\delta_{\hat{1}} P_{k,1}(x|\tilde{t}) = P_k(x|\tilde{t})$ for $k \geqslant 3$.*

Proof. Substituting $\tilde{t}$ for t into (4.2) we have

$$P_{k,1}(x|\tilde{t}) = P_k(x|\tilde{t}) P_1(x|\tilde{t}) - P_{k+1}(x|\tilde{t}) - (u_{k-1} + u_1) P_k(x|\tilde{t}).$$

By the explicit formula $P_1(x|\tilde{t}) = P_1(x)$, we have $\delta_{\hat{1}} P_1(x|\tilde{t}) = 1$. We also have $\delta_{\hat{1}} P_k(x|\tilde{t}) = \delta_{\hat{1}} P_{k+1}(x|\tilde{t}) = 0$ by Sublemma 9.9. Then we use the Leibnitz rule $\delta_{\hat{1}}(fg) = \delta_{\hat{1}}(f)g + (s_{\hat{1}} f)\delta_{\hat{1}}(g)$ to get $\delta_{\hat{1}} P_{k,1}(x|\tilde{t}) = P_k(x|\tilde{t})$. $\square$

Proof of Lemma 9.8. From the definition of the Pfaffian it follows that

$$P_{\lambda_1,\dots,\lambda_r,1}(x|\tilde{t}) = \sum_{j=1}^{r} (-1)^{r-j} P_{\lambda_j,1}(x|\tilde{t}) P_{\lambda_1,\dots,\widehat{\lambda_j},\dots,\lambda_r}(x|\tilde{t}).$$

Then the Leibnitz rule combined with Sublemma 9.9 and Sublemma 9.10 implies

$$\delta_{\hat{1}} P_{\lambda_1,\dots,\lambda_r,1}(x|\tilde{t}) = \sum_{j=1}^{r} (-1)^{r-j} P_{\lambda_j}(x|\tilde{t}) P_{\lambda_1,\dots,\widehat{\lambda_j},\dots,\lambda_r}(x|\tilde{t}) = P_{\lambda_1,\dots,\lambda_r}(x|\tilde{t}),$$

where in the last equality we used the expansion formula of Pfaffian again. $\square$

9.3.2. Proof of Lemma 9.6

Proof. Consider the case when n is even. By applying the same method of calculation as in type C case, we have

$$\delta_1 \delta_2 \cdots \delta_{n-1} K'_{2\rho_{n-1}} = P_{\rho_{n-1}+\rho_{n-2}}(x| - z_1, t_1, t_2, -z_2, t_3, -z_3, \dots).$$

The problem here is that $\rho_{n-1} + \rho_{n-2}$ is not 1-unremovable when n is even. So we cannot rewrite the function as $K'_{\rho_{n-1}+\rho_{n-2}}$. Nevertheless, by using Lemma 9.8, we can show

$$\delta_{\hat{1}} P_{\rho_{n-1}+\rho_{n-2}}(x| - z_1, t_1, t_2, -z_2, t_3, -z_3, \dots) = K'_{\rho_{n-1}+\rho_{n-2}-0^{n-2}1}.$$

The rest of calculation is similar to type C case. If n is odd, we can show this equation using only Proposition 9.3 as in type C case. $\square$

9.3.3. Proof of Lemma 9.7

Proof. Similar to the proof of Lemma 9.5 using Lemmas 4.8, 4.9, 4.11, and 4.12.

We calculate $\Phi_v(K'_{2\rho_{n-1}})$ for $v \in W'_n$. We have

$$\Phi_v\big(K'_{2\rho_{n-1}}\big) = P_{2\rho_{n-1}}(t_{v,1},\dots,t_{v,n}|t_1,-t_{v(1)},\dots,t_n,-t_{v(n)}).$$

Note that $t_{v,i}=0$ for $i>n$ since v is an element in W'_n.

Assume now that n is even. From the factorization formula (Lemma 4.11), this is equal to

$$\prod_{1\leqslant i<j\leqslant n}(t_{v,i}+t_{v,j}) \times s_{\rho_{n-1}}(t_{v,1},\dots,t_{v,n}|t_1,-t_{v(1)},\dots,t_n,-t_{v(n)}).$$

The factorial Schur polynomial factorizes further into linear terms by Lemma 4.8, and we finally obtain

$$\Phi_v\big(K'_{2\rho_{n-1}}\big) = \prod_{1\leqslant i<j\leqslant n}(t_{v,i}+t_{v,j}) \prod_{1\leqslant i<j\leqslant n}(t_j+t_{v(i)}).$$

We set $v=(\overline{\sigma(1)},\dots,\overline{\sigma(n)})$ for some $\sigma\in S_n$ since otherwise $t_{v,i}=t_{v,j}=0$ for some i,j with $i\neq j$ and then $\prod_{1\leqslant i<j\leqslant n}(t_{v,i}+t_{v,j})$ vanishes. Then the factor $\prod_{1\leqslant i<j\leqslant n}(t_j+t_{v(i)})$ is $\prod_{1\leqslant i<j\leqslant n}(t_j-t_{\sigma(i)})$. This is zero except for the case $\sigma=\mathrm{id}$, namely $v=w_0^{(n)}$. If $w=w_0^{(n)}$, we have $\Phi_v(K'_{\rho_{n-1}})=\prod_{1\leqslant i<j\leqslant n}(t_i+t_j)\prod_{1\leqslant i<j\leqslant n}(t_j-t_i)=\sigma^{(n)}_{w_0^{(n)}}|_{w_0^{(n)}}$.

Next we consider the case when n is odd. Note that the longest element $w_0^{(n)}$ in this case is $1\bar{2}\bar{3}\cdots\bar{n}$. Let $s(v)$ denote the number of non-zero entries in $t_{v,1},\dots,t_{v,n}$. Then we have $s(v)\leqslant n-1$ since $v\in W'_n$ and n is odd. We use the following identity:

$$\begin{aligned}&P_{2\rho_{n-1}}(x_1,\dots,x_{n-1}|t_1,-z_1,\dots,t_{n-1},-z_{n-1})\\ &\quad=\prod_{1\leqslant i<j\leqslant n-1}(x_i+x_j)\times s_{\rho_{n-1}+1^{n-1}}(x_1,\dots,x_{n-1}|t_1,-z_1,\dots,t_{n-1},-z_{n-1}).\end{aligned}\tag{9.3}$$

If $s(v)<n-1$, then $s(v)\leqslant n-3$ because $v\in W'_n$. This means that there are at least 3 zeros in $t_{v,1},\dots,t_{v,n}$. Because there is the factor $\prod_{1\leqslant i<j\leqslant n-1}(x_i+x_j)$ in (9.3) we have $\Phi_v(K'_{2\rho_{n-1}})=0$. So we suppose $s(v)=n-1$. By a calculation using the definition of the factorial Schur polynomial, we see that $s_{\rho_{n-1}+1^{n-1}}(x_1,\dots,x_{n-1}|t_1,-z_1,\dots,t_{n-1},-z_{n-1})$ is divisible by the factor $\prod_{i=1}^{n-1}(t_1-x_i)$. By this fact we may assume $t_{v,1},\dots,t_{v,n}$ is a permutation of $0,t_2,t_3,\dots,t_n$ since otherwise $\Phi_v(K'_{2\rho_{n-1}})$ is zero. Thus under the assumption, we have

$$\Phi_v\big(K'_{2\rho_{n-1}}\big) = \prod_{2\leqslant i<j\leqslant n}(t_i+t_j)s_{\rho_{n-1}+1^{n-1}}(t_2,\dots,t_n|t_1,-t_{v(1)},t_2,-t_{v(2)},\dots,t_{n-1},-t_{v(n-1)}).$$

By Lemma 4.9 this factorizes into $\prod_{2\leqslant i<j\leqslant n}(t_i+t_j)\prod_{j=2}^{n}(t_j-t_1)\prod_{1\leqslant i<j\leqslant n}(t_j+t_{v(i)})$. Now our assumption is that the negative elements in $\{v(1),\dots,v(n)\}$ are exactly $\{2,3,\dots,n\}$. Among these elements, only $w_0^{(n)}$ gives a non-zero polynomial, which is shown to be $\prod_{1\leqslant i<j\leqslant n}(t_i+t_j)(t_j-t_i)=\sigma^{(n)}_{w_0^{(n)}}|_{w_0^{(n)}}$. $\square$

10. Geometric construction of the universal localization map

In this section, we construct the morphism of $\mathbb{Z}[t]$-algebras $\pi_\infty : \boldsymbol{R}_\infty \to \varprojlim H^*_{T_n}(\mathcal{F}_n)$ defined in Corollary 7.8 above from a geometric point of view. We start this section by describing the embedding $\mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}$ explicitly, and calculate the localization of the Chern roots of tautological bundles. Then we introduce certain cohomology classes β_i in $H^*_{T_n}(\mathcal{F}_n)$, by using the geometry of isotropic flag varieties. These classes satisfy the relations of the Schur Q-functions $Q_i(x)$, and mapping $Q_i(x)$ to β_i ultimately leads to a homomorphism $\tilde{\pi}_\infty$, which turns out to equal π_∞. In particular, this provides an explanation on why the Schur Q-functions enter into our theory (cf. Proposition 10.4). The final goal is to establish the connection between π_∞ and the universal localization map Φ (Theorem 10.8).

The arguments in the preceding sections are logically independent from this section. However, we believe that the results in this section provide the reader with some insight into the underlying geometric idea of the algebraic construction.

10.1. Flag varieties of isotropic flags

The groups G_n are the group of automorphisms preserving a non-degenerate, bilinear form $\langle\cdot,\cdot\rangle$ on a complex vector space V_n. The pair $(V_n,\langle\cdot,\cdot\rangle)$ is the following:

(1) In type C_n, $V_n=\mathbb{C}^{2n}$; fix $\boldsymbol{e}_n^*,\dots,\boldsymbol{e}_1^*,\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ a basis for V_n. Then $\langle\cdot,\cdot\rangle$ is the skew-symmetric form given by $\langle\boldsymbol{e}_i,\boldsymbol{e}_j^*\rangle=\delta_{i,j}$ (the ordering of the basis elements will be important later, when we will embed G_n into G_{n+1}).

(2) In types B_n and D_n, V_n is an odd, respectively even-dimensional complex vector space. Let $\boldsymbol{e}_n^*,\dots,\boldsymbol{e}_1^*,\boldsymbol{e}_0,\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ respectively $\boldsymbol{e}_n^*,\dots,\boldsymbol{e}_1^*,\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ be a basis of V_n. Then $\langle\cdot,\cdot\rangle$ is the symmetric form such that $\langle\boldsymbol{e}_i,\boldsymbol{e}_j^*\rangle=\delta_{i,j}$.

A subspace V of V_n will be called *isotropic* if $\langle\boldsymbol{u},\boldsymbol{v}\rangle=0$ for any $\boldsymbol{u},\boldsymbol{v}\in V$. Then $\mathcal{F}_n$ is the variety consisting of complete isotropic flags with respect to the appropriate bilinear form. For example, in type C_n, $\mathcal{F}_n$ consists of nested sequence of vector spaces

$$F_1\subset F_2\subset\cdots\subset F_n\subset V_n=\mathbb{C}^{2n},$$

such that each F_i is isotropic and $\dim F_i=i$. Note that the maximal dimension of an isotropic subspace of $\mathbb{C}^{2n}$ is n; but the flag above can be completed to a full flag of $\mathbb{C}^{2n}$ by taking $V_{n+i}=V_{n-i}^{\perp}$, using the non-degeneracy of the form $\langle\cdot,\cdot\rangle$. A similar description can be given in types B_n and D_n, with the added condition that, in type D_n,

$$\dim F_n\cap\langle\boldsymbol{e}_n^*,\dots,\boldsymbol{e}_1^*\rangle\equiv 0 \mod 2;$$

in this case we say that all F_n are *in the same family* (cf. [14, p. 68]).

The flag variety $\mathcal{F}_n$ carries a transitive left action of the group G_n, and can be identified with the homogeneous space G_n/B_n, where B_n is the Borel subgroup consisting of upper triangular matrices in G_n. Let T_n be the maximal torus in G_n consisting of diagonal matrices in G_n. Let $t = \mathrm{diag}(\xi_n^{-1}, \ldots, \xi_1^{-1}, \xi_1, \ldots, \xi_n)$ be a torus element in types $\mathrm{C}_n, \mathrm{D}_n$, and $t = \mathrm{diag}(\xi_n^{-1}, \ldots, \xi_1^{-1}, 1, \xi_1, \ldots, \xi_n)$ in type B_n. We denote by t_i the character of T_n defined by $t \mapsto \xi_i^{-1}$ $(t \in T_n)$. Then the weight of $\mathbb{C}\boldsymbol{e}_i$ is $-t_i$ and that of $\mathbb{C}\boldsymbol{e}_i^*$ is t_i. We identify $t_i \in H^2_{T_n}(pt)$ with $c_1^T(\mathbb{C}\boldsymbol{e}_i^*)$, where $\mathbb{C}\boldsymbol{e}_i^*$ is the (trivial, but not equivariantly trivial) line bundle over pt with fiber $\mathbb{C}\boldsymbol{e}_i^*$. For $v \in W_n$, the corresponding T_n-fixed point e_v is

$$e_v : \langle \boldsymbol{e}^*_{v(n)} \rangle \subset \langle \boldsymbol{e}^*_{v(n)}, \boldsymbol{e}^*_{v(n-1)} \rangle \subset \cdots \subset \langle \boldsymbol{e}^*_{v(n)}, \boldsymbol{e}^*_{v(n-1)}, \ldots, \boldsymbol{e}^*_{v(1)} \rangle \subset V_n.$$

10.2. Equivariant embeddings of flag varieties

There is a natural embedding $G_n \hookrightarrow G_{n+1}$, given explicitly by

$$g \to \left(\begin{array}{c|c|c} 1 & & \\ \hline & g & \\ \hline & & 1 \end{array}\right).$$

This corresponds to the embedding of Dynkin diagrams in each type. This also induces embeddings $B_n \hookrightarrow B_{n+1}$, $T_n \hookrightarrow T_{n+1}$, and ultimately $\varphi_n : \mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}$. The embedding φ_n sends the complete isotropic flag $F_1 \subset \cdots \subset F_n$ of V_n to the complete isotropic flag of $V_{n+1} = \mathbb{C}\boldsymbol{e}^*_{n+1} \oplus V_n \oplus \mathbb{C}\boldsymbol{e}_{n+1}$:

$$\mathbb{C}\boldsymbol{e}^*_{n+1} \subset \mathbb{C}\boldsymbol{e}^*_{n+1} \oplus F_1 \subset \cdots \subset \mathbb{C}\boldsymbol{e}^*_{n+1} \oplus F_n.$$

Clearly φ_n is equivariant with respect to the embedding $T_n \hookrightarrow T_{n+1}$.

10.3. Localization of Chern classes of tautological bundles

Consider the flag of tautological (isotropic) vector bundles

$$0 = \mathcal{V}_{n+1} \subset \mathcal{V}_n \subset \cdots \subset \mathcal{V}_1 \subset \mathcal{E}, \qquad \operatorname{rank} \mathcal{V}_i = n - i + 1,$$

where $\mathcal{E}$ is the trivial bundle with fiber V_n and $\mathcal{V}_i$ is defined to be the vector subbundle of $\mathcal{E}$ whose fiber over the point $F_\bullet = F_1 \subset \cdots \subset F_n$ in $\mathcal{F}_n$ is F_{n-i+1}. Let $z_i = c_1^T(\mathcal{V}_i/\mathcal{V}_{i+1})$[5] denote the equivariant Chern class of the line bundle $\mathcal{V}_i/\mathcal{V}_{i+1}$.

Proposition 10.1. *Let $v \in W_n$. Then the localization map $\iota_v^* : H^*_{T_n}(\mathcal{F}_n) \to H^*_{T_n}(e_v)$ satisfies $\iota_v^*(z_i) = t_{v(i)}$.*

Proof. The pull-back of the line bundle $\mathcal{V}_i/\mathcal{V}_{i+1}$ via ι_v^* is the line bundle over e_v with fiber $\mathbb{C}\boldsymbol{e}^*_{v(i)}$, which has (equivariant) first Chern class $t_{v(i)}$. □

[5] The bundle $\mathcal{V}_i/\mathcal{V}_{i+1}$ is in fact negative; for example, in type C, if $n = 1$, $\mathcal{F}_1 = \mathbb{P}^1$ and $\mathcal{V}_1 = \mathcal{O}(-1)$. The reason for choosing positive sign for z_i is to be consistent with the conventions used by Billey and Haiman in [6].

10.4. The cohomology class β_i

In this subsection we introduce the cohomology classes β_i, which will later be identified to Schur Q-functions $Q_i(x)$.

The torus action on V_n induces a T_n-equivariant splitting $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{L}_i \oplus \mathcal{L}_i^*$ ($\mathcal{E} = \bigoplus_{i=1}^n \mathcal{L}_i \oplus \mathcal{L}_i^* \oplus \mathcal{L}_0$ for type B_n) where $\mathcal{L}_i$ (resp. $\mathcal{L}_i^*$) is the trivial line bundle over $\mathcal{F}_n$ with fiber $\mathbb{C}\boldsymbol{e}_i$ (resp. $\mathbb{C}\boldsymbol{e}_i^*$). Recall from Section 10.1 that T_n acts on $\mathcal{L}_i^*$ by weight t_i and that $t_i = c_1^T(\mathcal{L}_i^*)$.

Let $\mathcal{F}_n$ be the flag variety of type C_n or D_n and set $\mathcal{V} = \mathcal{V}_1$. We have the following exact sequence of T_n-equivariant vector bundles:

$$0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{V}^* \to 0, \tag{10.1}$$

where $\mathcal{V}^*$ denotes the dual bundle of $\mathcal{V}$ in $\mathcal{E}$ with respect to the bilinear form. Let $\mathcal{L} = \bigoplus_{i=1}^n \mathcal{L}_i$ and $\mathcal{L}^* = \bigoplus_{i=1}^n \mathcal{L}_i^*$. Since $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^*$, we have $c^T(\mathcal{E}) = c^T(\mathcal{L})c^T(\mathcal{L}^*)$. Define the class $\beta_i \in H_{T_n}^*(\mathcal{F}_n)$ by

$$\beta_i = c_i^T\big(\mathcal{V}^* - \mathcal{L}\big),$$

where $c_i^T(\mathcal{A} - \mathcal{B})$ is the term of degree i in the formal expansion of $c^T(\mathcal{A})/c^T(\mathcal{B})$. Using the relation $c^T(\mathcal{L})c^T(\mathcal{L}^*) = c^T(\mathcal{V})c^T(\mathcal{V}^*)$, we also have the expression:

$$\beta_i = c_i^T\big(\mathcal{L}^* - \mathcal{V}\big).$$

In terms of the Chern classes z_i, t_i, the class β_i has the following two equivalent expressions:

$$\sum_{i=0}^{\infty} \beta_i u^i = \prod_{i=1}^{n} \frac{1 - z_i u}{1 - t_i u} = \prod_{i=1}^{n} \frac{1 + t_i u}{1 + z_i u}. \tag{10.2}$$

Lemma 10.2. *The classes β_i satisfy the same relations as the Schur Q-functions of $Q_i(x)$, i.e.*

$$\beta_i^2 + 2\sum_{j=1}^{i} (-1)^j \beta_{i+j}\beta_{i-j} = 0 \quad \textit{for } i \geqslant 1.$$

Proof. We have the following two expressions:

$$\sum_{i=0}^{\infty} \beta_i u^i = \prod_{i=1}^{n} \frac{1 - z_i u}{1 - t_i u}, \qquad \sum_{j=0}^{\infty} (-1)^j \beta_j u^j = \prod_{i=1}^{n} \frac{1 - t_i u}{1 - z_i u}.$$

The lemma follows from multiplying both sides, and then extracting the degree $2i$ parts. $\square$

Minor modifications need to be done if $\mathcal{F}_n$ is the flag variety of type B_n. In this case the tautological sequence of isotropic flag subbundles consists of $0 = \mathcal{V}_{n+1} \subset \mathcal{V}_n \subset \cdots \subset \mathcal{V}_1 \subset \mathcal{E} = \mathbb{C}^{2n+1} \times \mathcal{F}_n$, but the dual bundle $\mathcal{V}_1^*$ of $\mathcal{V}_1$ is not isomorphic to $\mathcal{E}/\mathcal{V}_1$, which has rank $n+1$. However, the line bundle $\mathcal{V}_1^{\perp}/\mathcal{V}_1$ is equivariantly isomorphic to $\bigwedge^{2n+1} \mathcal{E}$ – cf. [14, p. 75] – so

$c_1^T(\mathcal{V}_1^{\perp}/\mathcal{V}_1) = 0$; here $\mathcal{V}_1^{\perp}$ denotes the bundle whose fiber over $V_1 \subset \cdots \subset V_n$ is the subspace of vectors in $\mathbb{C}^{2n+1}$ perpendicular to those in V_n with respect to the non-degenerate form $\langle \cdot,\cdot \rangle$. It follows that the bundle $\mathcal{E}/\mathcal{V}_1$ has (equivariant) total Chern class $(1 - z_1 u) \cdots (1 - z_n u)$, which is the same as the total Chern class of $\mathcal{V}_1^*$. Similarly, the total Chern class of $\mathcal{E}/\mathcal{L}$ with $\mathcal{L} = \bigoplus_{i=1}^n \mathcal{L}_i$ is $(1 + t_1 u) \cdots (1 + t_n u)$ and equals $c^T(\mathcal{L}^*)$. So the definition of β_i and the proofs of its properties remain unchanged.

Recall that in Section 7.5 we introduced $\pi_n : \boldsymbol{R}_\infty \to H^*_{T_n}(\mathcal{F}_n)$ by using the universal localization map Φ. The following is the key fact used in the proof of the main result of this section.

Lemma 10.3. *We have $\pi_n(Q_i(x)) = \beta_i$.*

Proof. It is enough to show that $\iota_v^*(\beta_i) = Q_i(t_v)$ for $v \in \boldsymbol{W}_n$. By Proposition 10.1 and the definition of β_i, we have

$$\iota_v^*\left(\sum_{i=0}^{\infty} \beta_i u^i\right) = \iota_v^*\left(\prod_{i=1}^{n} \frac{1 - z_i u}{1 - t_i u}\right) = \prod_{i=1}^{n} \frac{1 - t_{v(i)} u}{1 - t_i u}.$$

The factors $1 - t_{v(i)} u$ cancel out if $v(i)$ is positive, and the last expression becomes

$$\prod_{v(i)\ \text{negative}} \frac{1 - t_{v(i)} u}{1 + t_{v(i)} u} = \sum_{i=0}^{\infty} Q_i(t_v) u^i$$

where the last equality follows from the definition of $Q_i(x)$ and that of t_v. $\square$

10.5. Homomorphism $\tilde{\pi}_n$

We consider $\mathcal{F}_n$ of one of the types $\mathrm{B}_n, \mathrm{C}_n$, and D_n. We will define next the projection homomorphism from $\boldsymbol{R}_\infty$ to $H^*_{T_n}(\mathcal{F}_n)$, which will be used to construct the geometric analogue $\tilde{\pi}_n$ of π_n. Note that R_∞ is a proper subalgebra of $\boldsymbol{R}_\infty$ in types B and D. We regard $H^*_{T_n}(\mathcal{F}_n)$ as $\mathbb{Z}[t]$-module via the natural projection $\mathbb{Z}[t] \to \mathbb{Z}[t_1, \ldots, t_n]$.

Proposition 10.4. *There exists a homomorphism of graded $\mathbb{Z}[t]$-algebras $\tilde{\pi}_n : R_\infty \to H^*_{T_n}(\mathcal{F}_n)$ such that*

$$\tilde{\pi}_n\bigl(Q_i(x)\bigr) = \beta_i \quad (i \geqslant 1), \qquad \tilde{\pi}_n(z_i) = z_i \quad (1 \leqslant i \leqslant n) \quad \textit{and} \quad \tilde{\pi}_n(z_i) = 0 \quad (i > n).$$

Proof. This follows from the fact that R_∞ is generated as a $\mathbb{Z}[t]$-algebra by $Q_i(x), z_i$ $(i \geqslant 1)$, and that the ideal of relations among $Q_i(x)$ is generated by those in (4.1) (see [34, III, Section 8]). Since the elements β_i satisfy also those relations by Lemma 10.2, the result follows. $\square$

10.6. Types B *and* D

In this subsection, we extend $\tilde{\pi}_n$ from R'_∞ to $H^*_{T_n}(\mathcal{F}_n)$. The key to that is the identity $P_i(x) = \frac{1}{2} Q_i(x)$.

Proposition 10.5. *Let $\mathcal{F}_n$ be the flag variety of type* B_n *or* D_n*. Then there is an (integral) cohomology class γ_i such that $2\gamma_i = \beta_i$. Moreover, the classes γ_i satisfy the following quadratic relations*:

$$\gamma_i^2 + 2\sum_{j=1}^{i-1}(-1)^j \gamma_{i+j}\gamma_{i-j} + (-1)^i \gamma_{2i} = 0 \quad (i > 0).$$

Proof. Define $\gamma_i = \frac{1}{2}\beta_i$. Then, as in the proof of Lemma 10.3, the localization $\iota_v^*(\gamma_i) = \frac{1}{2}Q_i(t_v) = P_i(t_v)$ which is a polynomial with integer coefficients. The quadratic relations follow immediately from Lemma 10.2. □

The proposition implies immediately the following:

Proposition 10.6. *Let $\mathcal{F}_n$ be the flag variety of type* B_n *or* D_n*. There exists a homomorphism of graded $\mathbb{Z}[t]$-algebras $\tilde{\pi}_n : R'_\infty \to H^*_{T_n}(\mathcal{F}_n)$ such that*

$$\tilde{\pi}_n\big(P_i(x)\big) = \gamma_i \quad (i \geqslant 1) \quad \text{and} \quad \tilde{\pi}_n(z_i) = z_i \quad (1 \leqslant i \leqslant n) \quad \text{and} \quad \tilde{\pi}_n(z_i) = 0 \quad (i > n).$$

Remark 10.7. It is easy to see (cf. [14, Section 6.2]) that the morphism $\tilde{\pi}_n : R_\infty \to H^*_{T_n}(\mathcal{F}_n)$ is surjective in type C, and also in types B, D, but with coefficients over $\mathbb{Z}[1/2]$. But in fact, using that $\Phi : R'_\infty \to H_\infty$ is an isomorphism, one can show that surjectivity holds over $\mathbb{Z}$ as well.

10.7. The geometric interpretation of the universal localization map Φ

From Proposition 10.4 and Proposition 10.6, we have $\mathbb{Z}[t]$-algebra homomorphism $\tilde{\pi}_n : \boldsymbol{R}_\infty \to H^*_{T_n}(\mathcal{F}_n)$ for all types B, C, D. Since $\tilde{\pi}_n$ is compatible with maps $\varphi_n^* : H^*_{T_n}(\mathcal{F}_{n+1}) \to H^*_{T_n}(\mathcal{F}_n)$ induced by embeddings $\mathcal{F}_n \to \mathcal{F}_{n+1}$ there is an induced homomorphism

$$\tilde{\pi}_\infty : \boldsymbol{R}_\infty \to \varprojlim H^*_{T_n}(\mathcal{F}_n).$$

Recall from Section 7.5 that we have the natural embedding $\pi_\infty : R_\infty \hookrightarrow \varprojlim H^*_{T_n}(\mathcal{F}_n)$, defined via the localization map Φ. Then:

Theorem 10.8. *We have that $\tilde{\pi}_\infty = \pi_\infty$.*

Proof. It is enough to show that $\tilde{\pi}_n = \pi_n$. To do that, we compare both maps on the generators of $\boldsymbol{R}_\infty$. We know that $\tilde{\pi}_n(Q_i(x)) = \pi_n(Q_i(x)) = \beta_i$ by Lemma 10.3 and this implies that $\tilde{\pi}_n(P_i(x)) = \pi_n(P_i(x))$ for types B_n and D_n. It remains to show $\pi_n(z_i) = \tilde{\pi}_n(z_i)$. In this case, for $v \in \boldsymbol{W}_n$,

$$\iota_v^*\pi_n(z_i) = \Phi_v(z_i)^{(n)} = t_{v(i)}^{(n)} = \iota_v^*\tilde{\pi}_n(z_i).$$

This completes the proof. □

11. Kazarian's formula for Lagrangian Schubert classes

In this section, we give a brief discussion of a "multi-Schur Pfaffian" expression for the Schubert classes of the Lagrangian Grassmannian. This formula appeared in a preprint of Kazarian, regarding a degeneracy loci formula for the Lagrangian vector bundles [23].

11.1. Multi-Schur Pfaffian

We recall the definition of the multi-Schur Pfaffian from [23]. Let $\lambda = (\lambda_1 > \cdots > \lambda_r \geqslant 0)$ be any strict partition with r even. Consider an r-tuple of infinite sequences $c^{(i)} = \{c_k^{(i)}\}_{k=0}^{\infty}$ $(i = 1, \ldots, r)$, where each $c_k^{(i)}$ is an element in a commutative ring with unit. For $a \geqslant b \geqslant 0$, we set

$$c_{a,b}^{(i),(j)} := c_a^{(i)} c_b^{(j)} + 2\sum_{k=1}^{b} (-1)^k c_{a+k}^{(i)} c_{b-k}^{(j)}.$$

Assume that the matrix $(c_{\lambda_i,\lambda_j}^{(i),(j)})_{i,j}$ is skew-symmetric, i.e. $c_{\lambda_i,\lambda_j}^{(i),(j)} = -c_{\lambda_j,\lambda_i}^{(j),(i)}$ for $1 \leqslant i, j \leqslant r$. Then we consider its Pfaffian

$$\mathrm{Pf}_\lambda\big(c^{(1)}, \ldots, c^{(r)}\big) = \mathrm{Pf}\big(c_{\lambda_i,\lambda_j}^{(i),(j)}\big)_{1 \leqslant i<j \leqslant r},$$

called *multi-Schur Pfaffian*.

11.2. Factorial Schur functions as a multi-Schur Pfaffian

We introduce the following versions of factorial Schur Q-functions $Q_k(x|t)$:

$$\sum_{k=0}^{\infty} Q_k^{(l)}(x|t) u^k = \sum_{i=1}^{\infty} \frac{1 + x_i u}{1 - x_i u} \prod_{j=1}^{l-1} (1 - t_j u).$$

Note that, by definition, $Q_k^{(k)}(x|t) = Q_k(x|t)$ and $Q_k^{(1)}(x|t) = Q_k(x)$.

Proposition 11.1. *Let $\lambda = (\lambda_1 > \cdots > \lambda_r \geqslant 0)$ be any strict partition with r even. Set $c_k^{(i)} = Q_k^{(\lambda_i)}(x|t)$ for $i = 1, \ldots, r$. Then the matrix $(c_{\lambda_i,\lambda_j}^{(i),(j)})_{i,j}$ is skew-symmetric and we have*

$$\mathrm{Pf}_\lambda\big(c^{(1)}, \ldots, c^{(r)}\big) = Q_\lambda(x|t).$$

Proof. In view of the Pfaffian formula for $Q_\lambda(x|t)$ (Proposition 4.6), it suffices to show the following identity:

$$Q_{k,l}(x|t) = Q_k^{(k)}(x|t) Q_l^{(l)}(x|t) + 2\sum_{i=1}^{l} (-1)^i Q_{k+i}^{(k)}(x|t) Q_{l-i}^{(l)}(x|t). \tag{11.1}$$

By induction we can show that for $k \geqslant 0$

$$Q_j^{(j+k)}(x|t) = \sum_{i=0}^{j} (-1)^i e_i(t_{j+k-1}, t_{j+k-2}, \dots, t_{j-i+1}) Q_{j-i}(x|t),$$

$$Q_j^{(j-k)}(x|t) = \sum_{i=0}^{k} h_i(t_{j-k}, t_{j-k+1}, \dots, t_{j-i}) Q_{j-i}(x|t).$$

Substituting these expressions into (11.1), we get a quadratic expression in $Q_i(x|t)$'s. The obtained expression coincides with a formula for $Q_{k,l}(x|t)$ proved in [16, Proposition 7.1]. $\square$

11.3. Schubert classes in the Lagrangian Grassmannian as multi-Pfaffians

We use the notations from Section 10. The next formula expresses the equivariant Schubert class $\sigma_{w_\lambda}^{(n)}$ in a flag variety of type C in terms of a multi-Pfaffian. Recall that this is also the equivariant Schubert class for the Schubert variety indexed by λ in the Lagrangian Grassmannian, so this is a "Giambelli formula" in this case. Another such expression, in terms of ordinary Pfaffians, was proved by the first author in [16].

Proposition 11.2. *(Cf. [23, Theorem 1.1].) Set $\mathcal{U}_k = \bigoplus_{j=k}^{n} \mathcal{L}_i$. Then $\sigma_{w_\lambda}^{(n)} = \mathrm{Pf}_\lambda(c^T(\mathcal{E} - \mathcal{V} - \mathcal{U}_{\lambda_1}), \dots, c^T(\mathcal{E} - \mathcal{V} - \mathcal{U}_{\lambda_r}))$.*

Proof. By Theorem 6.6, we know $\pi_n(Q_\lambda(x|t)) = \sigma_{w_\lambda}^{(n)}$. On the other hand the formula of Proposition 11.1 writes $Q_\lambda(x|t)$ as a multi-Pfaffian. So it is enough to show that:

$$c_i^T(\mathcal{E} - \mathcal{V} - \mathcal{U}_k) = \pi_n\bigl(Q_i^{(k)}(x|t)\bigr).$$

We have

$$c^T(\mathcal{E} - \mathcal{V} - \mathcal{U}_k) = \frac{\prod_{i=1}^{n}(1 - t_i^2 u^2)}{\prod_{j=1}^{n}(1 + z_j u)\prod_{j=k}^{n}(1 - t_j u)} = \prod_{i=1}^{n} \frac{1 + t_i u}{1 + z_i u} \prod_{j=1}^{k-1}(1 - t_j u).$$

The first factor of the right-hand side is the generating function for $\beta_i = \pi_n(Q_i(x))$ $(i \geqslant 0)$. So the last expression is

$$\sum_{i=0}^{\infty} \pi_n\bigl(Q_i(x)\bigr)u^i \prod_{j=1}^{k-1}(1 - t_j u) = \sum_{i=0}^{\infty} \pi_n\bigl(Q_i^{(k)}(x|t)\bigr)u^i.$$

Hence the proposition is proved. $\square$

12. Type C double Schubert polynomials for $w \in W_3$

123	1
$\bar{1}23$	Q_1
213	$Q_1 + (z_1 - t_1)$
$\bar{2}13$	$Q_2 + Q_1(-t_1)$
$2\bar{1}3$	$Q_2 + Q_1 z_1$
$\bar{2}\bar{1}3$	Q_{21}
$1\bar{2}3$	$Q_3 + Q_2(z_1 - t_1) + Q_1(-z_1 t_1)$
$\bar{1}\bar{2}3$	$Q_{31} + Q_{21}(z_1 - t_1)$
132	$Q_1 + (z_1 + z_2 - t_1 - t_2)$
$\bar{1}32$	$2Q_2 + Q_1(z_1 + z_2 - t_1 - t_2)$
312	$Q_2 + Q_1(z_1 - t_1 - t_2) + (z_1 - t_1)(z_1 - t_2)$
$\bar{3}12$	$Q_3 + Q_2(-t_1 - t_2) + Q_1 t_1 t_2$
$3\bar{1}2$	$Q_3 + Q_{21} + Q_2(2z_1 - t_1 - t_2) + Q_1(z_1)(z_1 - t_1 - t_2)$
$\bar{3}\bar{1}2$	$Q_{31} + Q_{21}(-t_1 - t_2)$
$1\bar{3}2$	$Q_4 + Q_3(z_1 - t_1 - t_2) + Q_2(t_1 t_2 - z_1(t_1 + t_2)) + Q_1 z_1 t_1 t_2$
$\bar{1}\bar{3}2$	$Q_{41} + Q_{31}(z_1 - t_1 - t_2) + Q_{21}(t_1 t_2 - z_1(t_1 + t_2))$
231	$Q_2 + Q_1(z_1 + z_2 - t_1) + (z_1 - t_1)(z_2 - t_1)$
$\bar{2}31$	$Q_3 + Q_{21} + Q_2(z_1 + z_2 - 2t_1) + Q_1(-t_1)(z_1 + z_2 - t_1)$
321	$Q_3 + Q_{21} + Q_2(2z_1 + z_2 - 2t_1 - t_2) + Q_1(z_1 + z_2 - t_1)(z_1 - t_1 - t_2) + (z_1 - t_1)(z_1 - t_2)(z_2 - t_1)$
$\bar{3}21$	$Q_4 + Q_{31} + Q_3(z_1 + z_2 - 2t_1 - t_2) + Q_{21}(-t_1 - t_2)$ $+ Q_2(t_1 t_2 - (t_1 + t_2)(z_1 + z_2 - t_1)) + Q_1 t_1 t_2(z_1 + z_2 - t_1)$
$3\bar{2}1$	$Q_{31} + Q_3(z_1 - t_1) + Q_{21}(z_1 - t_1) + Q_2(z_1 - t_1)^2 + Q_1 z_1(-t_1)(z_1 - t_1)$
$\bar{3}\bar{2}1$	$Q_{32} + Q_{31}(-t_1) + Q_{21} t_1^2$
$2\bar{3}1$	$Q_{41} + Q_4(z_1 - t_1) + Q_{31}(z_1 - t_1 - t_2) + Q_3(z_1 - t_1)(z_1 - t_1 - t_2) + Q_{21}(t_1 t_2 - z_1(t_1 + t_2))$ $+ Q_2(z_1 - t_1)(t_1 t_2 - z_1(t_1 + t_2)) + Q_1(z_1 - t_1) z_1 t_1 t_2$
$\bar{2}\bar{3}1$	$Q_{42} + Q_{32}(z_1 - t_1 - t_2) + Q_{41}(-t_1) + Q_{31}(-t_1)(z_1 - t_1 - t_2) + Q_{21} t_1^2(z_1 - t_2)$
$23\bar{1}$	$Q_3 + Q_2(z_1 + z_2) + Q_1 z_1 z_2$
$\bar{2}3\bar{1}$	$Q_{31} + Q_{21}(z_1 + z_2)$
$32\bar{1}$	$Q_4 + Q_{31} + Q_3(2z_1 + z_2 - t_1 - t_2) + Q_{21}(z_1 + z_2)$ $+ Q_2((z_1 + z_2)(z_1 - t_1 - t_2) + z_1 z_2) + Q_1 z_1 z_2(z_1 - t_1 - t_2)$
$\bar{3}2\bar{1}$	$Q_{32} + Q_{41} + Q_{31}(z_1 + z_2 - t_1 - t_2) + Q_{21}(z_1 + z_2)(-t_1 - t_2)$
$3\bar{2}\bar{1}$	$Q_{32} + Q_{31} z_1 + Q_{21} z_1^2$
$\bar{3}\bar{2}\bar{1}$	Q_{321}
$2\bar{3}\bar{1}$	$Q_{42} + Q_{32}(z_1 - t_1 - t_2) + Q_{41} z_1 + Q_{31} z_1(z_1 - t_1 - t_2) + Q_{21} z_1^2(-t_1 - t_2)$
$\bar{2}\bar{3}\bar{1}$	$Q_{421} + Q_{321}(z_1 - t_1 - t_2)$
$13\bar{2}$	$Q_4 + Q_3(z_1 + z_2 - t_1) + Q_2(z_1 z_2 - t_1(z_1 + z_2)) + Q_1(-t_1) z_1 z_2$
$\bar{1}3\bar{2}$	$Q_{41} + Q_{31}(z_1 + z_2 - t_1) + Q_{21}(z_1 z_2 - t_1(z_1 + z_2))$
$31\bar{2}$	$Q_{41} + Q_4(z_1 - t_1) + Q_{31}(z_1 + z_2 - t_1) + Q_3(z_1 - t_1)(z_1 + z_2 - t_1) + Q_{21}(z_1 z_2 - t_1(z_1 + z_2))$ $+ Q_2(z_1 - t_1)(z_1 z_2 - t_1(z_1 + z_2)) + Q_1(z_1 - t_1) z_1 z_2(-t_1)$
$\bar{3}1\bar{2}$	$Q_{42} + Q_{32}(z_1 + z_2 - t_1) + Q_{41}(-t_1) + Q_{31}(z_1 + z_2 - t_1)(-t_1) + Q_{21} t_1^2(z_1 + z_2)$
$3\bar{1}\bar{2}$	$Q_{42} + Q_{41} z_1 + Q_{32}(z_1 + z_2 - t_1) + Q_{31} z_1(z_1 + z_2 - t_1) + Q_{21} z_1^2(z_2 - t_1)$
$\bar{3}\bar{1}\bar{2}$	$Q_{421} + Q_{321}(z_1 + z_2 - t_1)$
$1\bar{3}\bar{2}$	$Q_{43} + Q_{42}(z_1 - t_1) + Q_{32}(z_1^2 + t_1^2 - z_1 t_1) + Q_{41}(-z_1 t_1) + Q_{31} z_1(-t_1)(z_1 - t_1) + Q_{21}(z_1^2 t_1^2)$
$\bar{1}\bar{3}\bar{2}$	$Q_{431} + Q_{421}(z_1 - t_1) + Q_{321}(z_1^2 - z_1 t_1 + t_1^2)$

(*continued on next page*)

$12\bar{3}$	$Q_5 + Q_4(z_1+z_2-t_1-t_2) + Q_3(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$
	$+ Q_2(z_1z_2(-t_1-t_2)+t_1t_2(z_1+z_2)) + Q_1z_1z_2t_1t_2$
$\bar{1}2\bar{3}$	$Q_{51} + Q_{41}(z_1+z_2-t_1-t_2) + Q_{31}(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$
	$+ Q_{21}(z_1z_2(-t_1-t_2)+t_1t_2(z_1+z_2))$
$21\bar{3}$	$Q_{51} + Q_5(z_1-t_1) + Q_{41}(z_1+z_2-t_1-t_2) + Q_4(z_1-t_1)(z_1+z_2-t_1-t_2)$
	$+ Q_{31}(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2)) + Q_3(z_1-t_1)(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$
	$+ Q_{21}(z_1z_2(-t_1-t_2)+t_1t_2(z_1+z_2)) + Q_2(z_1-t_1)(z_1z_2(-t_1-t_2)+t_1t_2(z_1+z_2))$
	$+ Q_1z_1z_2t_1t_2(z_1-t_1)$
$\bar{2}1\bar{3}$	$Q_{52} + Q_{42}(z_1+z_2-t_1-t_2) + Q_{32}(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2)) + Q_{51}(-t_1)$
	$+ Q_{41}(-t_1)(z_1+z_2-t_1-t_2) + Q_{31}(-t_1)(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$
	$+ Q_{21}(t_1)^2(z_1z_2-(z_1+z_2)t_2)$
$2\bar{1}\bar{3}$	$Q_{52} + Q_{42}(z_1+z_2-t_1-t_2) + Q_{32}(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$
	$+ Q_{51}z_1 + Q_{41}z_1(z_1+z_2-t_1-t_2) + Q_{31}z_1(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$
	$+ Q_{21}z_1^2(t_1t_2-z_2(t_1+t_2))$
$\bar{2}\bar{1}\bar{3}$	$Q_{521} + Q_{421}(z_1+z_2-t_1-t_2) + Q_{321}(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$
$1\bar{2}\bar{3}$	$Q_{53} + Q_{52}(z_1-t_1) + Q_{51}(-z_1t_1) + Q_{43}(z_1+z_2-t_1-t_2) + Q_{42}(z_1-t_1)(z_1+z_2-t_1-t_2)$
	$+ Q_{41}(-z_1t_1)(z_1+z_2-t_1-t_2) + Q_{32}(z_1(z_1-t_1)(z_2-t_1-t_2)+t_1^2(z_2-t_2))$
	$+ Q_{31}(-z_1t_1)(z_1(z_2-t_1-t_2)-t_1(z_2-t_2)) + Q_{21}z_1^2t_1^2(z_2-t_2)$
$\bar{1}\bar{2}\bar{3}$	$Q_{531} + Q_{431}(z_1+z_2-t_1-t_2) + Q_{521}(z_1-t_1) + Q_{421}(z_1-t_1)(z_1+z_2-t_1-t_2)$
	$+ Q_{321}((z_1^2-z_1t_1+t_1^2)(z_2-t_2)+z_1t_1(t_1-z_1))$

13. Double Schubert polynomials in type D for $w \in W_3'$

123	1
213	$P_1 + (z_1-t_1)$
$\bar{2}\bar{1}3$	P_1
$\bar{1}\bar{2}3$	$P_2 + P_1(z_1-t_1)$
132	$2P_1 + (z_1+z_2-t_1-t_2)$
312	$P_2 + P_1(2z_1-t_1-t_2) + (z_1-t_1)(z_1-t_2)$
$\bar{3}\bar{1}2$	$P_2 + P_1(-t_1-t_2)$
$\bar{1}\bar{3}2$	$P_3 + P_2(z_1-t_1-t_2) + P_1(t_1t_2-z_1t_1-z_1t_2)$
231	$P_2 + P_1(z_1+z_2-2t_1) + (z_1-t_1)(z_2-t_1)$
321	$P_3 + P_{21} + P_2(2z_1+z_2-2t_1-t_2) + P_1(z_1^2+2z_1z_2+t_1^2+2t_1t_2-3t_1z_1-t_1z_2-t_2z_1-t_2z_2)$
	$+ (z_1-t_1)(z_1-t_2)(z_2-t_1)$
$\bar{3}\bar{2}1$	$P_{21} + P_2(-t_1) + P_1t_1^2$
$\bar{2}\bar{3}1$	$P_{31} + P_{21}(z_1-t_1-t_2) + P_3(-t_1) + P_2(-t_1)(z_1-t_1-t_2) + P_1t_1^2(z_1-t_2)$
$\bar{2}3\bar{1}$	$P_2 + P_1(z_1+z_2)$
$3\bar{2}\bar{1}$	$P_{21} + P_2z_1 + P_1z_1^2$
$\bar{3}2\bar{1}$	$P_3 + P_{21} + P_2(z_1+z_2-t_1-t_2) + P_1(z_1+z_2)(-t_1-t_2)$
$2\bar{3}\bar{1}$	$P_{31} + P_3z_1 + P_{21}(z_1-t_1-t_2) + P_2z_1(z_1-t_1-t_2) + P_1z_1^2(-t_1-t_2)$
$\bar{1}3\bar{2}$	$P_3 + P_2(z_1+z_2-t_1) + P_1(z_1z_2-t_1(z_1+z_2))$
$3\bar{1}\bar{2}$	$P_{31} + P_{21}(z_1+z_2-t_1) + P_3z_1 + P_2z_1(z_1+z_2-t_1) + P_1z_1^2(z_2-t_1)$
$\bar{3}1\bar{2}$	$P_{31} + P_{21}(z_1+z_2-t_1) + P_3(-t_1) + P_2(-t_1)(z_1+z_2-t_1) + P_1(z_1+z_2)t_1^2$
$1\bar{3}\bar{2}$	$P_{32} + P_{31}(z_1-t_1) + P_3(-z_1t_1) + P_{21}(z_1^2-z_1t_1+t_1^2) + P_2(-z_1t_1)(z_1-t_1) + P_1z_1^2t_1^2$

$\bar{1}2\bar{3}$	$P_4 + P_3(z_1+z_2-t_1-t_2) + P_2(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2)) + P_1(z_1z_2(-t_1-t_2)+t_1t_2(z_1+z_2))$
$2\bar{1}\bar{3}$	$P_{41} + P_4z_1 + P_{31}(z_1+z_2-t_1-t_2) + P_3(z_1)(z_1+z_2-t_1-t_2) + P_{21}(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$ $+ P_2z_1(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2)) + P_1z_1^2(t_1t_2-z_2t_1-z_2t_2)$
$\bar{2}1\bar{3}$	$P_{41} + P_4(-t_1) + P_{31}(z_1+z_2-t_1-t_2) + P_3(-t_1)(z_1+z_2-t_1-t_2)$ $+ P_{21}(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2)) + P_2(-t_1)(z_1z_2+t_1t_2-(z_1+z_2)(t_1+t_2))$ $+ P_1t_1^2(z_1z_2-z_1t_2-z_2t_2)$
$1\bar{2}\bar{3}$	$P_{42} + P_{32}(z_1+z_2-t_1-t_2) + P_{41}(z_1-t_1) + P_{31}(z_1-t_1)(z_1+z_2-t_1-t_2)$ $+ P_{21}(z_1^2z_2-t_1^2t_2+z_1t_1t_2-z_1z_2t_1+z_1^2(-t_1-t_2)+t_1^2(z_1+z_2)) + P_4(-z_1t_1)$ $+ P_3(-z_1t_1)(z_1+z_2-t_1-t_2) + P_2(-z_1t_1)(-z_1t_1-z_2t_1-z_1t_2+z_1z_2+t_1t_2) + P_1(z_1^2t_1^2)(z_2-t_2)$

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