Discrete Mathematics 310 (2010) 2271–2272

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

4-connected triangulations and 4-orderedness

Raiji Mukae [a,1], Kenta Ozeki [b,*,1]

[a] Faculty of Education and Human Sciences, Yokohama National University, 79-7 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan

[b] National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan

ARTICLE INFO

Article history:
Received 8 October 2009
Received in revised form 22 April 2010
Accepted 29 April 2010
Available online 4 June 2010

Keywords:
Triangulations of a surface
4-ordered

ABSTRACT

For a positive integer $k \geq 4$, a graph G is called *k-ordered*, if for any ordered set of k distinct vertices of G, G has a cycle that contains all the vertices in the designated order. Goddard (2002) [3] showed that every 4-connected triangulation of the plane is 4-ordered. In this paper, we improve this result; every 4-connected triangulation of any surface is 4-ordered. Our proof is much shorter than the proof by Goddard.

1. Introduction

A graph G is called *k-ordered* for an integer $4 \leq k \leq |V(G)|$, if for any ordered set of k distinct vertices of G, G has a cycle that contains all the vertices in the designated order. This topic has been extensively studied; see for example [1,2,4,5]. Considering the topic on the concept of "k-linked", it is known that the high connectivity guarantees the k-orderedness, in particular, every $10k$-connected graph is k-ordered [9]; see also [1]. However, little is known about the minimum connectivity that implies 4-ordered. Faudree [1] proposed the following question;

If G is a 6-connected graph, is G 4-ordered?

This question is still open. However, if we restrict ourselves to a triangulation of the plane, Goddard [3] showed that smaller connectivity assumption guarantees the 4-orderedness.

Theorem 1 (*Goddard [3]*)**.** *Let G be a 4-connected triangulation of the plane. Then G is 4-ordered.*

In this paper, we extend this result to other surfaces.

Theorem 2. *Let G be a 4-connected triangulation of any surface. Then G is 4-ordered.*

The proof of Theorem 2 is very different from that of Theorem 1. In [3], it is tried to find a contractible edge and to use the induction on $|V(G)|$. On the other hand, in this paper, we do not use the induction method. For given four vertices, we directly try to find a cycle containing such vertices in a given order. In fact, the proof of this paper is much shorter than the proof in [3].

2. Proof of Theorem 2

In order to prove Theorem 2, we use the following theorem.

* Corresponding author.
E-mail addresses: mkerij@gmail.com (R. Mukae), ozeki@nii.ac.jp, ozeki@comb.math.keio.ac.jp (K. Ozeki).

[1] Research Fellow of the Japan Society for the Promotion of Science.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.04.026

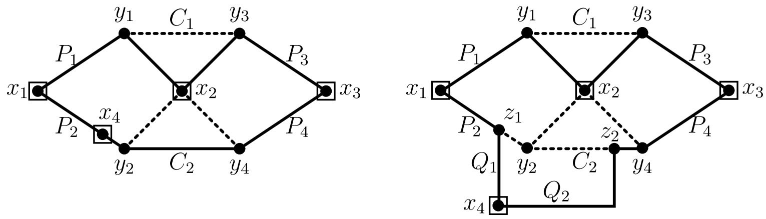

Fig. 1. The desired cycles.

Theorem 3 (*Seymour [7], Shiloach [8], Thomassen [10]*)**.** *Let G be a* 4*-connected graph, and suppose that four vertices* s_1, t_1, s_2, t_2 *are given. Then, either*

(1) *there are two disjoint paths* P_1, P_2 *such that* P_i *connects* s_i *and* t_i *for* $i = 1, 2$*; or*
(2) *there exists an embedding of G into the plane so that one face-boundary cycle contains four vertices* s_1, s_2, t_1, t_2 *in the clockwise order.*

Proof of Theorem 2. Let G be a 4-connected triangulation of a surface and let $\{x_1, x_2, x_3, x_4\}$ be an ordered set of four distinct vertices of G. We shall find a cycle containing x_1, x_2, x_3, x_4 in this order.

Case 1. For any $1 \le i \le 4$, $x_i x_{i-1} \in E(G)$ or $x_i x_{i+1} \in E(G)$.

In this case, we may assume that $x_4x_1, x_2x_3 \in E(G)$. Suppose that there exist no two disjoint paths P_1 and P_2 such that P_i connects x_{2i-1} and x_{2i} for $i = 1, 2$. It follows from Theorem 3 that G can be embedded into the plane so that one face-boundary contains x_1, x_3, x_2, x_4 in the clockwise order. If G is a triangulation of a surface which is not plane, then this is a contradiction, because G cannot be embedded into the plane. On the other hand, if G is a triangulation of the plane, then any embedding of G into the plane cannot have a non-triangular face, which is also a contradiction. In either case, we have a contradiction. Therefore there exist two disjoint paths P_1 and P_2 such that P_i connects x_{2i-1} and x_{2i} for $i = 1, 2$. So, $P_1 \cup x_2x_3 \cup P_2 \cup x_4x_1$ is a cycle containing x_1, x_2, x_3, x_4 in this order. □

Case 2. For some $1 \le i \le 4$, $x_i x_{i-1} \notin E(G)$ and $x_i x_{i+1} \notin E(G)$.

We may assume that $i = 2$, that is, $x_1x_2 \notin E(G)$ and $x_2x_3 \notin E(G)$. Since G is a triangulation, we can take a cycle C through all the vertices in $N_G(x_2)$, which is called the *link of* x_2; see Theorem 3 in [6]. By the assumption of Case 2, note that $x_1, x_3 \notin V(C)$. Since G is 4-connected, we can find four pairwise internally disjoint paths P_1, P_2, P_3, P_4 such that P_1 and P_2 connect x_1 and $V(C)$, and P_3 and P_4 connect x_3 and $V(C)$. We may assume that $|V(P_i) \cap V(C)| = 1$, say $\{y_i\} = V(P_i) \cap V(C)$. Notice that $y_i \neq y_j$ for any $1 \le i < j \le 4$. Since C is the link of x_2, note also that $x_2 \notin V(P_i)$ for every $1 \le i \le 4$. Moreover, replacing the indices of P_1 and P_2 if necessary, we may also assume that the subpath of C from y_1 to y_3, say C_1, does not intersect with the subpath of C from y_2 to y_4, say C_2. Let H be the cycle of G which consists of $P_1 \cup C_1 \cup P_3 \cup P_4 \cup C_2 \cup P_2$.

Since C is the link of x_2, $x_2y_i \in E(G)$ for any $1 \le i \le 4$. Hence x_2 can be "inserted" into H instead of C_1, that is, we can find the cycle $\left(H - C_1\right) \cup y_1x_2 \cup x_2y_2$. By the same way, we can also insert x_2 into H instead of C_2.

If $x_4 \in V(H)$, say $x_4 \in V(P_2 \cup C_2 \cup P_4)$ by the symmetry, then the cycle obtained from H by inserting x_2 instead of C_1 is the desired cycle. (See the left side of Fig. 1). Thus we may assume that $x_4 \notin V(H)$. Since G is 4-connected, we can find three pairwise internally disjoint paths Q_1, Q_2, Q_3 from x_4 to $V(H)$ in $G - x_2$. We may also assume that for $i = 1, 2$, $V(Q_i) \cap V(P_2 \cup C_2 \cup P_4) \neq \emptyset$, say $\{z_i\} = V(Q_i) \cap V(P_2 \cup C_2 \cup P_4)$. Let H' be the subpath of $P_2 \cup C_2 \cup P_4$ between z_1 and z_2. Then $(H - H') \cup Q_1 \cup Q_2$ is the cycle containing x_1, C_1, x_3, x_4 in this order. Hence the cycle obtained by inserting x_2 instead of C_2 is also the desired cycle. (See the right side of Fig. 1). This completes the proof of Theorem 2. □

Acknowledgements

The authors would like to thank Professor Ota and Professor Nakamoto for stimulating discussions and important suggestions.

References

[1] R.J. Faudree, Survey of results on k-ordered graphs, Discrete Math. 229 (2001) 73–87.
[2] R.J. Faudree, R.J. Gould, A.V. Kostochka, L. Lesniak, I. Schiermeyer, A. Saito, Degree conditions for k-ordered hamiltonian graphs, J. Graph Theory 42 (2003) 199–210.
[3] W. Goddard, 4-connected maximal planar graphs are 4-ordered, Discrete Math. 257 (2002) 405–410.
[4] K. Mészáros, On 3-regular 4-ordered graphs, Discrete Math. 308 (2008) 2149–2155.
[5] L. Ng, M. Schultz, k-ordered hamiltonian graphs, J. Graph Theory 24 (1997) 45–57.
[6] G. Ringel, Non-existence of graph embeddings, in: Theory and Applications of Graphs, in: Lecture Notes Math., vol. 642, Springer-Verlag, Berlin, 1978, pp. 465–476.
[7] P.D. Seymour, Disjoint paths in graphs, Discrete Math. 29 (1980) 293–309.
[8] Y. Shiloach, A polynomial solution to the undirected two paths problem, J. ACM 27 (1980) 445–456.
[9] R. Thomas, P. Wollan, An improved linear edge bound for graph linkages, European J. Combin. 26 (2005) 309–324.
[10] C. Thomassen, 2-linked graph, European J. Combin. 1 (1980) 371–378.