Clifford Theory for $G$-Functors

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When does a $G$-functor admit a Clifford theory? In this paper we give a simple axiomatic property which characterizes the existence of such a theory. Satisfaction of this condition in the different contexts leads automatically to the satisfaction of the corresponding theorems of Clifford. We apply it to several examples, some of them already known including the character and Burnside rings.

1. INTRODUCTION

Clifford’s theorems for characters are usually proved by using Frobenius reciprocity, since the original paper [1]. But it is also possible to develop a Clifford theory in the Burnside ring, although this reciprocity does not hold on it (see [7]).

These apparently distinct theories are actually particular cases of a more general setting. The suitable framework to establish it is provided by the $G$-functors, introduced by Green [3]. (An equivalent development can be made in terms of Mackey functors; see [2].)

In this paper, $G$ denotes a finite group. Every transversal (of right cosets or double cosets) considered shall contain the unit element of the group.

Definition 1.1 (see Green [3]). Let $\mathcal{A} = (M, T, R, C, \mathcal{B})$ be a quintuple of families $M, T, R, C, \mathcal{B}$ of the following kinds:

$M = (M_H)_{H \leq G}$ gives, for each subgroup $H$ of $G$, a free abelian semigroup $M_H$.

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Let $T = (T_{H,K})$ and $R = (R_{K,H})$ give, for each pair $(H, K)$ of subgroups of $G$ such that $H \leq K$, the respective homomorphisms of semigroups, $T_{H,K} : M_H \to M_K$ and $R_{K,H} : M_K \to M_H$.

$C = (C_{H,g})$ gives, for each pair $(H, g)$ where $H$ is a subgroup and $g$ an element of $G$, the semigroup homomorphism $C_{H,g} : M_H \to M_{H^g}$, where $H^g = g^{-1} H g$.

$B = (B_H)_{H \leq G}$ gives, for each subgroup $H$ of $G$, the basis $B_H$ of $M_H$.

We say that $B = (B, T, R, C, B)$ is a based $G$-functor if these families of semigroups and maps satisfy the following axioms, where $D, H, K$, and $L$ are subgroups of $G$ and $g, g'$ are elements of $G$:

(a) $T_{H,H} = \text{id}(M_H), T_{H,K} = T_{K,L} = T_{H,L}$ if $H \leq K \leq L$.
(b) $R_{H,H} = \text{id}(M_H), R_{K,H} = R_{H,D} = R_{K,D}$ if $D \leq H \leq K$.
(c) $C_{H,g} C_{H,g'} = C_{H,gg'}$.
(d) $C_{H,h} = \text{id}(M_H)$ if $h \in H$.
(e) $T_{H,K} C_{K,g} = C_{H,g} T_{H,K^g}$.
(f) $R_{K,H} C_{K,g} = C_{K,g} R_{K^g,H^g}$.
(g) Mackey axiom. If $H \leq L$, $K \leq L$ and if $\Gamma$ is a transversal of the $(H, K)$-double cosets in $L$, then

$$T_{H,L} R_{L,K} = \sum_{g \in \Gamma} C_{H,g} R_{H^g,H^g \cap K} T_{H^g \cap K,K}.$$ 

Since $C_{H,g}$ is an isomorphism, $\beta C_{H,g} \in B_{H^g}$ for any $\beta \in B_H$.

For each $H \leq G$, we consider the symmetric bilinear form $\langle \cdot, \cdot \rangle : M_H \times M_H \to \mathbb{N}$ such that $B_H$ is an orthonormal basis of $M_H$.

**Proposition 1.2.** Let $B = (B, T, R, C, B)$ be a based $G$-functor. We have:

(i) $\langle \delta C_{H,K}, \gamma C_{H,g} \rangle = \langle \delta, \gamma \rangle$ for all $\delta, \gamma \in M_H, g \in G$.

(ii) If $H \leq K$, then $\langle \alpha R_{K,H}, \beta \rangle = \langle \alpha R_{H,H}, \beta C_{H,g} \rangle$ for all $\alpha \in B_K, \beta \in B_H, g \in K$.

**Proof.** (i) Let $\delta = \sum_{\beta \in B} a_\beta \beta$ and $\gamma = \sum_{\beta \in B} b_\beta \beta$, where $a_\beta, b_\beta \in \mathbb{N}$. Then $\langle \delta, \gamma \rangle = \sum_{\beta \in B} a_\beta b_\beta$. Therefore, we have

$$B_H C_{H,g} = \{ \beta C_{H,g} : \beta \in B_H \} = B_{H^g}.$$
because $C_{H,g}$ is an isomorphism. Then

$$
\langle \delta C_{H,g}, \gamma C_{H,g} \rangle = \left( \sum_{\beta \in \mathcal{B}_H} a_\beta \beta C_{H,g}, \sum_{\beta \in \mathcal{B}_H} b_\beta \beta C_{H,g} \right) = \sum_{\beta \in \mathcal{B}_H} a_\beta b_\beta.
$$

(ii) Let $g \in K$. By (i) and axioms (f) and (d),

$$
\langle \alpha R_{K, H}, \beta \rangle = \langle \alpha R_{K, H} C_{H,g}, \beta C_{H,g} \rangle = \langle \alpha C_{K,g} R_{K, H^t}, \beta C_{H,g} \rangle = \langle \alpha R_{K, H}, \beta C_{H,g} \rangle.
$$

\[\] PROPOSITION 1.3. Let $H \leq K$. If $\alpha \in \mathcal{B}_H$, then there exists $\gamma \in \mathcal{B}_K$ with $\langle \alpha T_{H,K}, \gamma \rangle \neq 0$ such that $\langle \gamma R_{K, H}, \alpha \rangle \neq 0$.

Proof. By the Mackey axiom,

$$
\alpha T_{H,K} R_{K, H} = \sum_{g \in \Gamma} \alpha C_{H,g} R_{H^t, H^t \cap H} T_{H^t \cap H, H},
$$

where $\Gamma$ is a transversal of the $(H, H)$-double cosets in $K$. The term corresponding to $g = 1$ is $\alpha$. So $\langle \alpha T_{H,K} R_{K, H}, \alpha \rangle \neq 0$ and there exists $\gamma \in \mathcal{B}_K$ such that $\langle \alpha T_{H,K}, \gamma \rangle \neq 0$ and $\langle \gamma R_{K, H}, \alpha \rangle \neq 0$.

DEFINITION 1.4. Let $\mathcal{B} = (M, T, R, C, \mathcal{B})$ be a based $G$-functor, $H \leq K \leq G$, and $\beta \in \mathcal{B}_H$. The subgroup $\{g \in K \mid \beta C_{H,g} = \beta\}$ is called the inertia subgroup of $\beta$ in $K$. We write $\beta^i$ instead of $\beta C_{H,g}$ and, for each $g \in K$, we say that $\beta^i$ is a $K$-conjugate of $\beta$.

THEOREM 1.5 (see [5, Theorem VII 9.6]). Let $\mathcal{B} = (M, T, R, C, \mathcal{B})$ be a based $G$-functor. If $H \leq K \leq G$, $\beta \in \mathcal{B}_H$, $Y$ is the inertia subgroup of $\beta$ in $K$, and $\beta T_{H, Y} = \delta_1 \cdots + \delta_r$ with $\delta_i \in \mathcal{B}_Y$ ($1 \leq i \leq r$), then for all $1 \leq i, j \leq r$:

(i) $\delta_i T_{Y,K} \in \mathcal{B}_K$.

(ii) $\langle \delta_i T_{Y,K} R_{K, H}, \beta \rangle = \langle \delta_i R_{Y,H}, \beta \rangle$.

(iii) $\delta_i T_{Y,K} = \delta_j T_{Y,K}$ if and only if $\delta_i = \delta_j$.

Proof. By the Mackey axiom, $\beta T_{H, Y} R_{Y, H} = |Y : H| \beta$. Then $\delta_i R_{Y, H} = n_i \beta$ for some $n_i \in \mathbb{N}$, $1 \leq i \leq r$. Further $\sum_{i=1}^r n_i = |Y : H|$.

By Proposition 1.3, for each $1 \leq i \leq r$, there exists $\alpha_i \in \mathcal{B}_K$ such that

$$
\langle \alpha_i R_{K, Y}, \delta_i \rangle \neq 0 \neq \langle \delta_i T_{Y,K}, \alpha_i \rangle.
$$
Set \( \delta_i T_{Y,K} = \alpha_i + x_i, x_i \in M_K \). If \( \Gamma \) is a right transversal of \( Y \) in \( K \) and \( g \in \Gamma \), by Proposition 1.2 and axiom (b),

\[
\langle \alpha_i R_{K,H}, \beta^g \rangle = \langle \alpha_i R_{K,H}, \beta \rangle \\
= \langle \alpha_i R_{K,Y R_{Y,H}}, \beta \rangle \geq \langle \delta_i R_{Y,H}, \beta \rangle = n_i.
\]

Therefore \( \alpha_i R_{K,H} = m_i \sum_{g \in \Gamma} \beta^g + z_i \), where \( m_i \geq n_i \) and \( z_i \in M_H \).

Now, by the Mackey axiom

\[
n_i \sum_{g \rightarrow t} \beta^g = \delta_i T_{Y,K} R_{K,H} = \alpha_i R_{K,H} + x_i R_{K,H}
\]

and then \( m_i = n_i \), \( x_i = 0 \), and \( z_i = 0 \). Therefore \( \delta T_{Y,K} = \alpha_i \), and (i) is proved.

Further \( n_i = \langle \delta_i T_{Y,K} R_{K,H}, \beta \rangle = \langle \alpha_i R_{K,H}, \beta \rangle \). Since \( \langle \delta_i R_{Y,H}, \beta \rangle = n_i \), assertion (ii) follows.

Assume finally that \( \alpha_i = \alpha_j \) and \( \delta_i \neq \delta_j \) for some \( 1 \leq i, j \leq r \). Then both \( \langle \alpha_i R_{K,Y}, \delta_i \rangle \) and \( \langle \alpha_i R_{K,Y}, \delta_j \rangle \) are strictly positive. Therefore \( \langle \alpha_i R_{K,H}, \beta \rangle \geq \langle \delta_i R_{Y,H}, \beta \rangle + \langle \delta_j R_{Y,H}, \beta \rangle > n_i \), which contradicts (ii). Thus (iii) holds.

2. CLIFFORD THEORY FOR \( G \)-FUNCTORS

We say that the Clifford theory holds for a based \( G \)-functor \( A \) if the axiomatic version of the first and second theorems of Clifford (see Clifford [1]) are valid in \( A \) (see Theorems 2.6 and 2.10).

The axioms which define a based \( G \)-functor are not enough to ensure that Clifford theory holds for it (see, for instance, Example 2 of Section 3). If we add another axiom similar to the Frobenius reciprocity of characters, then an axiomatic version of Clifford’s theorems is valid (see Remark 2.2). However, there exist some examples of based \( G \)-functors, for instance the Burnside ring (see [7] or Example 4 in Section 3), which do not satisfy the Frobenius reciprocity and do satisfy Clifford’s theorems.

We give another condition, weaker than Frobenius reciprocity, which is valid in both character and Burnside ring functors, and which is sufficient to ensure that Clifford theory holds on a based \( G \)-functor. Later we shall see that this condition is also necessary.

DEFINITION 2.1. Let \( A = (M, T, R, C, \emptyset) \) be a based \( G \)-functor, \( H \leq K \leq G \), and \( J \leq K \).
We say that \( \alpha \in \mathcal{B}_K \) is \((K, H)\)-homogeneous if \( \alpha R_{K, H} = n \beta \) for some \( \beta \in \mathcal{B}_H \) and \( n \in \mathbb{N} \).

We say that \( \alpha \in \mathcal{B}_K \) is \((K, J)\)-projective if \( \langle \delta T_{J, K}, \alpha \rangle \neq 0 \) for some \( \delta \in \mathcal{B}_J \).

We say that \( \mathcal{A} \) satisfies property (P) if

\[
(P) \quad H \trianglelefteq K \leq G, \quad \alpha \in \mathcal{B}_K \\
\implies \left\{ \begin{array}{l}
\alpha \text{ is } (K, H)\text{-homogeneous or} \\
\alpha \text{ is } (K, J)\text{-projective for some } J, H \leq J < K.
\end{array} \right.
\]

Observe that property (P) cannot be deduced from the axioms which define a based \(G\)-functor, as it is shown in Example 2 of the last section of this paper.

**Remark 2.2.** The character ring functor satisfies property (P) because of the following result: If a based \(G\)-functor \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) satisfies the Frobenius reciprocity \( \langle \alpha, \beta T_{H, K} \rangle = \langle \alpha R_{K, H}, \beta \rangle \) for any \( H \leq K, \alpha \in \mathcal{B}_K, \beta \in \mathcal{B}_H \), then \( \mathcal{A} \) satisfies property (P).

To see it, if \( H \triangleleft K \) and \( \alpha \in \mathcal{B}_K \), take \( J = H \) and any \( \delta \in \mathcal{B}_J \) such that \( \langle \alpha R_{K, H}, \delta \rangle \neq 0 \). Then \( \alpha \) is \((K, J)\)-projective.

**Remark 2.3.** In the last section of this paper we see that the Burnside ring functor and the cohomological based \(G\)-functors (see Green [3]) also satisfy property (P).

**Lemma 2.4.** Let \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) be a based \(G\)-functor. Then the two following assertions are equivalent:

(i) The functor \( \mathcal{A} \) satisfies property (P).

(ii) If \( H \trianglelefteq K \leq G \) and \( \alpha \in \mathcal{B}_K \), then for any \( \beta \in \mathcal{B}_H \) such that \( \langle \alpha R_{K, H}, \beta \rangle \neq 0 \), there exist \( S, H \leq S \leq K \), and \( \gamma \in \mathcal{B}_S \) such that \( \langle \gamma T_{S, K}, \alpha \rangle \neq 0 \) and \( \gamma R_{S, H} = n \beta \) for some \( n \in \mathbb{N} \).

**Proof.** We use induction on \(|K: H|\) to prove that (i) implies (ii). Let \( H \trianglelefteq K, \alpha \in \mathcal{B}_K, \beta \in \mathcal{B}_H \) with \( \langle \alpha R_{K, H}, \beta \rangle \neq 0 \). If \( \alpha R_{K, H} = n \beta, n \in \mathbb{N} \), then it suffices to take \( S = K \) and \( \gamma = \alpha \). Otherwise, there exist \( J, H \leq J < K \), and \( \delta \in \mathcal{B}_J \) such that \( \langle \delta R_{J, K}, \alpha \rangle \neq 0 \). So, \( \alpha R_{K, H} \) is a submand of \( \delta T_{J, K} R_{K, H} = \sum_{\Gamma \in J} \delta^{\Gamma} R_{J, H} \), where \( \Gamma \) is a right transversal of \( J \) in \( K \). Hence \( \langle \delta^{g} T_{J, K} R_{K, H}, \beta \rangle \neq 0 \) for some \( g \in K \). By the inductive hypothesis, there exist \( S, H \leq S \leq J^{g} \), and \( \gamma \in \mathcal{B}_S \) such that \( \langle \gamma T_{S, K}, \delta^{g} \rangle \neq 0 \) and \( \gamma R_{S, H} = m \beta \) for some \( m \in \mathbb{N} \). Also \( \langle \gamma T_{S, K}, \alpha \rangle \neq 0 \) because \( \langle \gamma T_{S, J^{g} K}, \delta^{g} \rangle \neq 0 \) and \( \langle \delta^{g} T_{J^{g}, K}, \alpha \rangle = \langle \delta T_{J, K}, \alpha \rangle \neq 0 \).

The converse is easily proved.
Remark 2.5. (i) If a particular \( \alpha \in \mathcal{B}_K \) satisfies the condition (ii) of Lemma 2.4, it does not in general follow that \( \alpha \) satisfies the condition of property (P) (see Proposition 2.7 below).

(ii) Lemma 2.4 remains true if we write some \( \beta \) instead of any \( \beta \) in the second assertion, as the obvious proof of (ii) implies (i) shows.

The next result is an axiomatic version of the first theorem of Clifford for representations (see Clifford [1]).

**Theorem 2.6.** Let \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) be a based \( G \)-functor satisfying property (P), \( H \leq K \), and \( \alpha \in \mathcal{B}_K \). If \( \beta \in \mathcal{B}_H \) and \( \langle \alpha R_{K, H}, \beta \rangle \neq 0 \), then \( \alpha R_{K, H} = \epsilon(\beta^{g_1} + \cdots + \beta^{g_m}) \) for some \( e \in \mathbb{N} \), where \( \{g_1, \ldots, g_m\} \) is a right transversal of the inertia subgroup of \( \beta \) in \( K \) (so \( \beta^{g_1}, \ldots, \beta^{g_m} \) are the distinct \( K \)-conjugates of \( \beta \)).

**Proof.** By Proposition 1.2(ii), all the \( \beta^g \) have the same multiplicity in \( \alpha R_{K, H} \).

By Lemma 2.4, we may take \( S, H \leq S \leq K \), and \( \gamma \in \mathcal{B}_S \) satisfying \( \langle \gamma T_{S, K}, \alpha \rangle \neq 0 \) and \( \gamma R_{S, H} = n \beta \) for some \( n \in \mathbb{N} \).

Now, if \( \delta \in \mathcal{B}_H \) and \( \langle \alpha R_{K, H}, \delta \rangle \neq 0 \), then \( \langle \gamma T_{S, K} R_{K, H}, \delta \rangle \neq 0 \). By the Mackey axiom, \( \delta \) is a \( K \)-conjugate of \( \beta \).

When property (P) is not valid for every \( H \leq K \) and each \( \alpha \in \mathcal{B}_K \), but there exists some \( \alpha \in \mathcal{B}_K \) satisfying (ii) of Lemma 2.4, then we can state the following weak version of Theorem 2.6.

**Proposition 2.7.** Let \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) be a based \( G \)-functor. Let \( H \leq K \leq G \), \( H \leq S \leq K \), \( \alpha \in \mathcal{B}_K \), \( \gamma \in \mathcal{B}_S \), and \( \beta \in \mathcal{B}_H \) satisfying \( \langle \gamma T_{S, K}, \alpha \rangle \neq 0 \) and \( \gamma R_{S, H} = n \beta \) for some \( n \in \mathbb{N} \). Then \( \alpha R_{K, H} = \epsilon(\beta^{g_1} + \cdots + \beta^{g_m}) \) for some \( e \in \mathbb{N} \), where \( \{g_1, \ldots, g_m\} \) is a right transversal of the inertia subgroup of \( \beta \) in \( K \).

As a consequence of Proposition 2.7 (by taking \( S = H \) and \( \gamma = \beta \)), we obtain an axiomatic version of a Nakayama’s theorem (see [5, Theorem VII 9.3]):

**Corollary 2.8.** Let \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) be a based \( G \)-functor. Let \( H \leq K \leq G \) and \( \alpha \in \mathcal{B}_K \). If \( \langle \alpha R_{K, H} T_{H, K}, \alpha \rangle \neq 0 \), then \( \alpha R_{K, H} = \epsilon(\beta^{g_1} + \cdots + \beta^{g_m}) \) for some \( e \in \mathbb{N} \) and some \( \beta \in \mathcal{B}_H \), where \( \{g_1, \ldots, g_m\} \) is a right transversal of the inertia subgroup of \( \beta \) in \( K \).

**Remark 2.9.** Let \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) be a based \( G \)-functor satisfying property (P). Then the set of homomorphisms \( \{C_{l,g} : M_1 \to M_1, g \in G \} \) provides an action of \( G \) on the basis \( \mathcal{B}_1 \). Now, if \( \{\theta_j, j \in I\} \) is the set of
orbits of this action, $H \leq G$ and $j \in I$, set
\[ A_H^j = \{ \beta \in A_H, \langle \beta R_{H,1}, \delta \rangle \neq 0 \text{ for some } \delta \in \Theta_j \} \]
and denote $M_H^j$ as the subsemigroup of $M_H$ generated by $A_H^j$. Then $\mathfrak{A}^j = (M^j, T, R, C, \mathfrak{B})$ is a based $G$-subfunctor of $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{A}^j = \prod_{j \in I} \mathfrak{A}^j$.

**Proof.** By Theorem 2.6, each $\beta \in A_H^j$ belongs to a unique $A_H^j$, $j \in I$. Thus $M_H = \prod_{j \in I} M_H^j$. We next show that each $A_H^j$ is a subfunctor of $\mathfrak{A}$.

It is clear that if $g \in G$, then $C_{M^j, g} : M_H \to M_H^j$ maps the elements of $A_H^j$ into $A_H^j$, because $\langle \beta R_{H,1}, \delta \rangle \neq 0$, $\delta \in \Theta_j$, implies $\langle \beta C_{M^j, g} R_{H,1}, \delta \rangle \neq 0$ and $\delta \in \Theta_j$.

Let $D \leq H$, and let $\beta \in A_H^j$, $\gamma \in A_D$, such that $\langle \beta R_{H,1}, \gamma \rangle \neq 0$. If $\delta \in A_j$ satisfies $\langle \gamma R_{D,1}, \delta \rangle \neq 0$, then $\langle \beta R_{H,1}, \delta \rangle \neq 0$. So $\gamma \in A_H^j$ and $R_{H,1}$ maps $M_H^j$ into $M_D^j$.

Let now $H \leq K$, $\beta \in A_H^j$, and $\alpha \in A_K$ such that $\langle \beta T_{H,K}, \alpha \rangle \neq 0$. For every $\delta \in A_j$ with $\langle \alpha R_{K,1}, \delta \rangle \neq 0$, we get $\langle \beta T_{H,K} R_{K,1}, \delta \rangle \neq 0$ and by the Mackey axiom, $\delta \in \Theta_j$. So $\beta T_{H,K} \in M_K^j$.

The next result is an axiomatic version of the second theorem of Clifford for representations. (See Clifford [1].)

**Theorem 2.10.** Let $\mathfrak{A} = (M, T, R, C, \mathfrak{B})$ be a based $G$-functor satisfying property (P). If $H \leq K \leq G$, $\beta \in A_H$, and $Y$ is the inertia subgroup of $\beta$ in $K$, set
\[ \mathfrak{A} = \{ \delta \in A_Y; \langle \delta R_{Y,H}, \beta \rangle \neq 0 \} \]
and
\[ \mathfrak{B} = \{ \alpha \in A_K; \langle \alpha R_{K,H}, \beta \rangle \neq 0 \}. \]
Then

(i) $\delta T_{Y,K} \in \mathfrak{B}$ for all $\delta \in \mathfrak{A}$.

(ii) $\langle \delta R_{Y,H}, \beta \rangle = \langle \delta T_{Y,K} R_{K,H}, \beta \rangle$ for all $\delta \in \mathfrak{A}$.

(iii) If $\delta T_{Y,K} = \alpha$, $\delta \in \mathfrak{A}$, then $\delta$ is the unique element of $\mathfrak{A}$ occurring in $\alpha R_{K,Y}$.

(iv) The map $f : \mathfrak{A} \to \mathfrak{B}$ given by $f(\delta) = \delta T_{Y,K}$ is well-defined and bijective.

**Proof.** (i) Let $\delta \in \mathfrak{A}$. By Proposition 1.3 there exists $\alpha \in A_K$ such that $\langle \delta T_{Y,K}, \alpha \rangle \neq 0 \neq \langle \alpha R_{K,Y}, \delta \rangle$. Obviously $\alpha \in \mathfrak{B}$ and, by Theorem 2.6, $\alpha R_{K,H} = e(\beta^{g_1} + \cdots + \beta^{g_m})$, where $\{g_1, \ldots, g_m\}$ is a right transversal of $Y$ in $K$.

Now, $\langle \alpha R_{K,Y}, \delta \rangle \neq 0$ and $\delta R_{Y,H} = d \beta$ for some $d \in \mathbb{N}$. Hence necessarily $d \leq e$. 
On the other hand, \( \delta T_{Y,K} R_{K,H} = \sum_{i=1}^{m} \delta R_{Y,H} C_{H,g_i} \) by the Mackey axiom. So \( \delta T_{Y,K} R_{K,H} = d(\beta^\delta + \cdots + \beta^{\delta_n}) \), and then

\[
e \geq d = \langle \delta T_{Y,K} R_{K,H}, \beta \rangle \geq \langle \alpha R_{K,H}, \beta \rangle = e.
\]

Therefore \( d = e \). It follows that \( \delta T_{Y,K} R_{K,H} = \alpha R_{K,H} \) and \( \delta T_{Y,K} = \alpha \in \mathcal{C} \).

(ii) We have verified that both expressions take the same value, \( e \), when \( \delta \in \mathcal{A} \). Otherwise, they both are equal to zero.

(iii) Let \( \delta, \delta' \in \mathcal{A}, \delta \neq \delta' \), and assume that \( \langle \delta T_{Y,K} R_{K,Y}, \delta' \rangle \neq 0 \). Then

\[
\langle \delta T_{Y,K} R_{K,H}, \beta \rangle = \langle \delta T_{Y,K} R_{K,Y}, \beta \rangle \\
\geq \langle \delta R_{Y,H}, \beta \rangle + \langle \delta' R_{Y,H}, \beta \rangle,
\]

where \( \langle \delta' R_{Y,H}, \beta \rangle \neq 0 \), which contradicts (ii).

(iv) For any \( \delta \in \mathcal{A} \), we know that \( f(\delta) \in \mathcal{C} \).

Let \( \alpha \in \mathcal{C} \). By Lemma 2.4, there exist \( S, H \leq S \leq K \), and \( \gamma \in \mathcal{B} \) such that

\[
\gamma R_{S,H} = n\beta \quad \text{for some} \quad n \in \mathbb{N} \quad \text{and} \quad \langle \gamma T_{S,K}, \alpha \rangle \neq 0.
\]

It is clear that \( S \leq Y \), so \( \gamma T_{S,Y} \) makes sense. Since \( \langle \gamma T_{S,Y} T_{Y,K}, \alpha \rangle \neq 0 \), there is \( \delta \in \mathcal{B} \) such that

\[
\langle \gamma T_{S,Y}, \delta \rangle \neq 0 \neq \langle \delta T_{Y,K}, \alpha \rangle.
\]

Now \( \delta \in \mathcal{A} \) because \( \delta R_{Y,H} \) occurs in

\[
\gamma T_{S,Y} R_{Y,H} = \sum_{g \in \Gamma} \gamma R_{S,H} C_{H,g} = n|Y : S|\beta,
\]

where \( \Gamma \) is a right transversal of \( S \) in \( Y \).

It follows from (i) that \( \delta T_{Y,K} \in \mathcal{C} \), and then \( \delta T_{Y,K} = \alpha \) since \( \langle \delta T_{Y,K}, \alpha \rangle \neq 0 \). Therefore \( f \) is surjective.

Now let \( \delta, \delta' \in \mathcal{A} \) such that \( f(\delta) = f(\delta') = \alpha \). Then \( \langle \alpha R_{K,Y}, \delta \rangle \neq 0 \) and \( \langle \alpha R_{K,Y}, \delta' \rangle \neq 0 \). By (ii), \( \delta = \delta' \) and \( f \) is injective.

We show now that the Clifford theory holds for a based \( G \)-functor if and only if it satisfies property (P).

**Theorem 2.11.** If \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) is a based \( G \)-functor, then the following assertions are equivalent

(i) The functor \( \mathcal{A} \) satisfies property (P).

(ii) If \( H \triangleleft K \leq G \) and \( \beta \in \mathcal{B}_H \), then the map \( f \) of Theorem 2.10 is well-defined and bijective.

**Proof.** It is enough to see that (ii) implies (i). If \( H \triangleleft K \) and \( \alpha \in \mathcal{B}_K \) is not \((K,H)\)-homogeneous, then take any \( \beta \in \mathcal{B}_H \) such that \( \langle \alpha R_{K,H}, \beta \rangle \neq 0 \).
0 and \( J = Y < K \), the inertia subgroup of \( \beta \) in \( K \). Since the map \( f \) is bijective, there exists a unique \( \gamma \in \mathcal{B}_Y \) such that \( \langle \gamma R_{Y,H}, \beta \rangle \neq 0 \) satisfying \( \gamma_{T_{Y,K}} = \alpha \). 

Most of the based \( G \)-functors that we mention in the last section of this paper satisfy the following finiteness condition:

(F) For each \( \delta \in \mathcal{B}_1 \), there exist only finitely many elements \( \alpha \in \mathcal{B}_G \) such that \( \langle \alpha R_{G,1}, \delta \rangle \neq 0 \).

Notice that condition (F) cannot be deduced from the axioms which define a based \( G \)-functor, as it is shown in Example 2 of Section 3.

**Proposition 2.12.** Let \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) be a based \( G \)-functor satisfying condition (F). If \( H \leq K \) and \( \beta \in \mathcal{B}_H \), then there exist only finitely many elements \( \gamma \in \mathcal{B}_G \) such that \( \langle \gamma R_{K,H}, \beta \rangle \neq 0 \).

**Proof.** First assume \( K = G \) and take \( \beta \in \mathcal{B}_H \). If \( \delta \in \mathcal{B}_1 \) and \( \langle \beta R_{H,1}, \delta \rangle \neq 0 \), then the set \( \mathcal{B}_G(\beta) = \{ \alpha \in \mathcal{B}_G; \langle \alpha R_{G,H}, \beta \rangle \neq 0 \} \) is contained in \{ \alpha \in \mathcal{B}_G; \langle \alpha R_{G,1}, \delta \rangle \neq 0 \}, which is finite by hypothesis.

Now assume \( K < G \), \( \beta \in \mathcal{B}_H \). Let \( \alpha \in \mathcal{B}_K \) such that \( \langle \alpha R_{K,H}, \beta \rangle \neq 0 \). By Proposition 1.3, there is \( \gamma \in \mathcal{B}_G \) such that \( \langle \alpha T_{K,G}, \gamma \rangle \neq 0 \neq \langle \gamma R_{G,K}, \alpha \rangle \). Thus \( \langle \gamma R_{G,H}, \beta \rangle \neq 0 \). Since \( \mathcal{B}_G(\beta) \) is a finite set, it is clear that \{ \alpha \in \mathcal{B}_K; \langle \gamma R_{G,K}, \alpha \rangle \neq 0 \} \) is also a finite set, and the proof is complete.

This result allows us to give the following

**Definition 2.13.** Let \( \mathcal{A} = (M, T, R, C, \mathcal{B}) \) be a based \( G \)-functor satisfying condition (F). For each \( H \leq K \leq G \), we consider the homomorphism

\[
E_{H,K} : M_H \rightarrow M_K
\]

given by \( \langle \beta E_{H,K}, \alpha \rangle = \langle \beta, \alpha R_{K,H} \rangle \), for every \( \beta \in \mathcal{B}_H \), \( \alpha \in \mathcal{B}_K \).

There are natural examples of based \( G \)-functors, such as the character ring functor, satisfying \( E_{H,K} = T_{H,K} \) when \( H \leq K \leq G \).

**Remark 2.14.** It is easy to prove that the family \( (E_{H,K})_{H \leq K} \) satisfies, for all \( H \leq K \leq L \) and \( g \in G \):

(i) \( E_{H,K} E_{K,L} = E_{H,L} \).

(ii) \( E_{H,K} C_{K,g} = C_{H,g} E_{H,K} \).

(iii) If \( H \leq K \), \( \beta \in \mathcal{B}_H \) and \( g \in K \), then \( \beta E_{H,K} = \beta C_{H,g} E_{H,K} \).

If condition (F) holds, we may add another assertion to Theorem 2.10:

**Proposition 2.15.** If a based \( G \)-functor satisfies conditions (P) and (F), then, with the notations of Theorem 2.10, we have

\[
\delta E_{Y,K} = \delta T_{Y,K} \quad \text{for all } \delta \in \mathcal{A}.
\]
Proof. Let \( \delta \in \mathcal{A} \) and consider \( \alpha \in \mathcal{B}_K \) such that \( \langle \alpha R_{K,Y}, \delta \rangle \neq 0 \). Then \( \langle \alpha R_{K,H}, \beta \rangle \neq 0 \), so \( \alpha \in \mathcal{B} \). Now, since \( f \) is bijective, there exists a unique \( \delta' \in \mathcal{A} \) such that \( f(\delta') = \alpha \). Therefore \( \delta' \) occurs in \( \alpha R_{K,Y} = \delta'T_{Y,K}R_{K,Y} \). By Theorem 2.10 (iii) we obtain \( \delta = \delta' \), and then \( \alpha = f(\delta) = \delta T_{Y,K} \). So \( \alpha \) is the unique element \( \gamma \in \mathcal{B}_K \) such that \( \langle \delta E_{Y,K}, \gamma \rangle \neq 0 \). Also, \( \langle \delta E_{Y,K}, \alpha \rangle = \langle \delta, \alpha R_{K,Y} \rangle = 1 \) by Theorem 2.10 (ii).

3. EXAMPLES

Example 1. The character ring functor. For any \( H \leq G \), \( M_H \) is the abelian semigroup of the complex characters of \( H \). The families \( T, R, \) and \( C \) are given by the classical induction, restriction, and conjugation of characters, respectively.

If \( \mathcal{B}_H \) is the set of all irreducible characters of \( H \), then \( (M_T, R, C, \mathcal{B}) \) is a based \( G \)-functor satisfying property (P) by Remark 2.2.

Example 2. The Green ring functor. Let \( A \) be a complete commutative noetherian local ring (for example, a field or a complete discrete valuation ring). For each \( H \leq G \), let \( M_H \) be the additive abelian semigroup generated by all isomorphism classes \( (U) \) of finitely generated left \( AH \)-modules subject to the relations \( (U) + (U') = (U \oplus U') \) for any \( AH \)-modules \( U, U' \).

The Krull–Schmidt–Azumaya theorem shows that \( M_H \) is free over the basis \( \mathcal{B}_H \) of all isomorphism classes of indecomposable \( AH \)-modules.

If the families \( T, R, \) and \( C \) are given by the classical induction, restriction, and conjugation of modules, respectively, then \( \mathcal{A}_A = (M_T, R, C, \mathcal{B}) \) is a based \( G \)-functor.

In some cases this based \( G \)-functor does not satisfy property (P). For instance, if \( A \) is a field of characteristic \( p \), \( p \) being a prime dividing \( |G| \), it is well known that Theorem 2.6 does not hold for indecomposable modules (see, for instance [5, Example VII 9.1]). However, if \( H \leq K \leq G \) and \( p \) does not divide \( |K : H| \), then the hypothesis of Corollary 2.8 holds for any indecomposable \( AK \)-module (see [5, Theorem VII 7.7 b]).

Further, by Higman’s result, if the Sylow \( p \)-subgroups of \( G \) are non-cyclic, then \( \mathcal{A}_A \) is a based \( G \)-functor which does not satisfy condition (F) (see, for instance, [5, Theorem VII 5.4]).

Example 3. The \( G \)-functor defined by the group ring \( \mathbb{Z}[G] \) (see Green [3, Ex. 5.5 b]). We set \( \mathbb{N}[G] = \{ \sum n_g g \mid g \in G, n_g \in \mathbb{N} \} \). For each \( H \leq G \), let \( M_H = \{ \alpha \in \mathbb{N}[G] \mid \alpha^h = \alpha \ \text{for all} \ h \in H \} \). For each \( H \leq K \), \( T_{H,K} \) takes \( \alpha \in M_H \) to \( \sum \alpha^e \), summed over a right transversal \( \{ t \} \) of \( H \) in \( K \); \( R_{K,H} \) is the inclusion of \( M_K \) in \( M_H \) and \( C_{H,K} \) takes \( \alpha \in M_H \) to \( \alpha^g \in M_H \).

Consider in each \( M_H \) the basis \( \mathcal{B}_H \) consisting of all different sums \( \sum x^u \), where \( x \in G \) and \( u \) ranges over a right transversal of the centralizer of \( x \).
in $H$. Notice that the orbits of the conjugation action of $G$ on the elements of $\mathcal{B}_1$ are precisely the conjugacy classes of $G$.

Now, $(M, T, R, C, \mathcal{B})$ is a based $G$-functor which is cohomological (see Green [3, 1, 4]). We have:

**Proposition 3.1.** Each cohomological based $G$-functor $\mathcal{A} = (M, T, R, C, \mathcal{B})$ satisfies property (P).

**Proof.** Let $H \triangleleft K \leq G$. Since $\mathcal{A}$ is cohomological, we have $\alpha R_{K, H} T_{H, K} = |K : H| \alpha$ for all $\alpha \in \mathcal{B}_K$. Therefore, if $\delta \in \mathcal{B}_H$ is such that $\langle \alpha R_{K, H}, \delta \rangle$ is not 0, then the unique $\gamma \in \mathcal{B}_K$ satisfying $\langle \delta T_{H, K}, \gamma \rangle$ is not 0 is $\gamma = \alpha$. So it is enough to take $J = H$.

The following result is straightforward and shows that the based $G$-functors of Examples 4 and 5 satisfy property (P).

**Proposition 3.2.** Let $\mathcal{A} = (M, T, R, C, \mathcal{B})$ be a based $G$-functor. Suppose that $\mathcal{A}$ satisfies the following property: If $H \subseteq K \leq G$ and $\alpha \in \mathcal{B}_K$, then for some $\beta \in \mathcal{B}_H$ such that $\langle \alpha R_{K, H}, \beta \rangle$ is not 0, there exist $S, H \leq S \leq K$, and $\gamma \in \mathcal{B}_S$ satisfying $\gamma R_{S, H} = \beta$ and $\gamma T_{S, K} = \alpha$.

Then $\mathcal{A}$ satisfies property (P).

**Example 4.** The Burnside ring functor (see [7]). Each $M_H$ is the abelian semigroup of all isomorphism classes of $H$-sets; $T$, $R$, and $C$ are given by the classical induction, restriction, and conjugation of (isomorphism classes of) transitive sets, respectively (see Dress [2]).

We have in each $M_H$ the basis $\mathcal{B}_H$ consisting of all $H$-isomorphism classes of transitive $H$-sets.

If $H \subseteq K$ and $H/U$ is a transitive $H$-set occurring in the restriction of a transitive $K$-set $K/V$, then, by choosing a suitable representative of the equivalence class of $K/V$, we may assume that $H \cap V = U$. Now take $S = H V$. Then the $S$-set $S/V$ induced to $K$ is $K/V$ and restricted to $H$ is $H/U$.

By Proposition 3.2, $(M, T, R, C, \mathcal{B})$ is a based $G$-functor which satisfies property (P).

**Example 5.** Relatively free based $G$-functors (see Green [3, Ex. 5.6]). Consider a semifunctor $(N, R, C)$ on $G$, that is, a triple of families $N$, $R$, and $C$, where $N = (N_H)_{H \leq G}$ gives, for each subgroup $H$ of $G$, a free abelian semigroup $N_H$, and $R$ and $C$ are as in Definition 1.1, satisfying axioms (b), (c), (d), and (f).

Assume as well that the family $\mathcal{B}^0 = (\mathcal{B}_H^0)_{H \leq G}$, where every $\mathcal{B}_H^0$ is the basis of the abelian semigroup $N_H$, satisfies the two following conditions:

(i) $\beta R_{H, D} \in \mathcal{B}_H^0$ for any $D \leq H$ and any $\beta \in \mathcal{B}_H^0$ and

(ii) $\beta C_{H, g} \in \mathcal{B}_H^0$, for any $\beta \in \mathcal{B}_H^0$. 
Now construct the $G$-functor $(\overline{M}, \overline{T}, \overline{R}, \overline{C})$ as follows.

If $g \in G$ and $H, K \leq G$, consider the symbols $[H, g, K]$ in such a way that $[H, g, K] = [H', g', K']$ if and only if $H = H'$, $K = K'$ and $g^{-1}g' \in K$. For each $H \leq K$, set $M_H = \oplus N_H \otimes [D, x, H]$, summed over all distinct symbols $[D, x, H]$ with $D^x \leq H$. The sum in each summand is given by
\[
\beta \otimes [D, x, H] + \beta' \otimes [D, x, H] = (\beta + \beta') \otimes [D, x, H]
\]
if $\beta, \beta' \in N_H$.

Let $\overline{M}_H$ be the quotient of $M_H$ obtained by identifying $\beta \otimes [D, xg, H]$ with $\beta C_{D, x} \otimes [D^x, g, H]$ for all $\beta \in N_H$. Denote $(\beta \otimes [D, xg, H])$ as the corresponding class.

The maps $\overline{T}_{H, K}, \overline{R}_{H, F},$ and $\overline{C}_{H, g} (F \leq H \leq K, g \in G)$ are defined by
\[
\overline{T}_{H, K} : (\beta \otimes [D, x, H]) \mapsto (\beta \otimes [D, x, K])
\]
\[
\overline{R}_{H, F} : (\beta \otimes [D, x, H]) \mapsto \left( \sum_{g \in \Gamma} \beta R_{D, D \cap F^{x^{-1}g^{-1}}} \otimes [D \cap F^{x^{-1}g^{-1}}, gx, F] \right)
\]
\[
\overline{C}_{H, g} : (\beta \otimes [D, x, H]) \mapsto (\beta \otimes [D, xg, H^g]),
\]
where $\Gamma$ is a transversal of $(D, F^{x^{-1}})$-double cosets of $H^{x^{-1}}$.

Observe that this construction differs slightly from the one given by Green, but this modification is necessary to make the restriction well-defined (which requires independence of the chosen transversal).

Now $\mathcal{B}_H = \{ (\beta \otimes [D, x, H]); \beta \in \mathcal{B}_H^0 \}$ is the basis of $\overline{M}_H$. It is easy to check that $(\overline{M}, \overline{T}, \overline{R}, \overline{C}, \mathcal{B})$ is a based $G$-functor.

It also satisfies property (P) by Proposition 3.2.

Let $H \leq K \leq G$ and let $\alpha = (\beta \otimes [D, x, K]) \in \mathcal{B}_K$. If
\[
\delta = \left( \beta R_{D, D \cap H^{x^{-1}g^{-1}}} \otimes \left[ D \cap H^{x^{-1}g^{-1}}, gx, H \right] \right)
\]
occurs in $\alpha\overline{R}_{K, H}$, then take $S = D^{x^S}H$. Now, take $\gamma = (\beta \otimes [D, gx, D^{x^S}H])$ and check that $\gamma T_{S, K} = \alpha$ and $\gamma \overline{R}_{S, H} = \delta$.

Special cases of this example are the Burnside ring functor and the functor of non-abelian $G$-groups, mentioned below.

**Example 6. Non-abelian $G$-groups.** This example is developed in [6].

Let $H \leq G$. An $H$-group $A$ is a group $A$ together with a group homomorphism $H \to \text{Aut}(A)$. Denote $a \mapsto a^h$ the induced action of $h \in H$ on $a \in A$. The $H$-group $A$ is said to be irreducible if $1 \neq B \leq A$ and $B^h = B$ for all $h \in H$ implies $B = A$. Two $H$-groups $A$ and $B$ are $H$-equivalent if there exist a group isomorphism $f : A \to B$ and a derivation $\delta : H \to B$ (that is, a map $\delta$ satisfying $(h_1h_2)^{\delta} = h_1^{\delta h_2}h_2^{\delta}$ for all $h_1, h_2 \in H$) such that $a^{hf} = a^{f(h)h}$ for all $a \in A$, $h \in H$. 
For each subgroup $H$ of $G$, $M_H$ is the free abelian semigroup generated by the $H$-equivalence classes of irreducible non-abelian $H$-groups. The families $T$, $R$, and $C$ are defined in a natural way. $\mathcal{B}_H$ consists of all $H$-equivalence classes of irreducible non-abelian $H$-groups. Then $\mathcal{A} = (M, T, R, C, \mathcal{B})$ is a based $G$-functor which satisfies property (P).

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