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The finite element method for a boundary value problem with strong singularity☆

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ABSTRACT

The existence and uniqueness of the R_ν-generalized solution for the third-boundary-value problem and the non-self-adjoint second-order elliptic equation with strong singularity are established. We construct a finite element method with a basis containing singular functions. The rate of convergence of the approximate solution to the R_ν-generalized solution in the norm of the Sobolev weighted space is established and, finally, results of numerical experiments are presented.

1. Introduction

There are several approaches to the treatment of boundary value problems with degenerate (singular) data. These approaches usually depend on the character of singularities arising in the problem. In the one-dimensional case, such problems and their finite element approximations were investigated in [1,2]. The finite element approximation for the special case of a right-hand side with the Dirac δ-distribution was analyzed in [3–6]. A class of problems where the coefficients and the right-hand side function have singularities on the boundary is treated in [7]. The differential properties of solutions for other elliptic boundary value problems with singularities were analyzed by standard techniques [8–11] that go back to Kondratev's technique of Mellin transformation.

Boundary value problems with strongly singular solutions (we say that u is strongly singular if $u \notin H^1$ or the Dirichlet integral of u is divergent) occur in the physics of plasmas and gas discharges, nuclear physics and other fields of physics (see for examples [12–15]).

In [16] a singular boundary value problem was investigated, for which a generalized (weak) solution in the Sobolev space $H^1(\Omega)$ could not be defined, or did not have enough regularity. Therefore it was proposed to define the solution of that boundary value problem as a R_ν-generalized one. Such a new concept of solution led to the distinction of two classes of boundary value problems: problems with coordinated and uncoordinated degeneracy of input data; it also made it possible to study the existence and uniqueness of solutions as well as its coercivity and differential properties in the weighted Sobolev spaces (see [16–20]).

In [21–23] h, p and h–p versions of the finite element method were constructed and investigated for a Dirichlet problem with strong singularity of solution.

In this paper we consider the third-boundary-value problem for a non-self-adjoint second-order elliptic equation with coordinated degeneracy of input data whose solution has strong singularities on a finite set of points belonging to the

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curvilinear boundary of a two-dimensional convex domain. For this problem we define the solution as a R_ν-generalized one; we prove its existence and uniqueness in a weighted Sobolev space. We construct and investigate the finite element method for this problem. For that purpose the domain is divided quasi-uniformly into triangles. The points of singularity of the solution of the formulated problem form a subset of the set of triangle vertices. We introduce a finite element space which contains singular functions whose form depends on the space, to which the R_ν-generalized solution of the problem belongs. It was established that the approximation to the exact R_ν-generalized solution has first-order convergence in the norm of the Sobolev weighted space. Finally, we present some results and analysis of numerical experiments for modeling singular boundary value problems using our finite element method.

2. Notation; auxiliary statements

We denote the two-dimensional Euclidean space by R^2 with $x = (x_1, x_2)$ and $dx = dx_1 dx_2$. Let $\Omega \subset R^2$ be a bounded convex domain with piecewise smooth boundary $\partial\Omega$, and let $\overline{\Omega}$ be the closure of Ω, i.e. $\overline{\Omega} = \Omega \cup \partial\Omega$. We denote by $\bigcup_{i=1}^n \tau_i$ a set of points τ_i $(i = 1, \ldots, n)$ belonging to $\partial\Omega$, including the points of intersection of its smooth pieces.

Let O_i^κ be a disk of radius $\kappa > 0$ with its center in τ_i $(i = 1, \ldots, n)$, i.e. $O_i^\kappa = \{x : \|x - \tau_i\| \le \kappa\}$, and suppose that $O_i^\kappa \cap O_j^\kappa = \emptyset,\ i \ne j$. Let $\Omega' = \Omega \bigcap \bigcup_{i=1}^n O_i^\kappa$.

Let $\rho(x)$ be a function that is infinitely differentiable, positive everywhere, except in $\bigcup_{i=1}^n \tau_i$, and satisfies the following conditions:

(a) $\rho(x) = \kappa$ for $x \in \Omega \setminus \bigcup_{i=1}^n O_i^\kappa$,

(b) $\rho(x) = \left((x_1 - x_1^{(i)})^2 + (x_2 - x_2^{(i)})^2\right)^{1/2}$, $(x_1^{(i)}, x_2^{(i)}) = \tau_i$ for $x \in \Omega \cap O_i^{\kappa/2}$,

(c) $\kappa/2 \le \rho(x) \le \kappa$ for $x \in \Omega \setminus O_i^{\kappa/2}$ $(i = 1, \ldots, n)$.

Moreover, it is assumed that

$$\left|\frac{\partial\rho}{\partial x_i}\right| \le \delta, \quad i = 1, 2. \tag{2.1}$$

We introduce the weighted spaces with norms:

$$\|u\|^2_{H^k_{2,\alpha}(\Omega)} = \sum_{|\lambda|\le k} \int_\Omega \rho^{2\alpha+2|\lambda|-2k} |D^\lambda u|^2 \, dx,$$

$$\|u\|^2_{W^k_{2,\alpha}(\Omega)} = \sum_{|\lambda|\le k} \int_\Omega \rho^{2\alpha} |D^\lambda u|^2 \, dx, \qquad \|u\|^2_{L_{2,\alpha}(\partial\Omega)} = \int_{\partial\Omega} \rho^{2\alpha} u^2 \, ds,$$

$$\|u\|_{W^k_{2,0}(\Omega)} = \|u\|_{W^k_2(\Omega)},$$

$$\|u\|^2_{V^1_{2,\alpha}(\Omega)} = \|u\|^2_{H^1_{2,\alpha}(\Omega)} + \|u\|^2_{L_{2,\alpha-1/2}(\partial\Omega)},$$

where $D^\lambda = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}}$, $\lambda = (\lambda_1, \lambda_2)$ and $|\lambda| = \lambda_1 + \lambda_2$, k is a nonnegative integer, and α is a real number.

Let $H^k_{\infty,-\alpha}(\Omega, C)$ $(k \ge 0, \alpha \in R)$ be the set of functions with the norm satisfying the inequality

$$\|u\|_{H^k_{\infty,-\alpha}(\Omega,C)} = \max_{|\lambda|\le k} \operatorname*{ess\,sup}_{x\in\Omega} |\rho^{-\alpha+|\lambda|} D^\lambda u| \le C,$$

with a positive constant C independent of u. For $k = 0$ we have $H^0_{\infty,-\alpha}(\Omega, C) = L_{\infty,-\alpha(\Omega,C)}$.

Lemma 1 ([16]). *Let k be a nonnegative integer:*
(A) If $u \in H^k_{2,\alpha}$, then $\rho^{\alpha-(k-s)} u \in W^s_{2,0} (s = 0, \ldots, k)$ and

$$|\rho^\alpha u|_{W^k_{2,0}(\Omega)} + |\rho^{\alpha-1} u|_{W^{k-1}_{2,0}(\Omega)} + \cdots + |\rho^{\alpha-k} u|_{L_{2,0}(\Omega)} \le C_1 \|u\|_{H^k_{2,\alpha}(\Omega)},$$

where C_1 is a positive constant independent of u.
(B) If $\rho^{\alpha-(k-s)} u \in W^s_{2,0}$ $(s = 0, \ldots, k)$, then $u \in H^k_{2,\alpha}(\Omega)$ and there exist positive constants $C_0^*, \ldots, C_k^*$ independent of u such that

$$C_k^* |\rho^\alpha u|_{W^k_{2,0}(\Omega)} + C_{k-1}^* |\rho^{\alpha-1} u|_{W^{k-1}_{2,0}(\Omega)} + \cdots + C_0^* |\rho^{\alpha-k} u|_{L_{2,0}(\Omega)} \ge \|u\|_{H^k_{2,\alpha}(\Omega)}.$$

Lemma 2 ([17]). *Let $\rho(x) = ((x_1 - x_1^{(i)})^2 + (x_2 - x_2^{(i)})^2)^{\alpha_2}$ for $\Omega \bigcap O_i^{\kappa/2}$, where $\alpha_2 \ge 1/2$, $(x_1^{(i)}, x_2^{(i)}) = \tau^{(i)}$, $i = 1, \ldots, n$. If $u \in H^1_{2,\alpha}(\Omega)$ and parameters α_1 and α_2 satisfy the inequalities $\alpha_1 \le 1/(2\alpha_2)$, $\alpha > \alpha_1 - 1/(4\alpha_2)$, then*

$$\|u\|_{L_{2,\alpha-\alpha_1}(\Omega')} \le \kappa^{1-2\alpha_1\alpha_2} \left(C_1^* |u|_{W^1_{2,\alpha}(\Omega')} + C_2^* \|u\|_{W^1_{2,\alpha}(\Omega\setminus\Omega')} \right),$$

where C_1^*, C_2^* are positive constants independent of u and mes Ω'.

Lemma 3. *For any function u in $H^1_{2,\alpha}(\Omega)$ the trace $u|_{\partial\Omega}$ belongs to the space $L_{2,\alpha-1/2}(\partial\Omega)$ and the inequality*

$$\|u\|_{L_{2,\alpha-1/2}(\partial\Omega)} \le C_2 \|u\|_{H^1_{2,\alpha}(\Omega)} \tag{2.2}$$

holds, where C_2 is a positive constant independent of u.

Proof. We divide the boundary $\partial\Omega$ of the domain Ω into a set of simple pieces $\{\Gamma_j\}_{j=1}^{j=2n+p}$, where one of the ends of each Γ_j ($j = 1, \ldots, 2n$) is the point of singularity τ_i, $i = \left[\frac{j+1}{2}\right]$ (where $[x]$ denotes the integer part of x), and Γ_j ($j = 2n+1, \ldots, 2n+p$) does not contain such points.

Moreover the partition $\{\Gamma_j\}_{j=1}^{j=2n}$ is such that for every Γ_j the following conditions are satisfied:

(1) Each piece Γ_j ($j = 1, \ldots, 2n$) is laid in an upper half-plane of the local Cartesian coordinate system $O'\xi_1\xi_2$ introduced, with its origin at the point τ_i $\left(i = \left[\frac{j+1}{2}\right]\right)$.

(2) Each piece Γ_j is projected uniquely into some segment D of the axis $O'\xi_1$.

(3) For each point $\xi = (\xi_1, \xi_2) \in \Gamma_j$ the following conditions are satisfied:

$$\xi_2 = \varphi(\xi_1), \quad \xi_1 \in D, \qquad \varphi(\xi_1) \ne 0 \quad \text{for } \xi_1 \in D \setminus \{0\}; \tag{2.3}$$

$$\varphi(\xi_1) \in C^1(\overline{D}) \quad \text{and} \quad \varphi'(0) \ne 0. \tag{2.4}$$

(4) The domain $\Omega_j = \{\xi_1 \in D,\ 0 \le \xi_2 \le \varphi(\xi_1)\}$ ($j = 1, \ldots, 2n$) is a subdomain of Ω.

The transformation $(\xi_1, \xi_2) \leftrightarrow (x_1, x_2)$ is a linear transformation with constant coefficients, its determinant is equal to 1 and the partial derivatives of the function u with respect to x_i ($i = 1, 2$) are certain linear combinations of the partial derivatives of u with respect to ξ_i ($i = 1, 2$).

Taking this into account we establish the necessary estimate in the coordinate system $O'\xi_1\xi_2$ and then we return to the variables x_1, x_2.

We fix $\bar{\xi}_1$ ($\bar{\xi}_1 \ne 0$) in D and denote $\varphi(\bar{\xi}_1)$ by $\bar{\xi}_2$; then the equality

$$\rho^{\alpha-1/2}(\bar{\xi}_1, \bar{\xi}_2) u(\bar{\xi}_1, \bar{\xi}_2)\Big|_{\bar{\xi}\in\Gamma_j} = \rho^{\alpha-1/2}(\bar{\xi}_1, \xi_2) u(\bar{\xi}_1, \xi_2) + \int_{\xi_2}^{\bar{\xi}_2} \frac{\partial}{\partial t}\left(\rho^{\alpha-1/2}(\bar{\xi}_1, t) u(\bar{\xi}_1, t)\right) \mathrm{d}t$$

holds for any ξ_2 from segment $[0, \bar{\xi}_2]$.

We square it and integrate the equality obtained over ξ_2 on the segment $[0, \bar{\xi}_2]$. As a result we obtain

$$\left(\rho^{\alpha-1/2}(\bar{\xi}) u(\bar{\xi})\right)^2 \le \frac{2}{\bar{\xi}_2}\left(\int_0^{\bar{\xi}_2} \rho^{2\alpha-1}(\bar{\xi}_1, \xi_2) u^2(\bar{\xi}_1, \xi_2)\, \mathrm{d}\xi_2 + \int_0^{\bar{\xi}_2}\left(\int_{\xi_2}^{\bar{\xi}_2} \frac{\partial}{\partial t}(\rho^{\alpha-1/2}(\bar{\xi}_1, t) u(\bar{\xi}_1, t))\, \mathrm{d}t\right)^2 \mathrm{d}\xi_2\right). \tag{2.5}$$

We estimate the second integral in the right-hand side of (2.5). According to the Hardy inequality (see [28]) we have

$$\int_0^{\bar{\xi}_2}\left(\int_{\xi_2}^{\bar{\xi}_2} \frac{\partial}{\partial t}(\rho^{\alpha-1/2}(\bar{\xi}_1, t) u(\bar{\xi}_1, t))\, \mathrm{d}t\right)^2 \mathrm{d}\xi_2 \le C_3 \bar{\xi}_2 \int_0^{\bar{\xi}_2} \xi_2 \left(\frac{\partial(\rho^{\alpha-1/2}(\bar{\xi}_1, \xi_2) u(\bar{\xi}_1, \xi_2))}{\partial \xi_2}\right)^2 \mathrm{d}\xi_2. \tag{2.6}$$

Multiplying both sides of (2.5) by $\sqrt{1+\varphi^2(\xi_1)}$ and integrating it on D, using (2.6) and taking into account that $\xi_2 \le \rho(\xi)$, we conclude that

$$\|u\|^2_{L_{2,\alpha-1/2}(\Gamma_j)} \le C_4 \max_{\xi_1 \in D} \frac{\rho(\xi)}{\varphi(\xi_1)} \sqrt{1+\varphi^2(\xi_1)} \|u\|^2_{L_{2,\alpha-1}(\Omega_j)} + C_5 \|u\|^2_{H^1_{2,\alpha}(\Omega_j)}. \tag{2.7}$$

Due to conditions (2.3) and (2.4) $\max_{\xi_1\in D} \frac{\rho(\xi)}{\varphi(\xi_1)}\sqrt{1+\varphi^2(\xi_1)}$ is bounded; therefore the estimate

$$\|u\|^2_{L_{2,\alpha-1/2}(\Gamma_j)} \le C_6 \|u\|^2_{H^1_{2,\alpha}(\Omega_j)}$$

follows from (2.7). Returning to the initial coordinate system Ox_1x_2, we get

$$\|u\|_{L_{2,\alpha-1/2}(\Gamma_j)} \le C_7 \|u\|_{H^1_{2,\alpha}(\Omega)}. \tag{2.8}$$

On the pieces of the boundary Γ_j ($j = 2n+1, \ldots, 2n+p$), which do not contain any singular points, by means of the theorem about the estimate of the trace of a function we have the inequality

$$\|u\|_{L_{2,\alpha-1/2}(\Gamma_j)} \le C_8 \max_{x\in\Gamma_j} \rho^{-1/2}(x) \|u\|_{H^1_{2,\alpha}(\Omega)}, \tag{2.9}$$

in which the value $\max_{x\in\Gamma_j} \rho^{-1/2}(x)$ is finite because $\rho(x)$ is bounded from below on Γ_j ($j = 2n+1, \ldots, 2n+p$).

Since $\partial\Omega = \bigcup_{j=1}^{2n+p} \Gamma_j$, then from estimates (2.8) and (2.9) we establish (2.2). □

Remark 1. We did not succeed in extending the classical trace theorem $H_2^1(\Omega) \to H_2^{1/2}(\partial\Omega)$ to the case of the weighted Sobolev spaces $H_{2,\alpha}^1(\Omega) \to H_{2,\alpha}^{1/2}(\partial\Omega)$.

Theorem 1. *On the space $H_{2,\alpha}^1(\Omega)$ the norms $H_{2,\alpha}^1(\Omega)$ and $V_{2,\alpha}^1(\Omega)$ are equivalent, i.e.*

$$\|u\|_{H_{2,\alpha}^1(\Omega)} \sim \|u\|_{V_{2,\alpha}^1(\Omega)}.$$

The proof follows from Lemma 3 directly.

Lemma 4. *If a function u belongs to $H_{2,\alpha}^2(\Omega)$, then $\rho^\alpha(x)u(x)|_{x=\tau_i} = 0$ for $i = 1, \dots, n$.*

Proof. If $u \in H_{2,\alpha}^2(\Omega)$ then according to Lemma 1, $\rho^\alpha u$ belongs to $W_2^2(\Omega)$ and by the Sobolev embedding theorem $\rho^\alpha u \in C(\overline{\Omega})$.

Assume that $(\rho^\alpha(x)u(x))^2|_{x=\tau_i} > 0$ for some number i $(i = 1, \dots, n)$. Then for τ_i there exists such a δ that for all x in Ω_{τ_i} the inequality $(\rho^\alpha(x)u(x))^2 > 0$ will be valid, where $\Omega_{\tau_i} = \{x \in \Omega, |x - \tau_i| < \delta\}$. We have

$$\int_{\Omega_{\tau_i}} (\rho^\alpha u)^2 \rho^{-4}\, dx \geq \min_{x\in\Omega_{\tau_i}} (\rho^\alpha(x)u(x))^2 \int_{\Omega_{\tau_i}} \rho^{-4}\, dx = \infty.$$

By Lemma 4 u belongs to $H_{2,\alpha}^2(\Omega)$; therefore the integral in the left-hand side of the last inequality is finite. A contradiction is obtained; hence the assumption that $(\rho^\alpha(x)u(x))^2|_{x=\tau_i} > 0$ is not true. From this the statement of Lemma 4 follows. □

3. Problem formulation

Consider the differential equation

$$-\sum_{l,s=1}^{2} a_{ls}(x)\frac{\partial^2 u}{\partial x_l \partial x_s} + \sum_{l=1}^{2} a_l(x)\frac{\partial u}{\partial x_l} + a(x)u = f(x), \quad x \in \Omega \tag{3.1}$$

with the boundary condition

$$b(x)u + \frac{\partial u}{\partial N} = \varphi(x), \quad x \in \partial\Omega. \tag{3.2}$$

Here $\frac{\partial u}{\partial N} = \sum_{l,s=1}^{2} a_{ls}(x)\frac{\partial u}{\partial x_s}\cos(n, x_l)$ is a conormal derivative, and n is the outward normal vector.

Assume that the right-hand sides of (3.1) and (3.2) satisfy

$$f \in L_{2,\mu}(\Omega), \tag{3.3}$$

$$\varphi \in L_{2,\mu-1/2}(\partial\Omega), \quad \mu \geq 1/2. \tag{3.4}$$

Definition 1. The boundary value problem (3.1)–(3.2) is called the third-boundary-value problem with coordinated degeneracy of the input data, or Problem A, if (3.3) and (3.4) hold, $a_{ls}(x) = a_{sl}(x)$ $(l, s = 1, 2)$ and for some real number β

$$a_{ls} \in H_{\infty,-\beta}^1(\Omega, C_9), \qquad a_l \in L_{\infty,-(\beta-1)}(\Omega, C_{10}) \quad (l, s = 1, 2),$$

$$a \in L_{\infty,-(\beta-2)}(\Omega, C_{11}), \tag{3.5}$$

$$\sum_{l,s=1}^{2} a_{ls}(x)\xi_l\xi_s \geq C_{12}\rho^\beta(x)\sum_{s=1}^{2}\xi_s^2, \tag{3.6}$$

$$a(x) \geq C_{13}\rho^{\beta-2}(x) \quad \text{almost everywhere on } \Omega, \tag{3.7}$$

$$b \in L_{\infty,-(\beta-1)}(\partial\Omega, C_{14}), \tag{3.8}$$

$$b(x) \geq C_{15}\rho^{\beta-1}(x) \geq 0 \quad \text{almost everywhere on } \partial\Omega, \tag{3.9}$$

where C_i $(i = 9, \dots, 14)$ are positive constants independent of x, and ξ_1 and ξ_2 are any real parameters.

Denote by

$$a_\Omega(u, v) = \int_\Omega \left[\sum_{l,s=1}^{2} a_{ls}\rho^{2\nu}\frac{\partial u}{\partial x_l}\frac{\partial v}{\partial x_s} + a_{ls}\frac{\partial \rho^{2\nu}}{\partial x_l}\frac{\partial u}{\partial x_s}v + \frac{\partial a_{ls}}{\partial x_l}\rho^{2\nu}\frac{\partial u}{\partial x_s}v + a_l\rho^{2\nu}\frac{\partial u}{\partial x_l}v + a\rho^{2\nu}uv\right] dx,$$

$$a_{\partial\Omega}(u, v) = \int_{\partial\Omega} \rho^{2\nu} buv\, ds,$$

and

$$E(u,v)=a_\Omega(u,v)+a_{\partial\Omega}(u,v),\qquad(3.10)$$
$$l(v)=\int_\Omega \rho^{2\nu} f v\,\mathrm{d}x+\int_{\partial\Omega}\rho^{2\nu}\varphi v\,\mathrm{d}s$$

the bilinear and linear forms, respectively.

Definition 2. A function u_ν in $V^1_{2,\nu+\beta/2}(\Omega)$ is called an R_ν-generalized solution of the third-boundary-value problem with coordinated degeneracy of the input data if for any v in $V^1_{2,\nu+\beta/2}(\Omega)$ the identity

$$E(u_\nu,v)=l(v)$$

holds, where ν is arbitrary but fixed and satisfies the inequality

$$\nu\geq\mu+\beta/2-1.\qquad(3.11)$$

Theorem 2. *Let conditions (2.1), (3.3)–(3.9) and (3.11) and*

$$2\left(C_9(\delta\cdot 2|\nu|+1)+\frac{1}{2}C_{10}\right)^2<C_{12}C_{13}\qquad(3.12)$$

be satisfied.

Then there exists a unique R_ν-generalized solution u_ν of the third-boundary-value problem with coordinated degeneracy of the input data in the space $V^1_{2,\nu+\beta/2}(\Omega)$, and

$$\|u_\nu\|_{V^1_{2,\nu+\beta/2}(\Omega)}\leq C_{16}\left(\|f\|_{L_{2,\mu}(\Omega)}+\|\varphi\|_{L_{2,\mu-1/2}(\partial\Omega)}\right),\qquad(3.13)$$

where the positive constant C_{16} is independent of u_ν, f and φ.

Proof. First, we show that the forms $E(u,v)$ and $l(v)$ are continuous on $V^1_{2,\nu+\beta/2}(\Omega)$. In fact, by virtue of conditions (3.5), (2.1), (3.8) and the Cauchy–Schwarz inequality we have

$$|a_\Omega(u,v)|\leq C_{17}\|u\|_{H^1_{2,\nu+\beta/2}(\Omega)}\cdot\|v\|_{H^1_{2,\nu+\beta/2}(\Omega)},\qquad(3.14)$$

$$|a_{\partial\Omega}(u,v)|\leq C_{14}\|u\|_{L_{2,\nu+\beta/2-1/2}(\partial\Omega)}\cdot\|v\|_{L_{2,\nu+\beta/2-1/2}(\partial\Omega)},\quad\forall u,\quad v\in V^1_{2,\nu+\beta/2}(\Omega).\qquad(3.15)$$

From (3.10), (3.14) and (3.15) we obtain the continuity of the bilinear form

$$|E(u,v)|\leq C_{18}\|u\|_{V^1_{2,\nu+\beta/2}(\Omega)}\cdot\|v\|_{V^1_{2,\nu+\beta/2}(\Omega)},\quad\forall u,v\in V^1_{2,\nu+\beta/2}(\Omega).\qquad(3.16)$$

Then, taking into account that for the linear form $l(v)$ the inequality

$$|l(v)|^2\leq 2\left(\left|\int_\Omega\rho^{2\nu}f v\,\mathrm{d}x\right|^2+\left|\int_{\partial\Omega}\rho^{2\nu}\varphi v\,\mathrm{d}s\right|^2\right)\qquad(3.17)$$

is valid, and conditions (3.3), (3.4) and (3.11) hold, we estimate each term in the right-hand side of (3.17) using the Cauchy–Schwarz inequality and we have

$$\begin{aligned}\left|\int_{\partial\Omega}\rho^{2\nu}\varphi v\,\mathrm{d}s\right|^2 &\leq \max_{x\in\partial\Omega}\rho^{2\nu-2\mu-\beta+2}(x)\cdot\int_{\partial\Omega}\rho^{2\mu-1}\varphi^2\,\mathrm{d}s\cdot\int_{\partial\Omega}\rho^{2\nu+\beta-1}v^2\,\mathrm{d}s\\ &\leq C_{19}\|\varphi\|^2_{L_{2,\mu-1/2}(\partial\Omega)}\cdot\|v\|^2_{L_{2,\nu+\beta/2-1/2}(\partial\Omega)},\end{aligned}\qquad(3.18)$$

$$\left|\int_\Omega\rho^{2\nu}f v\,\mathrm{d}x\right|^2\leq C_{20}\|f\|^2_{L_{2,\mu}(\Omega)}\cdot\|v\|^2_{L_{2,\nu+\beta/2-1}(\Omega)}.\qquad(3.19)$$

From estimates (3.18), (3.19) and (3.17) we get

$$|l(v)|\leq C_{21}\|v\|_{V^1_{2,\nu+\beta/2}(\Omega)}\cdot(\|f\|_{L_{2,\mu}(\Omega)}+\|\varphi\|_{L_{2,\mu-1/2}(\partial\Omega)}).\qquad(3.20)$$

Let us now prove the $V^1_{2,\nu+\beta/2}(\Omega)$-ellipticity of the bilinear form $E(u,v)$; that is,

$$\exists C_{22}>0,\ \forall u\in V^1_{2,\nu+\beta/2}(\Omega)\quad E(u,u)\geq C_{22}\|u\|^2_{V^1_{2,\nu+\beta/2}(\Omega)}.\qquad(3.21)$$

Substituting v by u in (3.10) we have

$$E(u,u)=a_{\Omega}(u,u)+a_{\partial\Omega}(u,u).$$

By means of the Cauchy–Schwarz inequality, ε-inequality and conditions (2.1), (3.5)–(3.7) and (3.12), by analogy with [21], we establish the lower bound form $a_{\Omega}(u,u)$:

$$a_{\Omega}(u,u)\geq C_{23}\left(\int_{\Omega}\rho^{2\nu+\beta}\sum_{l=1}^{2}\left(\frac{\partial u}{\partial x_l}\right)^2\mathrm{d}x+\int_{\Omega}\rho^{2\nu+\beta-2}u^2\,\mathrm{d}x\right)=C_{23}\|u\|^2_{H^1_{2,\nu+\beta/2}(\Omega)},$$

with constant $C_{23}=\min(C_{12}-\sqrt{2}(C_9(\delta\cdot 2|\nu|+1)+\frac{1}{2}C_{10})\varepsilon, C_{13}-\sqrt{2}(C_9(\delta\cdot 2|\nu|+1)+\frac{1}{2}C_{10})\varepsilon^{-1})$.

Note that, if the condition (3.12) is satisfied, then there exists a positive constant ε such that

$$C_{12}-\sqrt{2}\left(C_9\left(\delta\cdot 2|\nu|+1\right)+\frac{1}{2}C_{10}\right)\varepsilon>0$$

and

$$C_{13}-\sqrt{2}\left(C_9\left(\delta\cdot 2|\nu|+1\right)+\frac{1}{2}C_{10}\right)\varepsilon^{-1}>0.$$

Using (3.9), for $a_{\partial\Omega}(u,u)$ we obtain

$$a_{\partial\Omega}(u,u)\geq C_{15}\int_{\partial\Omega}\rho^{2\nu+\beta-1}u^2\,\mathrm{d}s=C_{15}\|u\|^2_{L_{2,\nu+\beta/2-1/2}(\partial\Omega)}.$$

Then from the two last estimates we have (3.21) with constant $C_{22}=\min(C_{15},C_{23})$.

According to (3.16), (3.20) and (3.21) the bilinear form $E(u_\nu,v)$ is continuous and $V^1_{2,\nu+\beta/2}(\Omega)$-elliptical, and the linear form $l(v)$ is continuous on $V^1_{2,\nu+\beta/2}(\Omega)$; then the existence and uniqueness of an R_ν-generalized solution of Problem A follows from the Lax–Milgram theorem (see [24]).

Taking into account that

$$\begin{aligned}C_{22}\|u_\nu\|^2_{V^1_{2,\nu+\beta/2}(\Omega)}&\leq E(u_\nu,u_\nu)=l(u_\nu)\\&\leq C_{21}\|u_\nu\|_{V^1_{2,\nu+\beta/2}(\Omega)}\cdot(\|f\|_{L_{2,\mu}(\Omega)}+\|\varphi\|_{L_{2,\mu-1/2}(\partial\Omega)}),\end{aligned}$$

we get the estimate (3.13). □

Corollary 3.1. *If there exists at least one ν for which there exists a unique R_ν-generalized solution of the problem* A*, then one can always define a half-open interval $[\nu_1,\nu_2)$ such that for each $\nu\in[\nu_1,\nu_2)$, there exists a unique R_ν-generalized solution. Here*

$$\nu_1=\max\left\{\mu+\beta/2-1,\ \frac{1}{2\delta}\cdot\left(1-\frac{\sqrt{2C_{12}\cdot C_{13}}-C_{10}}{2C_9}\right)+\varepsilon\right\},$$
$$\nu_2=\frac{1}{2\delta}\left(\frac{\sqrt{2C_{12}\cdot C_{13}}-C_{10}}{2C_9}-1\right),$$

where ε is a given sufficiently small positive number.

Corollary 3.1 follows from the proof of Theorem 2.

Corollary 3.2. *If the assumptions of Theorem 2 are valid, then for all ν in the interval $[\nu_1,\nu_2)$, the R_ν-generalized solution of the problem A is unique.*

The proof of Corollary 3.2 is similar to that of Theorem 2 in [19].

Theorem 3. *Suppose that the conditions of Theorem 2 are satisfied.*

Then there exists a unique R_ν-generalized solution u_ν of the problem (3.1)–(3.2) with coordinated degeneracy of the input data in the space $H^1_{2,\nu+\beta/2}(\Omega)$ and

$$\|u_\nu\|_{H^1_{2,\nu+\beta/2}(\Omega)}\leq C_{24}(\|f\|_{L_{2,\mu}(\Omega)}+\|\varphi\|_{L_{2,\mu-1/2}(\partial\Omega)}),\tag{3.22}$$

where the positive constant C_{24} is independent of u_ν, f and φ.

The proof of this theorem is based on Theorems 1 and 2.

In this paper we do not propose to study the differentiability properties of the R_ν-generalized solution of problem A. Therefore we assume that if the assumptions of Theorem 2 are satisfied and the inequalities

$$\nu \geq \mu + \beta/2, \qquad \nu + \beta/2 > 2 \tag{3.23}$$

hold, then u_ν belongs to the space $H^2_{2,\nu+\beta/2}(\Omega)$.

Remark 2. To guarantee that the R_ν-generalized solution belongs to the Sobolev space $H^2_{2,\nu+\beta/2}(\Omega)$, the parameter ν must satisfy the condition (3.23), which is stronger than (3.11) (see [19]). The coercivity and differential properties of R_ν-generalized solutions for boundary value problems with strong singularity were analyzed in [16,19,20,25,26].

4. Construction of the scheme of the finite element method

We construct the scheme of the finite element method for determination of an R_ν-generalized solution of the third-boundary-value problem with coordinated degeneracy of the input data. For that purpose we perform a quasi-uniform triangulation of the domain Ω and we introduce a special system of basis functions.

Let Ω be embedded into a polygon $\hat{\Omega}$. We triangulate $\hat{\Omega}$ so that: (1) only sides or vertices can be common for the triangles $\{K\} = \{K_1, \dots, K_N\}$; (2) the points τ_i $(i = 1, \dots, n)$ belong to the set of vertices of the triangles K; (3) the smallest of the angles of the triangles is always strictly positive and independent of the triangulation; (4) all the triangles K_j have areas of identical order; (5) the values of mes (K_i) and mes $(K_j \cap \Omega)$, if $i \neq j$, also have identical order. We denote by Ω_h the union of all the triangles K_j $(\Omega_h = \bigcup_{i=1}^N K_i)$; h is the maximal length of the sides of the triangles. The vertices $P_1, \dots, P_{N_h}$ of the triangles K will be called the nodes of the triangulation. To each node P_i, except the nodes coincident with the points τ_i, we assign the function

$$\psi_i(x) = \rho^{-(\nu+\beta/2)}(x)\varphi_i(x), \quad i = 1, \dots, N_h - n,$$

where $\varphi_i(x)$ is linear on each triangle K, equal to 1 at the point P_i and zero at all the other nodes. We denote by $V^h(\Omega_h)$ the linear span $\{\psi_i\}_{i=1}^{N_h-n}$. Obviously, $V^h \subset H^1_{2,\nu+\beta/2}(\Omega_h)$. We shall approximate the R_ν-generalized solution of problem A on the restriction of this space to Ω.

A function u_ν^h in the space $V^h(\Omega)$ satisfying the equality

$$E(u_\nu^h, v^h) = l(v^h), \quad \forall v^h \in V^h$$

is called the approximate (finite element) R_ν-generalized solution of problem A.

An approximate solution will be found in the form

$$u_\nu^h(x) = \sum_{i=1}^{N_h-n} a_i\, \psi_i(x),$$

where $a_i = \rho^{\nu+\beta/2}(P_i)b_i$.

The coefficients a_i are defined from the system of equations

$$E(u_\nu^h, \psi_i) = l(\psi_i), \quad i = 1, \dots, N_h - n \tag{4.1}$$

or

$$\widehat{A}a = F,$$

where

$$a = (a_1, \dots, a_{N_h-n})^T, \qquad F = (F_1, \dots, F_{N_h-n})^T, \qquad \widehat{A} = (A_{ij}),$$
$$A_{ij} = A_{ji} = a_{\Omega_{ij}}(\psi_i, \psi_j) + a_{\partial\Omega_{ij}}(\psi_i, \psi_j),$$
$$F_i = \int_{\Omega_i} \rho^{2\nu} f\, \psi_i\, dx + \int_{\partial\Omega_j} \rho^{2\nu} \varphi \psi_i\, ds,$$
$$\partial\Omega_{ij} = \partial\Omega \cap \text{supp}\, \psi_i \cap \text{supp}\, \psi_j, \qquad \partial\Omega_i = \partial\Omega \cap \text{supp}\, \psi_i,$$
$$\Omega_{ij} = \Omega \cap \text{supp}\, \psi_i \cap \text{supp}\, \psi_j, \qquad \Omega_i = \Omega \cap \text{supp}\, \psi_i, \quad i, j = 1, \dots, N_h - n.$$

It is obvious that the approximate R_ν-generalized solution of problem A by the finite element method exists and is unique.

5. The estimate of the convergence rate of the finite element method in the space $H^1_{2,\nu+\beta/2}(\Omega)$

We establish an a priori estimate for the error $u_\nu - u_\nu^h$ in $H^1_{2,\nu+\beta/2}(\Omega)$ norm.

Lemma 5. *Let u_ν be the R_ν-generalized solution of problem* A, *and u_ν^h its approximate R_ν-generalized solution obtained by the finite element method. Then there exists a positive constant C_{26} independent of the space V^h such that*

$$\|u_\nu - u_\nu^h\|_{H^1_{2,\nu+\beta/2}(\Omega)} \leq C_{26} \inf_{v^h \in V^h} \|u_\nu - v^h\|_{H^1_{2,\nu+\beta/2}(\Omega)}.$$

By virtue of the continuity on $H^1_{2,\nu+\beta/2}(\Omega)$ and $H^1_{2,\nu+\beta/2}(\Omega)$-ellipticity of the bilinear form $E(u, v)$ the last inequality is established by analogy with [24, p. 109].

If the function $u_\nu \in H^2_{2,\nu+\beta/2}(\Omega)$ then $\rho^{\nu+\beta/2}u_\nu$ belongs to $W^2_2(\Omega)$ by Lemma 1. According to the theorem on the extension of function from the domain with piecewise smooth boundary (see, e.g., [27]) the function $\rho^{\nu+\beta/2}u_\nu$ can be extended to Ω_h so that its extension $(\rho^{\nu+\beta/2}u_\nu)^*$ belongs to $W^2_2(\Omega_h)$. Moreover, by Lemma 4 $(\rho^{\nu+\beta/2}(x)u_\nu(x))^*|_{x=\tau_i} = 0$ for $i = 1, \dots, n$.

For the function $(\rho^{\nu+\beta/2}u_\nu)^*$ in $W^2_2(\Omega_h)$ we use its values at the nodes of triangulation to construct the interpolant

$$u_{\nu,I}(x) = \sum_{i=1}^{N_h-n} (\rho^{\nu+\beta/2}(P_i)u_\nu(P_i))^*\psi_i(x).$$

Since $\inf_{v^h\in V^h} \|u_\nu - v^h\|_{H^1_{2,\nu+\beta/2}(\Omega)} \le \|u_\nu - u_{\nu,I}\|_{H^1_{2,\nu+\beta/2}(\Omega)}$ we obtain first the estimate for the error of the approximation of u_ν by the interpolant $u_{\nu,I}$.

Theorem 4. *Suppose that $u_\nu \in H^2_{2,\nu+\beta/2}(\Omega)$. Then the estimate*

$$\|u_\nu - u_{\nu,I}\|_{H^1_{2,\nu+\beta/2}(\Omega)} \le C_{27} \cdot h \cdot \|u_\nu\|_{H^2_{2,\nu+\beta/2}(\Omega)} \tag{5.1}$$

holds for a given triangulation of the domain Ω, and the positive constant C_{27} is independent of h and u_ν.

Proof. The functions u_ν and $u_{\nu,I}$ belong to the spaces $H^1_{2,\nu+\beta/2}(\Omega)$ and $H^1_{2,\nu+\beta/2}(\Omega_h)$. By Lemma 1 $\rho^{\nu+\beta/2}(u_\nu - u_{\nu,I}) \in W^1_2(\Omega)$, $\rho^{\nu+\beta/2-1}(u_\nu - u_{\nu,I}) \in L_2(\Omega)$ and the inequality

$$\|u_\nu - u_{\nu,I}\|_{H^1_{2,\nu+\beta/2}(\Omega)} \le C_{28}|\rho^{\nu+\beta/2}(u_\nu - u_{\nu,I})|_{W^1_2(\Omega)} + C_{29}\|\rho^{\nu+\beta/2-1}(u_\nu - u_{\nu,I})\|_{L_2(\Omega)} \tag{5.2}$$

holds.

We separately estimate each term in the right-hand side of (5.2). For the first term we have

$$\begin{aligned}|\rho^{\nu+\beta/2}(u_\nu - u_{\nu,I})|_{W^1_2(\Omega)} &= |(\rho^{\nu+\beta/2}u_\nu)^* - \rho^{\nu+\beta/2}u_{\nu,I}|_{W^1_2(\Omega)} \\ &\le |(\rho^{\nu+\beta/2}u_\nu)^* - \rho^{\nu+\beta/2}u_{\nu,I}|_{W^1_2(\Omega_h)}.\end{aligned} \tag{5.3}$$

We establish an estimate for the last term in (5.3).

Consider an arbitrary triangle K in Ω_h. Introduce a new local Cartesian coordinate system $O'\xi_1\xi_2$ making a translation and rotation of axes Ox_1 and Ox_2 in such a way that the triangle K becomes a subset of the square $\omega = [-\delta, \delta] \times [-\delta, \delta]$, where δ is a positive number of order $O(h)$.

As the transformation $(\xi_1, \xi_2) \leftrightarrow (x_1, x_2)$ is a linear transformation with constant coefficients, its determinant is equal to 1 and partial derivatives of the function $q(x)$ with respect to x_j $(j = 1, 2)$ are linear combinations of partial derivatives of $q(x)$ with respect to ξ_j $(j = 1, 2)$.

We establish an estimate for the error of the interpolation in $O'\xi_1\xi_2$ and then we return to variables x_1, x_2.

Denote by $\overline{(\rho^{\nu+\beta/2}u_\nu)^*}$ the extension of the function $(\rho^{\nu+\beta/2}u_\nu)^*$ from K to the square ω with the conservation of class; by $\overline{\rho^{\nu+\beta/2}u_{\nu,I}}$ we mean the expression $\sum_{i=1}^{3}(\rho^{\nu+\beta/2}(P_i)u_\nu(P_i))^*\bar{\varphi}_i(\xi)$, where $\bar{\varphi}_i(\xi)$ is such that $\bar{\varphi}_i(\xi) = 0$ is the equation of part of plane over ω, which is defined by three points $\bar{\varphi}_i(P_i) = 1$ and $\bar{\varphi}_i(P_j) = 0, j = 1, 2, 3,\ j \ne i$.

Then we map ω onto the square $\Pi = \{y : y = (y_1, y_2); -1 \le y_i \le 1, i = 1, 2\}$ by means of the transformation $y_i = \xi_i/\delta$ $(i = 1, 2)$ with Jacobian $I = \frac{\text{mes }(\omega)}{\text{mes }(\Pi)} = O(h^2)$.

If the function $q(\xi)$ is defined on ω, then we will denote by $\bar{q}(y)$ the function which is defined on Π by $\bar{q}(y) = q(y_1\delta, y_2\delta)$. Now we observe that

$$\begin{aligned}|(\rho^{\nu+\beta/2}u_\nu)^* - \rho^{\nu+\beta/2}u_{\nu,I}|_{W^1_2(K)} &\le |\overline{(\rho^{\nu+\beta/2}u_\nu)^*} - \overline{\rho^{\nu+\beta/2}u_{\nu,I}}|_{W^1_2(\omega)} \\ &= \frac{I^{1/2}}{\delta}|\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}} - \overline{\overline{\rho^{\nu+\beta/2}u_{\nu,I}}}|_{W^1_2(\Pi)}.\end{aligned} \tag{5.4}$$

It is obvious that the functional $g\left(\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}\right) = |\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}} - \overline{\overline{\rho^{\nu+\beta/2}u_{\nu,I}}}|_{W^1_2(\Pi)}$ is linear with respect to $\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}$ in $W^2_2(\Pi)$ and it satisfies the inequality

$$g\left(\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}\right) \le C_{30}\|\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}\|_{W^2_2(\Pi)}.$$

It is valid since

$$|\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}|_{W_2^1(\Pi)} \le \|\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}\|_{W_2^2(\Pi)},$$

$$|\overline{\overline{\rho^{\nu+\beta/2}u_{\nu,I}}}|_{W_2^1(\Pi)} \le \max_{y\in\Pi}|\overline{\overline{(\rho^{\nu+\beta/2}(y)u_\nu(y))^*}}|\sum_{i=1}^{3}|\overline{\overline{\varphi}}_i|_{W_2^1(\Pi)} \le C_{31}\|\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}\|_{W_2^2(\Pi)}.$$

Moreover, we note that the functional $g\left(\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}\right)$ vanishes on polynomials of degrees 0 and 1, i.e. $g(P) = 0$, where $P(y) = \sum_{|\alpha|<2} a_\alpha y^\alpha, \alpha = (\alpha_1, \alpha_2), y^\alpha = y_1^{\alpha_1}y_2^{\alpha_2}$. In fact, if $\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}$ is a polynomial of degree 1 or 0, then $\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}} = 0$ defines the part of plane over Π which coincides with the plane $\overline{\overline{\rho^{\nu+\beta/2}u_{\nu,I}}} = 0$, because

$$\overline{\overline{((\rho^{\nu+\beta/2}(Q_i)u_\nu(Q_i)))^*}} = \overline{\overline{\rho^{\nu+\beta/2}(Q_i)u_{\nu,I}(Q_i)}}, \quad i = 1, 2, 3,$$

where Q_i is the projection of the point P_i onto Π.

Thus all the conditions of the Bramble–Hilbert lemma for unweighted Sobolev spaces (see, e.g., [24]) are satisfied and the estimate

$$g\left(\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}\right) \le C_{32}|\overline{\overline{(\rho^{\nu+\beta/2}u_\nu)^*}}|_{W_2^2(\Pi)} \tag{5.5}$$

is true. By virtue of (5.4) and (5.5) and according to Lemma 1 we have

$$|(\rho^{\nu+\beta/2}u_\nu)^* - \rho^{\nu+\beta/2}u_{\nu,I}|_{W_2^1(K)} \le C_{33}h|\overline{(\rho^{\nu+\beta/2}u_\nu)^*}|_{W_2^2(\omega)} \le C_{34}h\|u_\nu\|_{H^2_{2,\nu+\beta/2}(K\cap\Omega)}. \tag{5.6}$$

Summing (5.6) for all K in Ω_h and taking into account (5.3) we get

$$|\rho^{\nu+\beta/2}(u_\nu - u_{\nu,I})|_{W_2^1(\Omega)} \le C_{35}h\|u_\nu\|_{H^2_{2,\nu+\beta/2}(\Omega)}. \tag{5.7}$$

If the function $u_\nu \in H^2_{2,\nu+\beta/2}(\Omega)$ then $\rho^{\nu+\beta/2-1}u_\nu$ belongs to $W^1_{2,0}(\Omega)$ by Lemma 1. By using the theorem about interpolation for unweighted Sobolev spaces (see, e.g., [24]) and Lemma 1 for the second term in the right-hand side of (5.2), we obtain the estimate

$$\|u_\nu - u_{\nu,I}\|_{L_{2,\nu+\beta/2-1}(\Omega)} \le C_{36}h|\rho^{\nu+\beta/2-1}u_\nu|_{W^1_{2,0}(\Omega)} \le C_{36}h\|u_\nu\|_{H^2_{2,\nu+\beta/2}(\Omega)}. \tag{5.8}$$

From (5.2), (5.7) and (5.8) the statement of Theorem 4 follows. □

On the basis of the theorem proved we establish the estimate of the convergence rate.

Theorem 5. *Suppose that the R_ν-generalized solution u_ν of the third-boundary-value problem with coordinated degeneracy of the input data belongs to the space $H^2_{2,\nu+\beta/2}(\Omega)$.*

Then there exists a constant C_{37} independent of u_ν, u_ν^h and h such that the convergence estimate

$$\|u_\nu - u_\nu^h\|_{H^1_{2,\nu+\beta/2}(\Omega)} \le C_{37}h\|u_\nu\|_{H^2_{2,\nu+\beta/2}(\Omega)} \tag{5.9}$$

holds for the triangulation of the domain Ω constructed.

Proof. The proof of this statement follows from Lemma 5 and Theorem 4. □

6. Numerical experiments

We have carried out a set of numerical tests for boundary value problems with singularity using our finite element method. The errors of the numerical approximations to the R_ν-generalized and generalized ($\nu = 0$) solution in the norm of the space $C(\Omega_h)$ in the mesh points were compared.

Example 1. Let

$$\Omega = \{x : x = (x_1, x_2), -1 < x_1 < 1, 0 < x_2 < 1\}$$

be a rectangle with boundary $\partial\Omega$ and $\bar{\Omega} = \Omega \cup \partial\Omega$. Let $\Gamma = \{x : x = (x_1, 0), 0 < x_1 < 1\}$, and O be a point with coordinates $(0, 0)$.

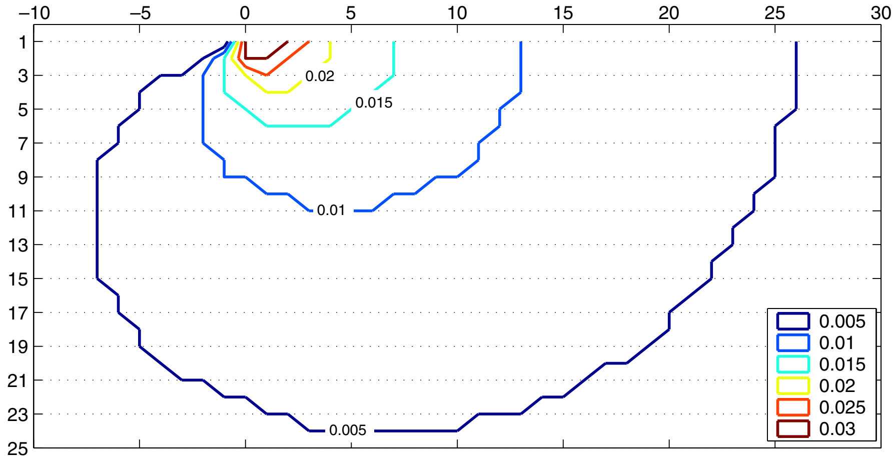

Fig. 1. Error of generalized solution ($\nu = 0$) in the neighborhood of the point of singularity.

Table 1
The influence of the mesh-size variations on the behaviour of the error of the generalized ($\nu = 0$) and R_ν-generalized solutions for $\nu = 0.516$, $\gamma = 0.35$, $\bar{\delta}_1 = 10^{-2}$, $\bar{\delta}_2 = 5 \cdot 10^{-3}$.

h	0.02564		0.01695		0.01266		0.010309		0.00847
	Generalized solution	R_ν-generalized solution	Generalized solution	R_ν-generalized solution	Generalized solution	R_ν-generalized solution	Generalized solution	R_ν-generalized solution	R_ν-generalized solution
Δ	0.07933	0.01665	0.06521	0.01359	0.05666	0.00798	0.05128	0.0072	0.0068
n_1	151	1	168	1	161	0	148	0	0
n_2	>400	5	>400	3	>400	8	>500	5	5

We consider the boundary value problem

$$Av \equiv -\sum_{l=1}^{2} \frac{\partial}{\partial x_1}\left(a_l(x)\frac{\partial v}{\partial x_l}\right) + a(x)v = f(x), \quad x \in \Omega,$$

$$\frac{\partial v}{\partial \eta} = \varphi_1(x), \quad x \in \Gamma, \qquad v = \varphi_2(x), \quad x \in \partial\Omega/\Gamma,$$

where

$$a_l = 1, \quad (l = 1, 2), \qquad a = 0, \qquad \varphi_1 = x_1^{-1/2}, \qquad \varphi_2 = r^{1/2}\sin\varphi,$$

$$f = -\frac{3}{4}r^{-3/2}\sin\varphi, \qquad r = (x_1^2 + x_2^2)^{1/2}, \qquad \varphi = \mathrm{arctg}\,(x_2/x_1).$$

The exact solution of this problem is $v = r^{1/2}\sin\varphi$.

We denote the set of the mesh points on the domain $\bar{\Omega}$ by

$$\bar{\Omega}_h = \{x_h : x_h = ((i_1 - 0.5\,\mathrm{sign}\,(i_1))h_1, i_2h_2), h_1 = 2/2N_1 - 1,$$
$$h_2 = 1/N_2,\ i_1 = -N_1, \ldots, -1, 1, \ldots, N_1, i_2 = 0, \ldots, N_2\}.$$

Let $\rho(x) = \min\{r_\gamma,\ \mathrm{dist}(x,\ O)\}$, where $r_\gamma = \max_{i=1,2} h_i^{1-\gamma/2}$, $0 < \gamma < 1$.

The system of algebraic equation (4.1) for the problem in Example 1 was solved using the Chebyshev method with the optimal set of the iteration parameters. For calculations of both the generalized ($\nu = 0$) and R_ν-generalized solutions we took the same initial approximation and the number of iterations sufficient for the stability of the process. For each approximate solution we calculated the following values: the error $\delta(x_h) = |v(x_h) - u^h(x_h)|$ at each node of the mesh $\bar{\Omega}_h$, the maximal error $\Delta = \max_{\bar{\Omega}_h} \delta(x_h)$, the numbers of nodes, n_1 and n_2, where the error exceeds the given limit values $\bar{\delta}_1$ and $\bar{\delta}_2$ respectively.

Figs. 1 and 2 show the nodes of the mesh where the error exceeds some given limit value for the generalized ($\nu = 0$) and R_ν-generalized solutions in the neighborhood of the point of singularity ($h = 0.0103$, $\nu = 0.49$, $\gamma = 0.3$). Here, for limit values of the error we use following values: 0.03, 0.025, 0.02, 0.015, 0.01, 0.005 (Tables 1–4).

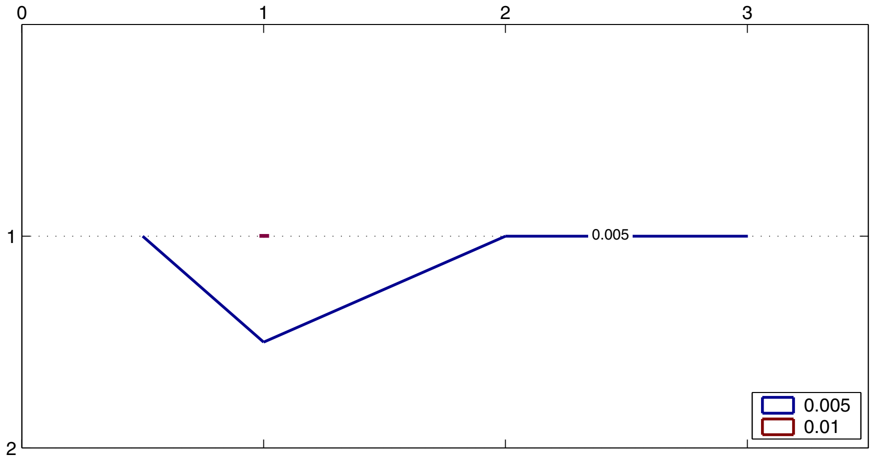

Fig. 2. Error of R_ν-generalized solution in the neighborhood of the point of singularity.

Table 2
The influence of the parameter γ on the behaviour of the error of the R_ν-generalized solution ($N_1 = 49$, $N_2 = 98$, $\nu = 0.516$, $\bar{\delta}_1 = 10^{-2}$, $\bar{\delta}_2 = 5 \cdot 10^{-3}$).

γ	0.1	0.25	0.31	0.35	0.41	0.45	0.5	Generalized solution
$h^{1-\gamma/2}$	0.013	0.0182	0.0209	0.0230	0.0263	0.0289	0.0324	
Δ	0.0187	0.0164	0.0163	0.00719	0.01093	0.01346	0.0178	0.0513
n_1	6	3	1	0	2	8	30	148
n_2	68	23	2	5	25	76	170	More than 500

Table 3
The influence of the parameter ν on the behaviour of the error of the R_ν-generalized solution ($N_1 = 49$, $N_2 = 98$, $\gamma = 0.35$, $\bar{\delta}_1 = 10^{-2}$, $\bar{\delta}_2 = 5 \cdot 10^{-3}$).

ν	0.3	0.4	0.515	0.516	0.517	0.55	0.6	0.7
Δ	0.02529	0.01682	0.00727	0.00719	0.00724	0.00863	0.01116	0.0171
n_1	6	1	0	0	0	0	4	12
n_2	41	4	4	5	5	10	26	95

Table 4
The errors $\|v_\nu - u_\nu^h\|_{H^1_{2,\nu}(\Omega)}$ for $\nu = 0.516$, $\gamma = 0.35$.

h	0.02564	0.01695	0.01266	0.010309	0.00847
$\|v_\nu - u_\nu^h\|_{H^1_{2,\nu}}$	0.01284	0.00850	0.00638	0.00518	0.00462

Example 2.

$$Av \equiv -\sum_{l=1}^{2} \frac{\partial}{\partial x_l}\left(a_l(x)\frac{\partial v}{\partial x_l}\right) + a(x)v = f(x), \quad x \in \Omega, \tag{6.1}$$

$$b(x)v + \frac{\partial v}{\partial N} = \varphi_1(x), \quad x \in \Gamma, \qquad v = \varphi_2(x), \quad x \in \partial\Omega \setminus \Gamma, \tag{6.2}$$

where

$$a_1 = a_2 = r^{1/2}, \qquad a = r^{-3/2}, \qquad b = r^{-1/2},$$
$$f = -0,25 \cdot r^{-3}\sin(\varphi/2), \qquad \varphi_2 = r^{-3/2}\sin(\varphi/2),$$
$$\varphi_1 = r^{-3}(\cos(3\varphi/2) + \sin(\varphi/2) - 0,5 \cdot \cos(\varphi/2)).$$

The exact solution of this problem is $v = r^{-3/2}\sin(\varphi/2)$ (Tables 5 and 6).

Calculations ($h = 0.0103$, $\gamma = 0.31$, $\nu = 6.0$) showed that

$$\Delta_{\text{g.s.}} = 893.7,$$

where $\Delta_{\text{g.s.}}$ is the maximal value of the errors in the mesh points for the approximate generalized solution ($\nu = 0$).

Table 5
The errors $\|v_\nu - u_\nu^h\|_{H^1_{2,\nu}(\Omega)}$ for $\nu = 2.0$, $\gamma = 0.31$.

h	0.02564	0.01695	0.01266	0.01031	0.00847
$\|v_\nu - u_\nu^h\|_{H^1_{2,\nu}}$	$1.12 \cdot 10^{-3}$	$6.7 \cdot 10^{-4}$	$4.7 \cdot 10^{-4}$	$2.9 \cdot 10^{-4}$	$2.2 \cdot 10^{-4}$

Table 6
The errors $\|v_\nu - u_\nu^h\|_{H^1_{2,\nu}(\Omega)}$ for $\nu = 6.5$, $\gamma = 0.3$.

h	0.02564	0.01695	0.01266	0.01031	0.00847
$\|v_\nu - u_\nu^h\|_{H^1_{2,\nu}}$	$1.01 \cdot 10^{-4}$	$5.26 \cdot 10^{-5}$	$1.83 \cdot 10^{-5}$	$7.62 \cdot 10^{-6}$	$4.26 \cdot 10^{-6}$

Example 3. Consider the boundary value problem (6.1) and (6.2), where

$$a_1 = a_2 = r^{-1/2}, \qquad a = r^{-5/2}, \qquad b = r^{-3/2}$$
$$f = -6,25 \cdot r^{-5}\sin(\varphi/2), \qquad \varphi_2 = r^{-5/2}\sin(\varphi/2),$$
$$\varphi_1 = r^{-4}(1,5 \cdot \cos(3\varphi/2) + \sin(\varphi/2) - \cos(\varphi/2))$$

and

$$v = r^{-5/2}\sin(\varphi/2).$$

Finally, a number of calculations showed that:

(1) the value of the error is always decreasing as the distance from the mesh points to the point of singularity increases;

(2) if we choose parameters ν and γ near to optimal, the accuracy of the approximation in the case of the R_ν-generalized solution is in general two orders higher than for the generalized solution;

(3) if the input data had strong singularity, it is impossible to find the generalized solution ($\nu = 0$), because the computation is interrupted by an exception, while the R_ν-generalized solution can be computed with high accuracy.

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