Journal of Differential Equations **171**, 233–250 (2001)
doi:10.1006/jdeq.2000.3847, available online at http://www.idealibrary.com on IDEAL®

Infinity of Subharmonics for Asymmetric Duffing Equations with the Lazer–Leach–Dancer Condition[1]

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Received December 10, 1998; revised February 25, 2000

In this paper, based on a generalized version of the Poincaré–Birkhoff twist theorem by Franks, we establish the existence of infinitely many subharmonics for the asymmetric Duffing equation with the classical Lazer–Leach–Dancer condition. As a consequence of our result, we obtain a sufficient and necessary condition for

Dancer condition; Poincaré–Birkhoff twist theorem.

1. INTRODUCTION

In this paper, we consider periodic solutions for Duffing equations

$$x'' + g(x) = p(t), \tag{1.1}$$

where $g: \mathbf{R} \to \mathbf{R}$ is a continuous function and $p: \mathbf{R} \to \mathbf{R}$ is a continuous 2π-periodic function. We are interested in Eq. (1.1) at resonance by means of potential, that is,

$$(G_0) \qquad \lim_{x \to +\infty} \frac{2G(x)}{x^2} = a^2, \qquad \lim_{x \to -\infty} \frac{2G(x)}{x^2} = b^2, \tag{1.2}$$

where $G(x) = \int_0^x g(s)\, ds$ and a, b are positive constants which satisfy that

$$\frac{1}{a} + \frac{1}{b} = \frac{2}{n}. \tag{1.3}$$

The resonance problem for Eq. (1.1) has been widely investigated in the literature. One of the classical results is given by Lazer and Leach [11] for the symmetric equation

$$x'' + n^2 x + h(x) = p(t), \tag{1.4}$$

[1] This work is supported by NSF of Jiangsu Province, China, Grant BK97136.

0022-0396/01 $35.00

where $h(x)$ is a continuous bounded function. It was shown in [11] that Eq. (1.4) has at least one 2π-periodic solution if

$$\left| \int_0^{2\pi} p(t)\, e^{int}\, dt \right| < 2(\liminf_{x \to +\infty} h(x) - \limsup_{x \to -\infty} h(x)). \tag{1.5}$$

On the other hand, the nonexistence of a 2π-periodic solution for (1.4) was given recently by Alonso and Ortega [2] when h is non-constant and moreover

$$2(\sup h - \inf h) \leqslant \left| \int_0^{2\pi} p(t)\, e^{int}\, dt \right|. \tag{1.6}$$

The asymmetric equation

$$x'' + a^2 x^+ - b^2 x^- + h(x) = p(t), \tag{1.7}$$

where $x^+ = \max\{x, 0\}$ and $x^- = \min\{-x, 0\}$, was considered first by Fŭcik [7] and by Dancer [4]. They called this equation one of "jumping nonlinearities."

Assume that

$$\lim_{|x| \to \infty} \frac{h(x)}{x} = 0, \tag{1.8}$$

and denote by

$$h_\pm = \lim_{x \to \pm\infty} h(x).$$

In [4], Dancer proved, by using rather delicate estimates, the existence of 2π-periodic solution for (1.7) provided that (1.8) and

(i) $h_\pm$ exist, at least one of h_+ and h_- is infinite, and $h_+ \neq h_-$; (1.9)

or

(ii) both h_+ and h_- are finite and

$$2n\left(\frac{h_+}{a^2} - \frac{h_-}{b^2}\right) \neq M(\tau) \qquad \text{for all} \quad \tau \in [0, 2\pi], \tag{1.10}$$

where $M(\tau) = \int_\tau^{\tau+2\pi} p(s)\, \phi(s-\tau)\, ds$ and $\phi(t)$ is the solution of

$$x'' + a^2 x^+ - b^2 x^- = 0$$

satisfying $\phi(0)=0$, $\phi'(0)=1$. This result can be considered as a generalization of Lazer and Leach's classical result, from symmetric Eq. (1.4) to asymmetric Eq. (1.7). Dancer showed in the same paper that for each (a, b) satisfying (1.3), there is a 2π-periodic function $p(t)$ such that the equation

$$x'' + a^2x^+ - b^2x^- = p(t) \tag{1.11}$$

has no 2π-periodic solution.

More interest on the asymmetric Eq. (1.7) came from Lazer and McKenna [12]. They showed in [12] that the asymmetric Eq. (1.7) models the motion of a particle subjected to an asymmetric restoring force and appeared, after separation of variables, as a simplified version of the model of a suspension bridge.

Recently, there have been many interesting results on understanding the resonance phenomenon for the asymmetric oscillator. Ortega [13] showed that all solutions of the small perturbated asymmetric equation

$$x'' + a^2x^+ - b^2x^- = 1 + p(t) \tag{1.12}$$

are bounded via Moser's twist theorem, where $p(t)$ is small and smooth enough. As a consequence, there are infinitely many 2π-periodic solutions and other subharmonic solutions for (1.12). Ortega's result has been generalized to Eq. (1.11) with $p(t)$ smooth enough and $M(\tau)\neq 0$ by Liu [10]. On the other hand, in another paper, Alonso and Ortega [3] showed that there is $R>0$, such that every solution $x(t)$ of (1.11) with

$$(x(t_0))^2 + (x'(t_0))^2 > R$$

for some $t_0 \in \mathbf{R}$, goes to infinity in the future or in the past provided that $M(\tau)=0$ for some τ and for these τ, $M'(\tau)\neq 0$. Alonso and Ortega's result is related to the phenomenon of instability at the roots of unity that appear in Hamiltonian mechanics and implies that there is no large amplitude periodic solution in that case.

We continue the work on the direction of asymmetric oscillations. We show in this paper that under Dancer's conditions there are infinitely many subharmonics of Eq. (1.7), especially, there are arbitrarily large amplitude subharmonics. Instead of (1.8), we assume (G_0) and

$$(g_0) \quad \liminf_{|x|\to\infty} \frac{g(x)}{x} \geqslant \alpha > 0.$$

Moreover, we introduce the conditions

$$\lim_{x \to +\infty} h(x) = +\infty, \qquad \limsup_{x \to -\infty} h(x) < +\infty; \tag{1.13}$$

$$\lim_{x \to -\infty} h(x) = -\infty, \qquad \liminf_{x \to +\infty} h(x) > -\infty; \tag{1.14}$$

$$\lim_{x \to +\infty} h(x) = -\infty, \qquad \liminf_{x \to -\infty} h(x) > -\infty; \tag{1.15}$$

$$\lim_{x \to -\infty} h(x) = +\infty, \qquad \limsup_{x \to +\infty} h(x) < +\infty. \tag{1.16}$$

$$h_\pm \text{ are finite}, \qquad 2n\left(\frac{h_+}{a^2} - \frac{h_-}{b^2}\right) > \max_{\tau \in [0, 2\pi]} M(\tau); \tag{1.17}$$

$$h_\pm \text{ are finite}, \qquad 2n\left(\frac{h_+}{a^2} - \frac{h_-}{b^2}\right) < \min_{\tau \in [0, 2\pi]} M(\tau). \tag{1.18}$$

If Dancer's condition (1.9) or (1.10) holds, then $h(x)$ satisfies one of the above conditions.

Our main theorem in this paper is the following

THEOREM 1.1. *Assume* (G_0), (g_0), *and one of the conditions* (1.13)–(1.18) *holds. Then Eq.* (1.7) *has a* 2π*-periodic solution and a* $2m\pi$*-periodic solution* $x_m(t)$, *for any sufficiently large positive integer* m. $x_m(t)$ *satisfies that*

$$\lim_{m \to \infty} \min_{t \in \mathbf{N}} ((x_m'(t))^2 + (x_m(t))^2) = +\infty.$$

$x_m(t)$ *has* $2(m+1)n$ *zeros in* $[0, 2m\pi)$ *provided that* (1.13) *or* (1.14) *or* (1.17) *holds and* $x_m(t)$ *has* $2(m-1)n$ *zeros in* $[0, 2m\pi)$ *provided that* (1.15) *or* (1.16) *or* (1.18) *holds.*

Moreover, if $p(t)$ *has* 2π*-least period and* m, n *are prime to each other,* $x_m(t)$ *is a* $2m\pi$*-least periodic solution.*

As a consequence of the above theorem and Proposition 3.1 in [3], we obtained the following necessary and sufficient condition for the existence of arbitrarily large amplitude periodic solutions.

THEOREM 1.2. *Assume the* $h_\pm$ *are finite and*

$$\left|2n\left(\frac{h_+}{a^2} - \frac{h_-}{b^2}\right) - M(\tau)\right| + |M'(\tau)| > 0, \qquad \text{for all} \quad \tau \in [0, 2\pi]. \tag{1.19}$$

Then, Eq. (1.7) *has arbitrarily largbe amplitude periodic solutions if and only if Dancer's condition* (1.10) *holds.*

Our approach is based on some estimates for the successor map and uses the Poincaré–Birkhoff twist theorem recent generalized by Franks [8].

The rest of the paper is organized as follows. In Section 2, we give some estimates for the successor map. These estimates will be used to prove twist properties for the successor map in Section 3. In the last section, we use the twist theorem to prove the existence of infinitely many subharmonics.

2. THE SUCCESSOR MAP

The successor map was used successfully in studying bounded perturbations of oscillators at resonance, see [1, 3, 4, 13, 14]. In this section we will give some estimates based on phase plane analysis. We assume (G_0) and (g_0) throughout the section. Moreover, we assume the uniqueness of the solution for the initial value problem associated to (1.1).

Let $x(t; \tau, v)$ be the solution of (1.1) satisfying

$$x(\tau; \tau, v) = 0, \qquad x'(\tau; \tau, v) = v. \tag{2.1}$$

Denote by τ_1 the first zero of $x(t; \tau, v)$ to the right of τ, that is,

$$\tau_1 > \tau, \qquad x(\tau_1; \tau, v) = 0, \qquad x(t; \tau, v) \neq 0, \qquad \text{for} \quad t \in (\tau, \tau_1).$$

We also use the notation $v_1 = x'(\tau_1; \tau, v)$. The successor map is defined by

$$S\colon (\tau, v) \mapsto (\tau_1, v_1).$$

To prove the definition of S well, we need the following argument.

Note that from assumption (G_0), we have $K > 0$, such that

$$G(x) \geqslant -K, \qquad \text{for} \quad x \in \mathbf{R}.$$

Let $x(t) = x(t; \tau, v)$ and $H(t) = \frac{1}{2}(x'(t))^2 + G(x(t))$. Then we have

$$\begin{aligned}|H'(t)| &= |x'(t)\, x''(t) + g(x(t))\, x'(t)| = |p(t)\, x'(t)| \\ &\leqslant P \max\{2, H(t) + K\},\end{aligned}$$

where $P = \max_{t \in [0, 2\pi]} |p(t)|$. By using the Gronwall inequality, we can prove that for any given $T > 0$, there is $v_0 > 0$, such that

$$e^{-PT}\frac{v^2}{2} - K(1 - e^{-PT}) \leqslant H(t) \leqslant e^{PT}\frac{v^2}{2} + K(e^{PT} - 1), \tag{2.2}$$

for $|t - \tau| \leqslant T$ and $v \geqslant v_0$. Moreover, using polar coordinates

$$x = r\cos\theta, \qquad x' = r\sin\theta,$$

we have the polar form associated to (1.1)

$$\theta' = \frac{-xg(x) - (x')^2 + p(t)\,x}{r^2}, \qquad r' = \frac{xx' + x'g(x) - p(t)\,x'}{r}. \tag{2.3}$$

Assumption (G_0) implies that there are $k_1, k_2 > 0$, such that

$$k_1 H(t) \leqslant r^2(t) \leqslant k_2 H(t) \qquad \text{for} \quad H(t) \gg 1. \tag{2.4}$$

From (2.2), (2.3), (2.4), and assumption (g_0), one can prove that for $T > 0$ and $r_0 > 0$, there is $v_0 > 0$, such that

$$\theta'(t) < -\tfrac{1}{2}\min\{1, \alpha\} \qquad \text{and} \qquad r(t) \geqslant r_0,$$
$$\text{for} \quad v \geqslant v_0 \quad \text{and} \quad |t - \tau| \leqslant T,$$

which implies that

$$\theta(\tau + T) - \theta(\tau) \leqslant -\tfrac{1}{2}\min\{1, \alpha\}\ T, \qquad \text{for} \quad v \gg 1$$

and then $x(t; \tau, v)$ has a zero in the right of τ. Thus S is well defined for $|v| \gg 1$ and the uniqueness of the solution for the initial value problem guarantees that S is continuous and one to one. The periodicity of $p(t)$ leads to that

$$S(\tau + 2\pi, v) = S(\tau, v) + (2\pi, 0).$$

In what follows, let $v > 0$. Denote by $\tau_1^{(0)}$ the first right zero of the solution $x(t; \tau, v)$ of the autonomous equation $x'' + g(x) = 0$ satisfying initial condition (2.1). Then

$$\tau_1^{(0)} = \tau + \int_0^{G^{-1}(v^2/2)} \frac{2\,ds}{\sqrt{v^2 - 2G(s)}}, \qquad \text{for} \quad v \gg 1.$$

Moreover, we have

Lemma 2.1. *Assume* (G_0) *and* (g_0). *Then*

$$\tau_1^{(0)} = \tau + \frac{\pi}{a} + \circ(1), \qquad \text{as} \quad v \to +\infty,$$

where $\circ(1) \to 0$ *for* $v \to +\infty$ *as usual.*

Proof. This is a consequence of Theorem 2 in [5] by using assumptions (G_0) and (g_0). ∎

It was proved in [15], under the assumption

$$\limsup_{x\to+\infty}\frac{G(x)}{g^2(x)}<+\infty,$$

that $\tau_1^{(0)}=\tau_1^{(0)}(v)$ is an improper integral which converges uniformly with respect to $v\gg 1$. Moreover $\tau_1^{(0)}(v)$ has the growing property $\forall\varepsilon>0$, $\exists v_0>0$, and $\delta>0$, such that

$$|\tau_1^{(0)}(v+\Delta))-\tau_1^{(0)}(v)|<\varepsilon, \qquad \text{for} \quad v\geqslant v_0 \quad \text{and} \quad |\Delta|<\delta v.$$

By using the above properties and some delicate phase plane analysis, the author proved the following basic lemma in [15]

Lemma 2.2. *Assume* (G_0) *and* (g_0). *Then*

$$\tau_1=\tau_1^{(0)}+\circ(1), \qquad \textit{as} \quad v\to+\infty.$$

Similar ideas are employed in [16]; the reader can consult [16] for details.

As a consequence of Lemma 2.1 and Lemma 2.2, we have immediately that

$$\tau_1=\tau+\frac{\pi}{a}+\circ(1), \qquad \text{as} \quad v\to+\infty. \tag{2.5}$$

For v_1, we have

Lemma 2.3. *Let* $l(t)=\sqrt{2H(t)}$. *Then we have, for given* $T>0$, *that*

$$|l(t)-v|\leqslant 2PT, \qquad \textit{for} \quad |t-\tau|\leqslant T \quad \textit{and} \quad v\gg 1.$$

Especially, $||v_1|-v|\leqslant 2P\,|\tau_1-\tau|$, *for* $v\gg 1$, *where* $P=\max_{t\in[0,2\pi]}|p(t)|$.

Proof. By using (2.2), we have

$$l(t)\geqslant 2\sqrt{K}, \qquad \text{for} \quad |t-\tau|\leqslant T \quad \text{and} \quad v\gg 1,$$

and then

$$l(t)\geqslant \tfrac{1}{2}\,|x'(t)|, \qquad \text{for} \quad |t-\tau|\leqslant T \quad \text{and} \quad v\gg 1.$$

Note that

$$\begin{aligned} l'(t)&=l(t)^{-1}\,(x'(t)\,x''(t)+g(x(t))\,x'(t))\\ &=l(t)^{-1}\,(x'(t)\,p(t)). \end{aligned}$$

Thus we have

$$|l(t)-l(\tau)|=\left|\int_\tau^t \frac{x'(s)}{l(s)}\, p(s)\, ds\right| \leqslant 2P\,|t-\tau| \leqslant 2PT,$$

which follows that

$$||v_1|-v| = |l(\tau_1)-l(\tau)| \leqslant P\,|\tau_1-\tau|. \quad \blacksquare$$

Our next step is to give more precise estimates for τ_1.

At first, we suppose one of the conditions (1.13)–(1.16) holds. We discuss only the case with the condition (1.13); the arguments for other cases are similar. Then we suppose

$$\lim_{x\to+\infty} (g(x)-a^2x) = +\infty. \tag{2.6}$$

We use the notation

$$E_M = \left\{\theta \,\middle|\, \theta=\theta(t)=\arctan\left(\frac{x'(t;\tau,v)}{x(t;\tau,v)}\right) \text{ for } |x(t;\tau,v)| \leqslant M,\ t\in[\tau,\tau_1]\right\}, \tag{2.7}$$

where M is a parameter. If M is fixed, then

$$mes(E_M) \to 0, \qquad \text{as} \quad v\to+\infty. \tag{2.8}$$

Note that for $k>0$ given, we have $M>0$, such that

$$g(x)-a^2x-p(t) \geqslant k>0, \qquad \text{for} \quad x\geqslant M. \tag{2.9}$$

Therefore, by using (2.3), (2.4), (2.7)–(2.9), and Lemma 2.3, we obtain

$$\begin{aligned}
\tau_1-\tau &= \int_{-\pi/2}^{\pi/2} \frac{r^2\, d\theta}{xg(x)+(x')^2-p(t)\,x} \\
&\leqslant \int_{[-\pi/2,\,\pi/2]\setminus E_M} \frac{r^2\, d\theta}{a^2x^2+(x')^2+kx} + \int_{E_M} \frac{r^2\, d\theta}{a^2x^2+(x')^2-\widetilde{M}} \\
&= \int_{-\pi/2}^{\pi/2} \frac{r^2\, d\theta}{a^2x^2+(x')^2} + \int_{[-\pi/2,\,\pi/2]\setminus E_M} \frac{r^2(-kx)\, d\theta}{(a^2x^2+(x')^2+kx)(a^2x^2+(x')^2)} \\
&\quad + \int_{E_M} \frac{r^2\widetilde{M}\, d\theta}{(a^2x^2+(x')^2-\widetilde{M})(a^2x^2+(x')^2)} \\
&\leqslant \frac{\pi}{a}-\frac{\delta}{v}, \qquad \text{for} \quad v\gg 1,
\end{aligned}$$

where $\tilde{M} = M(\max_{|x| \leqslant M} |g(x)| + aM + P)$ and δ is a constant, independent of v. Moreover

$$\delta \to +\infty, \qquad \text{as} \quad k \to +\infty. \tag{2.10}$$

Thus we have

LEMMA 2.4. *If* (2.6) *holds, then for any* $\delta > 0$ *there exists* $v_0 > 0$, *such that*

$$\tau_1 < \tau + \frac{\pi}{a} - \frac{\delta}{v}, \qquad \textit{for} \quad v \geqslant v_0.$$

Denote by $(\tau_2, v_2) = S(\tau_1, v_1)$. Then we have

$$\tau_2 = \tau + \frac{2\pi}{n} + \circ(1), \qquad \text{as} \quad v \to +\infty. \tag{2.11}$$

Moreover, since $\limsup_{x \to -\infty} (g(x) - b^2 x) < +\infty$, we assume $g(x) - b^2 x \leqslant \beta < +\infty$ for $x \leqslant 0$. Then

$$\begin{aligned}
\tau_2 - \tau_1 &= \int_{\pi/2}^{3\pi/2} \frac{r^2\, d\theta}{xg(x) + (x')^2 - p(t)x} \\
&\leqslant \int_{\pi/2}^{3\pi/2} \frac{r^2\, d\theta}{b^2x^2 + (x')^2 + (\beta + P)x} \\
&= \int_{\pi/2}^{3\pi/2} \frac{r^2\, d\theta}{b^2x^2 + (x')^2} \\
&\quad - \int_{\pi/2}^{3\pi/2} \frac{r^2(\beta + P)x\, d\theta}{(b^2x^2 + (x')^2 + (\beta + P)x)(b^2x^2 + (x')^2)}.
\end{aligned}$$

Thus we have

$$\tau_2 \leqslant \tau_1 + \frac{\pi}{b} + O\left(\frac{1}{v_1}\right), \qquad \text{as} \quad v_1 \to \infty, \tag{2.12}$$

where $O(1/v)$ denotes a function for which there are positive constants c_1 and v_0 such that

$$\left| O\left(\frac{1}{v}\right) \right| \leqslant \frac{c_1}{v}, \qquad \text{for} \quad v \geqslant v_0.$$

By using (2.10) and similar estimates for the other cases, we have proved

Lemma 2.5. *Assume* (G_0) *and* (g_0). *Then for any given* $\delta > 0$, *there exists* $v_0 > 0$, *such that*

$$0 < \tau_2 - \tau < \frac{2\pi}{n} - \frac{\delta}{|v|}, \qquad \text{for} \quad |v| \geqslant v_0,$$

provided that condition (1.13) *or* (1.14) *holds*,

$$\tau_2 - \tau > \frac{2\pi}{n} + \frac{\delta}{|v|}, \qquad \text{for} \quad |v| \geqslant v_0,$$

provided that condition (1.15) *or* (1.16) *holds.*

Second, let $h(x)$ be bounded and $h_\pm$ exist. In this case, we will give the following more precise estimation for v_2.

Lemma 2.6. *Assume* $h_\pm$ *exist. Let* $x(t; \tau, v)$ *be the solution of* (1.7) *satisfying*

$$x(\tau; \tau, v) = 0, \qquad x'(\tau; \tau, v) = v > 0.$$

Then

$$v_2 = v + \int_\tau^{\tau + 2\pi/n} p(s)\, \phi'(s - \tau)\, ds + O\left(\frac{1}{v}\right), \qquad \text{as} \quad v \to +\infty. \tag{2.13}$$

Proof. At first, we will prove that

$$v_1 = -v - \int_\tau^{\tau + \pi/a} p(s) \cos a(s - \tau)\, ds + O\left(\frac{1}{v}\right), \qquad \text{as} \quad v \to +\infty.$$

By the proof of Lemma 2.3, we have

$$l(t) - l(\tau) = \int_\tau^t \frac{x'(s)}{l(s)}\, p(s)\, ds.$$

Recall that

$$x'(s) = -v \cos a(s - \tau) - \int_\tau^s (p(\xi) - h) \cos a(s - \xi)\, d\xi.$$

Therefore

$$\begin{aligned}|v_1| - v &= l(\tau_1) - l(\tau)\\ &= \int_\tau^{\tau_1} \frac{-v\cos a(s-\tau)}{l(s)}\, p(s)\, ds\\ &\quad - \int_\tau^{\tau_1} \frac{\int_\tau^s (p(\xi)-h)\cos a(s-\xi)\, d\xi}{l(s)}\, p(s)\, ds\\ &= \int_{\tau+\pi/a}^{\tau} p(s)\cos a(s-\tau)\, ds + O\left(\frac{1}{v}\right), \qquad \text{as} \quad v \to +\infty,\end{aligned}$$

by using (2.5) and Lemma 2.3. Furthermore, a similar estimation for $v_2 - v_1$ yields the conclusion of the lemma. ▮

Next, we discuss the estimation for τ_2. By variation of the constant formula, we have

$$x(t; \tau, v) = \frac{v}{a}\sin a(t-\tau) + \int_\tau^t \frac{p(s)-h}{a}\sin a(t-s)\, ds.$$

Thus τ_1 satisfies that

$$\frac{v}{a}\sin a(\tau_1 - \tau) = \int_\tau^{\tau_1} \frac{h-p(s)}{a}\sin a(\tau_1 - s)\, ds$$

and

$$\tau_1 - \tau = \frac{\pi}{a} + O\left(\frac{1}{v}\right),$$

which follows that

$$\begin{aligned}\frac{v}{a}\sin a(\tau_1-\tau) &= \frac{v}{a}\sin a\left(\frac{\pi}{a} - \tau_1 + \tau\right)\\ &= v\left(\frac{\pi}{a} - \tau_1 + \tau + O\left(\frac{1}{v^3}\right)\right),\end{aligned}$$

and then

$$\begin{aligned}-\tau_1 + \tau + \frac{\pi}{a} &= \frac{1}{v}\int_\tau^{\tau_1} \frac{h-p(s)}{a}\sin a(\tau_1 - s)\, ds + O\left(\frac{1}{v^2}\right)\\ &= \frac{1}{v}\int_\tau^{\tau+\pi/a} \frac{h_+}{a}\sin a\left(\tau + \frac{\pi}{a} - s\right) ds\\ &\quad + \frac{1}{v}\int_\tau^{\tau+\pi/a} \frac{p(s)}{a}\sin a(s-\tau)\, ds + \circ\left(\frac{1}{v}\right).\end{aligned}$$

Same arguments show that

$$-\tau_2+\tau_1+\frac{\pi}{b}=\frac{1}{|v_1|}\int_{\tau+\pi/a}^{\tau+2\pi/n}\frac{h_-}{b}\sin b\left(\tau+\frac{\pi}{a}+\frac{\pi}{b}-s\right)ds$$
$$+\frac{1}{|v_1|}\int_{\tau+\pi/a}^{\tau+2\pi/n}\frac{p(s)}{a}\sin b\left(s-\frac{\pi}{a}-\tau\right)ds+\circ\left(\frac{1}{v}\right).$$

Hence, we can conclude that

LEMMA 2.7. *Assume $h_\pm$ exist. Let $x(t;\tau,v)$ be the solution of* (1.7) *satisfying*

$$x(\tau;\tau,v)=0,\qquad x'(\tau;\tau,v)=v>0.$$

Then

$$\tau_2=\tau+\frac{2\pi}{n}-\frac{2}{v}\left(\left(\frac{h_+}{a^2}-\frac{h_-}{b^2}\right)-\int_\tau^{\tau+2\pi/n}p(s)\,\phi(s-\tau)\,d\tau\right)+\circ\left(\frac{1}{v}\right),\qquad(2.14)$$

as $v\to+\infty$.

3. TWIST PROPERTY

In this section, we will show some twist properties for the successor map based on the estimations given in the last section.

Denote by $(\tau_m, v_m)=S^m(\tau,v)$, π_1, and π_2 the projections from (τ,v) to its first and second factor, respectively. Form Lemma 2.1, Lemma 2.2, and Lemma 2.3, we know that there is $v_0>0$, such that

$$||v_2|-|v||\leqslant\tilde{P},\qquad\text{for}\quad|v|\geqslant v_0,$$

where $\tilde{P}$ is a constant independent of v.

Then, by using Lemma 2.5, we have

LEMMA 3.1. *If* (G_0), (g_0), *and* (1.13) *or* (1.14) *hold, then there exists a* $m_*>0$, *such that for integer* $m\geqslant m_*$, *we have*

$$\tau_{2mn}-\tau<2(m-1)\,\pi,\qquad\textit{for}\quad v\in[v_0+(mn-1)\,\tilde{P},(3mn+1)\,\tilde{P}].$$

Proof. Choose $v_0\geqslant 0$, so that $\delta(\tilde{P})^{-1}>8\pi$, where δ is defined in Lemma 2.5. Moreover, let $m_*\geqslant v_0(2n\tilde{P})^{-1}$. Then for $m\geqslant m_*$ and

$$v\in[v_0+(mn-1)\,\tilde{P},(3mn+1)\,\tilde{P}],$$

we have

$$\pi_2(S^{2i}(\tau, v)) \geqslant v_0, \qquad i = 1, 2, ..., mn-1;$$

thus

$$\begin{aligned}
\tau_{2mn} - \tau &< mn \frac{2\pi}{n} - \left(\frac{\delta}{v} + \frac{\delta}{v_2} + \cdots + \frac{\delta}{v_{2(mn-1)}}\right) \\
&< 2m\pi - \left(\frac{\delta}{v} + \frac{\delta}{v + \widetilde{P}} + \cdots + \frac{\delta}{v + (mn-1)\widetilde{P}}\right) \\
&= 2m\pi - \frac{\delta}{\widetilde{P}} \sum_{j=0}^{mn-1} \frac{1}{v(\widetilde{P})^{-1} + j} \\
&\leqslant 2m\pi - 8\pi \sum_{j=0}^{mn-1} \frac{1}{3mn + 1 + j} \\
&< 2m\pi - 8\pi \frac{1}{4} = 2(m-1)\pi.
\end{aligned}$$

The last inequality is valid because

$$\frac{1}{3mn+1} + \frac{1}{3mn+2} + \cdots + \frac{1}{4mn} = \ln\left(\frac{4}{3}\right) + \circ(1), \qquad \text{as} \quad m \to +\infty,$$

and then

$$\frac{1}{3mn+1} + \frac{1}{3mn+2} + \cdots + \frac{1}{4mn} > \frac{1}{4}, \qquad \text{for} \quad m \gg 1. \quad \blacksquare$$

In what follows, we will give a similar estimation as shown in Lemma 3.1 under assumption (1.17).

From Lemma 2.6 and Lemma 2.7, we know S^{2n} is in the form

$$\begin{cases} \tau_{2n} = \tau + 2\pi + L(\tau)\dfrac{1}{v} + \circ\left(\dfrac{1}{v}\right), \\ v_{2n} = v - L'(\tau) + O\left(\dfrac{1}{v}\right), \end{cases} \tag{3.1}$$

where $L(\tau) = M(\tau) - (2n(h_+/a^2 - h_-/b^2))$. Our assumptions implies that $L(\tau) < 0$, 2π-periodic, and C^2.

Introduce new variables (τ, ρ) with

$$\tau = \tau, \qquad \rho = \frac{1}{\delta v},$$

where $\rho \in [\rho_-, \rho_+]$, $\rho_\pm$ are positive constants depending on $L(\tau)$ and $\delta > 0$ is a parameter to be determined later. Then S^{2n} becomes

$$\begin{cases} \tau_{2n} = \tau + 2\pi + \delta L(\tau)\,\rho + \circ(\delta^2), \\ \rho_{2n} = \rho + \delta L'(\tau)\,\rho^2 + O(\delta^2). \end{cases} \tag{3.2}$$

To give the estimations associated to (3.2), we employ a idea similar to that in [3]. Consider the differential equation

$$\begin{cases} \tau' = L(\tau)\,\rho, \\ \rho' = L'(\tau)\,\rho^2. \end{cases} \tag{3.3}$$

The integral of (3.3) is

$$\Gamma_I\colon -\frac{\rho}{L(\tau)} = I. \tag{3.4}$$

$I = 0$ corresponds to the origin in the phase plane. When $I > 0$, Γ_I is a star-shaped closed curve and the origin is in its interior. Moreover, any solution $(\tau(t; \tau_0, \rho_0), \rho(t; \tau_0, \rho_0))$ with the initial condition

$$\tau(0; \tau_0, \rho_0) = \tau_0, \qquad \rho(0; \tau_0, \rho_0) = \rho_0, \qquad (\tau_0, \rho_0) \in \Gamma_I$$

is a periodic solution with the period

$$k(I) = \int_0^{2\pi} \frac{ds}{IL^2(s)}.$$

Especially, the periodic solution starting from $(\tau_0, 1)$ has the period between $[\int_0^{2\pi} (ds/(I_+ L^2(s))), \int_0^{2\pi} (ds/(I_- L^2(s)))]$, where $I_+ = \max_{\tau \in [0, 2\pi]} \{-L^{-1}(\tau)\}$ and $I_- = \min_{\tau \in [0, 2\pi]} \{-L^{-1}(\tau)\}$.

The solution of (3.2) can be considered as a lift of a class of approximate solution of (3.3) by the one-step method with step-size δ. Since L is C^2 and the set

$$\tilde{E} := \{(\tau, \rho) \mid -L(\tau)\, I_- - \varepsilon_0 \leqslant \rho \leqslant -L(\tau)\, I_+ + \varepsilon_0,\ \tau \in [0, 2\pi]\}$$

is a compact set for $\varepsilon_0 > 0$ small enough, we can prove the following

Lemma 3.2. *For any given $T > 0$, there is a $\Delta > 0$ and a function $\omega\colon (0, \Delta) \to \mathbf{R}^+$ with $\lim_{\delta \to 0} \omega(\delta) = 0$, such that for $\delta \in (0, \Delta)$, the solution of*

(3.2) *with the initial condition* τ_0, $\rho_0 = 1$ *is defined for* $S^{2in}(\tau_0, 1)$, $i = 1, 2, ..., m$, *where* $m = [T/\delta]$. *Moreover,*

$$|\tau_{2in} - 2i\pi - \tau(i\delta)| + |\rho_{2in} - \rho(i\delta)| < \omega(\delta), \qquad \textit{for} \quad i = 1, 2, ..., \left[\frac{T}{\delta}\right], \tag{3.5}$$

where $\tau(t) = \tau(t; \tau_0, \rho_0)$, $\rho(t) = \rho(t; \tau_0, \rho_0)$.

Thus, for $T = (3/2) \int_0^{2\pi} (ds/(I\_L^2(s)))$, $\delta \ll 1$, and $m = [T/\delta]$, we have

$$(\tau_{2in}, \rho_{2in}) \in \tilde{E}, \qquad i = 1, 2, ..., m$$

$$\tau(m\delta) - \tau_0 < -2\pi$$

which implies that

$$\tau_{2mn} - \tau_0 < 2(m-1)\pi.$$

Turning to Eq. (3.1), we have

LEMMA 3.3. *Assume Dancer's assumption* (1.17). *Then we have* $m_* \in \mathbf{N}$, *such that for integer* $m \geqslant m_*$, *there is* $v = v_* \in (m/T, m + 1/T)$, *with*

$$\pi_2(S^{2in}(\tau, v)) \geqslant v_0, \qquad \textit{for} \quad i = 1, 2, ..., m \quad \textit{and} \quad \tau_{2mn} - \tau_0 < 2(m-1)\pi.$$

On the other hand, from the estimation (2.11), for any fixed $m \in N$, we have $v^* > 0$, such that

$$\tau_{2mn} - \tau > 2(m-1)\pi, \qquad \text{for} \quad v \geqslant v^*.$$

For the cases with the assumption (1.15) or (1.16) or (1.18), we can prove, with the obvious modifications, that for sufficiently large positive integer m, there are $v^* > v_* > 0$, such that

$$\tau_{2mn} - \tau > 2(m+1)\pi, \qquad \text{for} \quad v = v_*,$$

and

$$\tau_{2mn} - \tau < 2(m+1)\pi, \qquad \text{for} \quad v = v^*.$$

Hence, we have proved that the map S^{2mn} has a twist property on annulus $S^1 \times [v_*, v^*]$ under assumptions (G_0), (g_0), and one of the conditions (1.13)–(1.18).

4. THE EXISTENCE OF INFINITELY MANY SUBHARMONICS

In this section we will prove the existence of infinitely many subharmonic solutions for Eq. (1.1) by using a generalization of the Poincaré–Birkhoff twist theorem given by Franks [8].

Define the map T by $T(\tau, v) = (\tau_n', v_n')$. The periodicity of the time yields $T(\tau + 2\pi, v) = T(\tau, v) + (2\pi, 0)$. In Section 2, we have proved that for any given $v_B > 0$, there exists $v_A > 0$ such that $v_1 \geqslant v_B$, provided $v \geqslant v_A$.

Denote by

$$A = \{(\tau, v) \mid v \geqslant v_A\}, \qquad B = \{(\tau, v) \mid v \geqslant v_B\}.$$

Without loss of the generality, we suppose g, p be c^1 which guarantee that the solutions of (1.1) are unique with respect to initial data. By using an approximation approach as in [6], we can obtain a series of periodic solutions for an approximate family for Eq. (1.1) and these periodic solutions will converge to a solution of Eq. (1.1) with the same period under our assumptions. Thus we have

LEMMA 4.1. *$T: A \to B$ is a homeomorphism from A to $T(A) \subset B$. Moreover T preserves the element of area given by $v dv d\tau$.*

The proof of Lemma 4.1 is almost the same as Lemma 1 in [9], so we omit the detail here. Let $s = v^2/2$. Then T preserves the element of area given by $ds d\tau$. Suppose $\omega = (v^2/2)\, d\tau = s\, d\tau$. We have

LEMMA 4.2. *For any c^1 Jordan closed* arc *γ in A, we have*

$$\int_\gamma \omega = \int_{T \circ \gamma} \omega.$$

The proof of Lemma 3.2 is the same as Proposition 2.3 in [13] by using the Stokes theorem.

Now consider map $T_m = T^m$. As shown in Section 2, for any v_A sufficiently large, we have m, $v_A^* > 0$, such that T_m is the boundary twist on annulus

$$\tilde{A} = \{(\tau, v) \mid v_A \leqslant v \leqslant v_A^*\}.$$

Moreover, we can choose v_A such that $\tilde{A} \subset \tilde{B}$ and $T_m(\tilde{A}) \subset \tilde{B}$, where

$$\tilde{B} = \{(\tau, v) \mid v_0 \leqslant v_B \leqslant v \leqslant v_B^*\}.$$

As a consequence of Lemma 4.2, we have

$$area(\tilde{B} \backslash T_m(\tilde{A})) = area(\tilde{B} \backslash \tilde{A}).$$

Therefore, we meet all the assumptions of the Franks twist theorem (see Remark of Theorem 4.2 in [8]).

Hence, T_m possesses at least two fixed points which correspond two $2m\pi$-periodic solutions for Eq. (1.1) under assumptions (G_0), (g_0), and (1.13) or (1.14) or (1.17), and two $2(m+1)\pi$-periodic solutions for Eq. (1.1) under assumptions (G_0), (g_0), and (1.15) or (1.16) or (1.18).

These procedures are valid for sufficiently large positive integers m. Thus, with the obvious modifications, we have proved that there exists $m_* \in \mathbf{N}$, such that for integer $m \geqslant m_*$ Eq. (1.1) has at least two $2m\pi$-periodic solutions $x_m(t)$ with

$$\lim_{k \to \infty} ((x'_m(t))^2 + (x_m(t))^2) = +\infty.$$

Moreover, $x_m(t)$ has $2(m+1)n$ or $2(m-1)n$ zeros in $[0, 2m\pi)$. If m, n are prime to each other, then m, $(m+1)n$ and m, $(m-1)n$ are prime to each other; thus $x_m(t)$ is $2m\pi$-least periodic provided that $p(t)$ is 2π-least periodic.

The existence of the 2π-periodic solution is now a consequence by using the Massera theorem. The proof of Theorem 1.1 is thus completed.

Finally, we note that $x_m(t)$ turns around the origin; thus we have that

$$\lim_{m \to \infty} \left(\max_{t \in [0, 2m\pi]} x_m(t) - \min_{t \in [0, 2m\pi]} x_m(t) \right) = +\infty.$$

So we call these subharmonics with arbitrarily large amplitude. On the other hand, in the case with $h_\pm$ finite, if Dancer's condition (1.10) does not hold, then there is some $\tau \in [0, 2\pi]$, such that

$$2n \left(\frac{h_+}{a^2} - \frac{h_-}{b^2} \right) = M(\tau).$$

Moreover, $|M'(\tau)| > 0$ from the assumption (1.19). So we meet all the conditions of Proposition 3.1 in [3]. We conclude from this proposition that there is $R > 0$, such that every solution $x(t)$ of (1.7) with

$$(x(t_0))^2 + (x'(t_0))^2 > R$$

for some $t_0 \in \mathbf{R}$, goes to infinity in the future or in the past, which implies no large amplitude periodic solution. Thus we complete the proof of Theorem 1.2.

ACKNOWLEDGMENTS

The author thanks Professor Bin Liu for his helpful discussions. He also thanks the referee, who pointed out mistakes in the original manuscript and suggested giving a simplification for the proof of Lemma 3.3 by using an idea from [3].

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