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Computation of canonical matrices for chains and cycles of linear mappings

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Abstract

Van Dooren [Linear Algebra Appl. 27 (1979) 103] constructed an algorithm for the computation of all irregular summands in Kronecker's canonical form of a matrix pencil. The algorithm is numerically stable since it uses only unitary transformations.

We construct a unitary algorithm for computation of the canonical form of the matrices of a chain of linear mappings

$$V_1 \text{ --- } V_2 \text{ --- } \cdots \text{ --- } V_t$$

and extend Van Dooren's algorithm to the matrices of a cycle of linear mappings

$$\begin{tikzcd} V_1 \arrow[r, dash] \arrow[rrr, dash, bend right] & V_2 \arrow[r, dash] & \cdots \arrow[r, dash] & V_t \end{tikzcd}$$

where all V_i are complex vector spaces and each line denotes $\to$ or $\leftarrow$.

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1. Introduction

All matrices and vector spaces are considered over the field $\mathbb{C}$ of complex numbers.

By the theorem on pencils of matrices (see [8, Section V]), every pair of $p \times q$ matrices reduces by transformations of simultaneous equivalence

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$$(A_1,\ A_2) \mapsto (R^{-1}A_1S,\ R^{-1}A_2S) \tag{1}$$

(R and S are arbitrary nonsingular matrices) to a direct sum, determined uniquely up to permutation of summands, of pairs of the form

$$(I_n, J_n(\lambda)),\ (J_n(0), I_n),\ (F_n, G_n),\ (F_n^{\mathrm{T}}, G_n^{\mathrm{T}}), \tag{2}$$

where

$$F_n = \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix},\quad G_n = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix},\quad n \geqslant 1 \tag{3}$$

are $(n-1) \times n$ matrices, and $J_n(\lambda)$ is a Jordan block. The direct sum of pairs is defined by

$$(A, B) \oplus (C, D) = (A \oplus C,\ B \oplus D) = \left(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix}\right).$$

Note that F_1 and G_1 in (3) have size 0×1. It is agreed that there exists exactly one matrix, denoted by 0_{n0}, of size $n \times 0$ and there exists exactly one matrix, denoted by 0_{0n}, of size $0 \times n$ for every nonnegative integer n; they represent the linear mappings $0 \to \mathbb{C}^n$ and $\mathbb{C}^n \to 0$ and are considered as zero matrices. Then

$$M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{mq} & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{mq} \end{bmatrix}$$

and

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}$$

for every $p \times q$ matrix M_{pq}.

Van Dooren [19] constructed an algorithm that for every pair (A, B) of $p \times q$ matrices calculates a simultaneously equivalent pair

$$(A_1, B_1) \oplus \cdots \oplus (A_r, B_r) \oplus (C, D),$$

where all (A_i, B_i) are of the form

$$(I_n, J_n(0)),\ (J_n(0), I_n),\ (F_n, G_n),\ (F_n^{\mathrm{T}}, G_n^{\mathrm{T}}),$$

and the matrices C and D are nonsingular. The pair (C, D) is called a *regular part* of (A, B) and is simultaneously equivalent to a direct sum of pairs of the form $(I_n, J_n(\lambda))$ with $\lambda \neq 0$. This algorithm uses only transformations (1) with *unitary* R and S, which is important for its numerical stability.

In this article we construct a unitary algorithm for computation of the canonical form of the matrices of a chain of linear mappings

$$V_1 \xrightarrow{\ \mathscr{A}_1\ } V_2 \xrightarrow{\ \mathscr{A}_2\ } \cdots \xrightarrow{\ \mathscr{A}_{t-1}\ } V_t \tag{4}$$

(see Proposition 4.1) and extend Van Dooren's algorithm to the matrices of a cycle of linear mappings

$$\begin{tikzcd} \mathscr{A}: V_1 \arrow[r, dash, "\mathscr{A}_1"] \arrow[rrrr, dash, bend right, "\mathscr{A}_t"'] & V_2 \arrow[r, dash, "\mathscr{A}_2"] & \cdots \arrow[r, dash, "\mathscr{A}_{t-2}"] & V_{t-1} \arrow[r, dash, "\mathscr{A}_{t-1}"] & V_t, \end{tikzcd} \quad t \geqslant 2, \tag{5}$$

(see Theorem 6.1), where each line is the arrow $\longrightarrow$ or the arrow $\longleftarrow$ and $V_1, \dots, V_t$ are vector spaces.

For instance, the linear mappings $\mathscr{A}_1$ and $\mathscr{A}_2$ of a cycle

$$\begin{tikzcd} V_1 \arrow[r, shift left, "\mathscr{A}_1"] \arrow[r, shift right, "\mathscr{A}_2"'] & V_2 \end{tikzcd}$$

are represented by a pair of matrices (A_1, A_2) with respect to bases in V_1 and V_2, and a change of the bases reduces this pair by transformations of simultaneous equivalence (1); in this case our algorithm coincides with Van Dooren's algorithm.

Similarly, the linear mappings $\mathscr{A}_1$ and $\mathscr{A}_2$ of a cycle

$$\begin{tikzcd} V_1 \arrow[r, shift left, "\mathscr{A}_1"] & V_2 \arrow[l, shift left, "\mathscr{A}_2"] \end{tikzcd}$$

are represented by a pair (A_1, A_2), and a change of the bases in V_1 and V_2 reduces this pair by transformations of *contragredient equivalence*

$$(A_1,\ A_2) \mapsto (R^{-1}A_1S,\ S^{-1}A_2R).$$

The *direct sum* of the cycle (5) and a cycle

$$\begin{tikzcd} \mathscr{A}': V_1' \arrow[r, dash, "\mathscr{A}_1'"] \arrow[rrrr, dash, bend right, "\mathscr{A}_t'"'] & V_2' \arrow[r, dash, "\mathscr{A}_2'"] & \cdots \arrow[r, dash, "\mathscr{A}_{t-2}'"] & V_{t-1}' \arrow[r, dash, "\mathscr{A}_{t-1}'"] & V_t' \end{tikzcd}$$

with the same orientation of arrows is the cycle $\mathscr{A} \oplus \mathscr{A}'$:

$$\begin{tikzcd} V_1 \oplus V_1' \arrow[r, dash, "\mathscr{A}_1 \oplus \mathscr{A}_1'"] \arrow[rrr, dash, bend right, "\mathscr{A}_t \oplus \mathscr{A}_t'"'] & V_2 \oplus V_2' \arrow[r, dash, "\mathscr{A}_2 \oplus \mathscr{A}_2'"] & \cdots \arrow[r, dash, "\mathscr{A}_{t-1} \oplus \mathscr{A}_{t-1}'"] & V_t \oplus V_t' \end{tikzcd}$$

A cycle $\mathscr{A}$ of the form (5) is called *regular* if all $\mathscr{A}_i$ are bijections; otherwise it is called *singular*. By a *regularizing decomposition* of $\mathscr{A}$, we mean a decomposition

$$\mathscr{A} = \mathscr{D} \oplus \cdots \oplus \mathscr{G} \oplus \mathscr{P}, \tag{6}$$

where $\mathscr{D}, \dots, \mathscr{G}$ are direct-sum-indecomposable singular cycles and $\mathscr{P}$ is a regular cycle.

In Section 2 we recall notions of quiver representations; they allow to formulate our algorithms pictorially.

In Section 3 we recall the classification of chains (4) and cycles (5) of linear mappings. The classification of cycles of linear mappings was obtained by Nazarova [15] and, independently, by Donovan and Freislich [5] (see also [7], Theorem 11.1).

In Section 4 we construct an algorithm that gets the canonical form of the matrices of a chain of linear mappings using only unitary transformations.

In Sections 5 and 6 we construct an algorithm that gets a regularizing decomposition (6) of a cycle of linear mappings using only unitary transformations.[1] The singular summands $\mathscr{D}, \ldots, \mathscr{G}$ will be obtained in canonical form.

The canonical form of the (nonsingular) matrices $P_1, \ldots, P_t$ of the regular summand

$$\begin{tikzcd} \mathscr{P}: U_1 \arrow[r, dash, "\mathscr{P}_1"] \arrow[rrrr, dash, bend right, "\mathscr{P}_t"'] & U_2 \arrow[r, dash, "\mathscr{P}_2"] & \cdots \arrow[r, dash, "\mathscr{P}_{t-2}"] & U_{t-1} \arrow[r, dash, "\mathscr{P}_{t-1}"] & U_t \end{tikzcd}$$

in (6) is not determined by this algorithm. We may compute it as follows. We first reduce P_1 to the identity matrix changing the basis in the space U_2. Then we reduce P_2 to the identity matrix changing the basis in the space U_3, and so on until obtain

$$P_1 = \cdots = P_{t-1} = I_n. \tag{7}$$

At last, changing the bases of all spaces $U_1, \ldots, U_t$ by the same transition matrix S (this preserves the matrices (7)), we can reduce the remaining matrix P_t to a nonsingular Jordan canonical matrix Φ by similarity transformations $S^{-1}P_tS$. Clearly, the obtained sequence

$$(I_n, \ldots, I_n, \Phi)$$

is the canonical form of the matrices of $\mathscr{P}$.

2. Terminology of quiver representations

The notion of a quiver and its representations was introduced by Gabriel [6] (see also [7, Section 7]) and admits to formulate classification problems for systems of linear mappings. A *quiver* is a directed graph; loops and multiple arrows are allowed. Its *representation* $\mathscr{A}$ over $\mathbb{C}$ is given by assigning to each vertex v a complex vector space V_v and to each arrow $\alpha : u \to v$ a linear mapping $\mathscr{A}_\alpha : V_u \to V_v$ of the corresponding vector spaces.

For instance, a representation of the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\varepsilon"] & \\ 1 \arrow[loop left, "\alpha"] \arrow[ur, "\beta"] \arrow[rr, shift left, "\gamma"] \arrow[rr, shift right, "\delta"'] & & 3 \arrow[loop right, "\zeta"] \end{tikzcd}$$

is a system of linear mappings

[1] This improves the numerical stability of the algorithms. Nevertheless, this does not guarantee that the computed structure of the cycle coincides with its original structure.

$$
\begin{tikzcd}
 & V_2 \arrow[dr, "\mathscr{A}_\varepsilon"] & \\
V_1 \arrow[loop left, "\mathscr{A}_\alpha"] \arrow[ur, "\mathscr{A}_\beta"] \arrow[rr, shift left, "\mathscr{A}_\gamma"] \arrow[rr, shift right, "\mathscr{A}_\delta"'] & & V_3 \arrow[loop right, "\mathscr{A}_\zeta"]
\end{tikzcd}
$$

The number

$$\dim_v \mathscr{A} := \dim V_v$$

is called the *dimension of $\mathscr{A}$ at the vertex v*, the set of these numbers

$$\dim \mathscr{A} := \{\dim V_v\}_v$$

is called the *dimension of $\mathscr{A}$*.

Two representations $\mathscr{A}$ and $\mathscr{A}'$ are called *isomorphic* if there exists a set $\mathscr{S}$ of linear bijections $\mathscr{S}_v : \mathscr{A}_v \to \mathscr{A}'_v$ (assigned to all vertices v) transforming $\mathscr{A}$ to $\mathscr{A}'$. That is, the diagram

$$
\begin{tikzcd}
V_u \arrow[r, "\mathscr{A}_\alpha"] \arrow[d, "\mathscr{S}_u"'] & V_v \arrow[d, "\mathscr{S}_v"] \\
V'_u \arrow[r, "\mathscr{A}'_\alpha"] & V'_v
\end{tikzcd}
\tag{8}
$$

must be commutative ($\mathscr{A}'_\alpha \mathscr{S}_u = \mathscr{S}_v \mathscr{A}_\alpha$) for every arrow $\alpha : u \longrightarrow v$. In this case we write

$$\mathscr{S} = \{\mathscr{S}_v\} : \mathscr{A} \stackrel{\sim}{\to} \mathscr{A}' \quad \text{and} \quad \mathscr{A} \simeq \mathscr{A}'. \tag{9}$$

The *direct sum* of $\mathscr{A}$ and $\mathscr{A}'$ is the representation $\mathscr{A} \oplus \mathscr{A}'$ formed by $V_v \oplus V'_v$ and $\mathscr{A}_\alpha \oplus \mathscr{A}'_\alpha$.

The following theorem is a well-known corollary of the Krull–Schmidt theorem [1, Theorem I.3.6] and holds for representations over an arbitrary field.

Theorem 2.1. *Every representation of a quiver decomposes into a direct sum of indecomposable representations uniquely, up to isomorphism of summands.*

Every representation of a quiver over $\mathbb{C}$ is isomorphic to a representation, in which the vector spaces V_v assigned to the vertices all have the form $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$. Such a representation of dimension $\{d_v\}$ with $d_v \in \{0, 1, 2, \ldots\}$ is called a *matrix representation* [2] and is given by a set $\mathbb{A}$ of matrices $\mathbb{A}_\alpha \in \mathbb{C}^{d_v \times d_u}$ assigned to the arrows $\alpha : u \longrightarrow v$. We will consider mainly matrix representations.

For every matrix representation $\mathbb{A} = \{A_\alpha\}$ of a quiver $\mathscr{Q}$, we define the *transpose matrix representation*

$$\mathbb{A}^{\mathrm{T}} = \{A_\alpha^{\mathrm{T}}\} \tag{10}$$

[2] A matrix representation also arises when we fix bases in all the spaces of a representation. As follows from (8), two matrix representations are isomorphic if and only if they give the same representation but in possible different bases.

of the quiver $\mathscr{Q}^{\mathrm{T}}$ obtained from $\mathscr{Q}$ by changing the direction of each arrow. Clearly,

$$\mathbb{S} = \{S_v\} : \mathbb{A} \xrightarrow{\sim} \mathbb{B} \text{ implies } \mathbb{S}^{\mathrm{T}} = \{S_v^{\mathrm{T}}\} : \mathbb{B}^{\mathrm{T}} \xrightarrow{\sim} \mathbb{A}^{\mathrm{T}}. \tag{11}$$

The systems of linear mappings (4) and (5) may be considered as representations of the quivers

$$\mathscr{L} : 1 \overset{\alpha_1}{\text{———}} 2 \overset{\alpha_2}{\text{———}} \cdots \overset{\alpha_{t-2}}{\text{———}} (t-1) \overset{\alpha_{t-1}}{\text{———}} t \tag{12}$$

and

$$\mathscr{C} : \begin{tikzcd} 1 \arrow[r, dash, "\alpha_1"] \arrow[rrrr, dash, bend right, "\alpha_t"'] & 2 \arrow[r, dash, "\alpha_2"] & \cdots \arrow[r, dash, "\alpha_{t-2}"] & (t-1) \arrow[r, dash, "\alpha_{t-1}"] & t \end{tikzcd} \tag{13}$$

with the same orientations of arrows as in (4) and (5). The quiver (13) will be called a *cycle*; the symbol $\mathscr{C}$ will always denote the cycle (13).

If $\mathbb{A}$ is a matrix representation of a quiver with an indexed set of arrows $\{\alpha_i \mid i \in I\}$, we will write A_i instead of $\mathbb{A}_{\alpha_i}$. So a matrix representation $\mathbb{A}$ of the cycle $\mathscr{C}$ is given by a sequence of matrices

$$\mathbb{A} = (A_1, \ldots, A_t).$$

3. Classification theorems

In this section, we recall the classification of representations of the quivers (12) and (13), and mention articles considering special cases. Some of these articles are little known outside of representation theory.

We first consider the cycles of length 2. The representations of the cycle $1 \rightrightarrows 2$ were classified by Kronecker [12] in 1890 (see also [8, Section V] or [7, Section 1.8]): every pair of $p \times q$ matrices is simultaneously equivalent to a direct sum of pairs of the form (2). A simple and short proof of this result was obtained by Nazarova and Roiter [16].

A classification of representations of the cycle $1 \rightleftarrows 2$ was obtained by Dobrovol'skaya and Ponomarev [4] in 1965: every matrix representation is isomorphic to a direct sum, determined uniquely up to permutation of summands, of matrix representations of the form

$$(I_n, J_n(\lambda)),\ (J_n(0), I_n),\ (F_n, G_n^{\mathrm{T}}),\ (F_n^{\mathrm{T}}, G_n) \tag{14}$$

(see (3)). Over an arbitrary field, the Jordan block $J_n(\lambda)$ is replaced by a Frobenius block

$$\Phi_n = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1 \end{bmatrix},$$

where

$$x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1}x + \alpha_n = p(x)^t$$

for some irreducible polynomial $p(x)$ and some integer t. This result was proved again by Rubió and Gelonch [17] in 1992, Holtz [10] in 2000, and Horn and Merino [11] in 1995; the last article also contains many applications of this classification.

A classification of systems of linear mappings of the form

$$\begin{array}{ccc} V_1 & \longrightarrow & V_2 \\ \downarrow & & \uparrow \\ V_3 & \longleftarrow & V_4 \end{array}$$

was given by Nazarova [13] in 1961 over the field with two elements, and by Nazarova [14] in 1967 over an arbitrary field.

A quiver is said to be of *tame type* if the problem of classifying its representations does not contain the problem of classifying pairs of matrices up to simultaneous similarity. If a quiver $\mathcal{Q}$ is not of tame type, then a full classification of its representations is impossible since it must contain a classification of representations of all quivers (see [18, Section 3.1] or [3, Section 2]). Nevertheless, each particular representation of $\mathcal{Q}$ can be reduced to canonical form (see [2] or [18, Section 1.4]).

Nazarova [15] and, independently, Donovan and Freislich [5] in 1973 classified representations of all quivers of tame type (see also [7, Section 11]). In particular, they classified representations of the cycle (13), which is of tame type (see this classification also in [7, Theorem 11.1]). This classification is not mentioned in many articles on linear algebra and system theory that study its special cases (for instance, in the article by Gelonch [9] containing the classification of representations of the cycle (13) with orientation $1 \to 2 \to \cdots \to t \to 1$).

Gabriel [6] (see also [7, Section 11]) classified representations of all quivers having a finite number of nonisomorphic indecomposable representations. In particular, he classified representations of the quiver (12).

Now we formulate theorems that classify representations of the quivers (12) and (13).

For every pair of integers (i, j) such that $1 \leqslant i \leqslant j \leqslant t$, we define the matrix representation

$$\mathbb{L}_{ij} : 1 \xrightarrow{\;\;0\;\;} \cdots \xrightarrow{\;\;0\;\;} i \xrightarrow{\;\;I_1\;\;} \cdots \xrightarrow{\;\;I_1\;\;} j \xrightarrow{\;\;0\;\;} \cdots \xrightarrow{\;\;0\;\;} t \tag{15}$$

of dimension $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots 0)$ of the quiver (12). By the next theorem, which holds over an arbitrary field, the representations $\mathbb{L}_{ij}$ form a full set of nonisomorphic indecomposable matrix representations of (12).

Theorem 3.1 (see [6]). *For every system of linear mappings* (4), *there are bases of the spaces* $V_1, \ldots, V_t$, *in which the sequence of matrices of* $\mathscr{A}_1, \ldots, \mathscr{A}_{t-1}$ *is a direct sum of sequences* $(0, \ldots, 0, I_1, \ldots, I_1, 0, \ldots, 0)$ *of dimension* $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots 0)$. *This sum is determined by the system* (4) *uniquely up to permutation of summands.*

The classification of representations of a cycle (13) follows from Theorem 2.1 and the next fact: if a matrix representation of this cycle is direct-sum-indecomposable, then at least $t-2$ of its matrices are nonsingular. Clearly, these $t-2$ matrices reduce to the identity matrices and the remaining two matrices reduce to the form (2) or (14) depending on the orientation of their arrows. This gives the following theorem.

Theorem 3.2 (see [5] or [15]). *For every system of linear mappings* (5), *there are bases in the spaces* $V_1, \dots, V_t$, *in which the sequence of matrices of* $\mathscr{A}_1, \dots, \mathscr{A}_t$ *is a direct sum*, *determined by* (5) *uniquely up to permutation of summands*, *of sequences of the following form* (*the points denote sequences of identity matrices or* 0_{00}):

(i) $(J_n(\lambda), \dots)$ *with* $\lambda \neq 0$;
(ii) $(\dots, J_n(0), \dots)$ *with* $J_n(0)$ *at the place* $i \in \{1, \dots, t\}$;
(iii) $(\dots, A_i, \dots, A_j, \dots)$, *where* A_i *and* A_j *depend on the direction of the mappings* $\mathscr{A}_i$ *and* $\mathscr{A}_j$ *in the sequence*

$$V_1 \ \overset{\mathscr{A}_1}{\text{——}}\ V_2 \ \overset{\mathscr{A}_2}{\text{——}}\ \cdots \ \overset{\mathscr{A}_{t-1}}{\text{——}}\ V_t \ \overset{\mathscr{A}_t}{\text{——}}\ V_1$$

(*see* (5)) *as follows*:

$$(A_i, A_j) = \begin{cases} (F_n, G_n) \text{ or } (F_n^{\mathrm{T}}, G_n^{\mathrm{T}}) & \text{if } \mathscr{A}_i \text{ and } \mathscr{A}_j \text{ have opposite directions}, \\ (F_n, G_n^{\mathrm{T}}) \text{ or } (F_n^{\mathrm{T}}, G_n) & \text{otherwise}. \end{cases}$$

This theorem, with a nonsingular Frobenius block instead of $J_n(\lambda)$ in (i), holds over an arbitrary field.

In the remaining part of this section, we recall Gabriel and Roiter's construction [7, Section 11.1] of summands (ii) and (iii).

For every integer n, denote by $[n]$ the natural number such that

$$1 \leqslant [n] \leqslant t \quad \text{and} \quad [n] \equiv n \bmod t.$$

Let

$$l \text{ —— } (l+1) \text{ —— } (l+2) \text{ —— } \cdots \text{ —— } r, \qquad 1 \leqslant l \leqslant t, \tag{16}$$

be a "clockwise walk" on the cycle (13) that starts at the vertex l, passes through the vertices

$$[l+1], [l+2], \dots, [r-1],$$

and stops at the vertex $[r]$. This walk determines the representation $\mathscr{A}$ of $\mathscr{C}$ in which each space V_v is spanned by all $i \in \{l, l+1, \dots, r\}$ such that $[i] = v$:

$$V_v = \langle i \mid l \leqslant i \leqslant r,\ [i] = v \rangle,$$

and all the nonzero actions of linear mappings $\mathscr{A}_{\alpha_1}, \dots, \mathscr{A}_{\alpha_t}$ on the basis vectors are given by (16). The matrices of $\mathscr{A}_{\alpha_1}, \dots, \mathscr{A}_{\alpha_t}$ in these bases form a matrix representation denoted by

$$\mathbb{G}_{lr}. \tag{17}$$

Example 3.1. The walk

$$\begin{tikzcd}[row sep=tiny, column sep=small]
 & 2 & 3 \arrow[l] & \\
1 \arrow[ur] & 8 & 9 \arrow[l] & \\
7 \arrow[ur] \arrow[dr] & & & 4 \arrow[uul] \arrow[dl] \\
 & 6 & 5 \arrow[l] &
\end{tikzcd}$$

on the cycle

$$\mathscr{C}:\quad \begin{tikzcd}[row sep=tiny, column sep=small]
 & 2 & 3 \arrow[l] & \\
1 \arrow[ur] \arrow[dr] & & & 4 \arrow[ul] \arrow[dl] \\
 & 6 & 5 \arrow[l] &
\end{tikzcd}$$

determines the representation

$$\mathbb{G}_{1,9}:\quad \begin{tikzcd}
 & \langle 2,8\rangle & \langle 3,9\rangle \arrow[l, "I_2"'] & \\
\langle 1,7\rangle \arrow[ur, "I_2"] \arrow[dr, "{\begin{bmatrix}0 & 1\end{bmatrix}}"'] & & & \langle 4\rangle \arrow[ul, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] \arrow[dl, "I_1"] \\
 & \langle 6\rangle & \langle 5\rangle \arrow[l, "I_1"] &
\end{tikzcd}$$

Lemma 3.1 (see [7, Section 11.1]). *The set of all* $\mathbb{G}_{lr}$ *coincides with the set of matrix representations of the form* (ii) *and* (iii):

(a) $\mathbb{G}_{lr}$ *with* $r \not\equiv l-1 \bmod t$ *is the matrix representation* (iii) *of dimension* $(d_1, \ldots, d_t)$, *where* d_i *is the number of* $n \in \{l, l+1, \ldots, r\}$ *such that* $[n] = i$. (*Note that all representations of the form* (iii) *have distinct dimensions and so they are determined by their dimensions*.)

(b) $\mathbb{G}_{l,l-1+pt} = (I_p, \ldots, I_p, J_p(0), I_p, \ldots, I_p)$, *where* $J_p(0)$ *is at the* $[l-1]$*st place*.

4. Chains of linear mappings

In this section we give an algorithm that calculates the canonical form of the matrices of a chain of linear mappings (4) using only unitary transformations.

We may represent a system of linear mappings (4) by a sequence of matrices $\mathbb{A} = (A_1, \ldots, A_{t-1})$ choosing bases in the spaces $V_1, \ldots, V_t$. We will consider this sequence as the matrix representation

$$\mathbb{A}: 1 \xrightarrow{\ A_1\ } 2 \xrightarrow{\ A_2\ } \cdots \xrightarrow{\ A_{t-1}\ } t \tag{18}$$

of the quiver (12).

For every vertex i, a change of the basis in V_i changes $\mathbb{A}$. This transformation of $\mathbb{A}$ will be called a *transformation at vertex* i. It will be called a *unitary transformation* if the transition matrix to a new basis of V_i is unitary.

4.1. The algorithm for chains

Let $\mathbb{A}$ be a matrix representation (18) of dimension

$$\dim \mathbb{A} = (d_1, \dots, d_t)$$

of the quiver (12).

Step 1: By unitary transformations at vertices 1 and 2, we reduce A_1 to the form

$$B_1 = \begin{bmatrix} 0 & H \\ 0 & 0 \end{bmatrix}, \tag{19}$$

where H is a nonsingular matrix. These transformations change A_2; denote the new matrix by A_2'. Define the representation

$$\mathbb{A}^{(1)} : 1 \xrightarrow{\quad B_1 \quad} 2 \xrightarrow{\quad A_2' \quad} 3 \xrightarrow{\quad A_3 \quad} \cdots \xrightarrow{\quad A_{t-1} \quad} t$$

of the quiver (12).

Step r $(1 < r < t)$: Assume we have constructed in the step $r-1$ a representation

$$\mathbb{A}^{(r-1)} : 1 \xrightarrow{\ B_1\ } \cdots \xrightarrow{\ B_{r-1}\ } r \xrightarrow{\ A_r'\ } (r+1) \xrightarrow{\ A_{r+1}\ } \cdots \xrightarrow{\ A_{t-1}\ } t\,, \tag{20}$$

where $B_1, \dots, B_{r-1}$ are block matrices. Denote by $k_1, \dots, k_r$ the sizes of horizontal strips of B_{r-1} if $\alpha_{r-1} : (r-1) \longrightarrow r$ and the sizes of vertical strips of B_{r-1} if $\alpha_{r-1} : (r-1) \longleftarrow r$ (see (12)).

We will reduce $\mathbb{A}^{(r-1)}$ by unitary transformations at the vertices r and $r+1$:

(i) If $\alpha_r : r \longrightarrow r+1$, then we divide A_r' into r vertical strips of sizes $k_1, k_2, \dots, k_r$ and reduce A_r' to the form

$$B_r = \left[\begin{array}{cc|cc|cc|c|cc} 0 & H_1 & * & * & * & * & \cdots & * & * \\ 0 & 0 & 0 & H_2 & * & * & \cdots & * & * \\ 0 & 0 & 0 & 0 & 0 & H_3 & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & H_r \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \tag{21}$$

(where all H_i are nonsingular matrices and all $*$ are unspecified matrices) starting from the first vertical strip by unitary column-transformations within vertical strips and by unitary row-transformations.

(ii) If $\alpha_r : r \longleftarrow r+1$, then we partition A_r' into r horizontal strips of sizes $k_1, k_2, \dots, k_r$ and reduce A_r' to the form

$$
B_r = \left[\begin{array}{cc|c|c|c|c|c}
0 & H_1 & * & \cdots & * & * & * \\
0 & 0 & * & \cdots & * & * & * \\
\hline
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
0 & 0 & 0 & \cdots & H_{r-2} & * & * \\
0 & 0 & 0 & \cdots & 0 & * & * \\
\hline
0 & 0 & 0 & \cdots & 0 & H_{r-1} & * \\
0 & 0 & 0 & \cdots & 0 & 0 & * \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & H_r \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array}\right] \tag{22}
$$

(where all H_i are nonsingular matrices) starting from the lower strip, by unitary row-transformations within horizontal strips and by unitary column-transformations.

These transformations change A_{r+1} (if $r < t-1$); denote the obtained matrix by A'_{r+1} and define the representation

$$
\mathbb{A}^{(r)} : 1 \xrightarrow{\ B_1\ } \cdots \xrightarrow{\ B_r\ } (r+1) \xrightarrow{\ A'_{r+1}\ } (r+2) \xrightarrow{\ A_{r+2}\ } \cdots \xrightarrow{\ A_{t-1}\ } t\,, \tag{23}
$$

where $B_1, \ldots, B_{r-1}$ are the same as in (20).

4.2. The result

After step $t-1$, we have obtained the representation

$$
\mathbb{A}^{(t-1)} : 1 \xrightarrow{\ B_1\ } 2 \xrightarrow{\ B_2\ } \cdots \xrightarrow{\ B_{t-1}\ } t\,,
$$

where each B_r has the form (21) or (22). Denote by D_r the matrix obtained from B_r by replacement of each block H_i by the identity matrix of the same size and all blocks $*$ by 0. Define the representation

$$
\mathbb{D} : 1 \xrightarrow{\ D_1\ } 2 \xrightarrow{\ D_2\ } \cdots \xrightarrow{\ D_{t-1}\ } t\,, \tag{24}
$$

and transform it to a representation $\mathbb{Q}$ of another quiver as follows.

We first replace each vertex i by the vertices $i_1, \ldots, i_{d_i}$, where

$$
d_i = \dim_i \mathbb{D} = \dim_i \mathbb{A}.
$$

Then we replace each arrow $i \xrightarrow{\ D_i\ } (i+1)$ by arrows that are in one-to-one correspondence with the units of the matrix D_i: every unit at the place (p, q) in D_i determines the arrow

$$
i_q \xrightarrow{I_1} (i+1)_p \quad \text{if } \alpha_i : i \longrightarrow (i+1)
$$

or the arrow

$$i_p \xleftarrow{I_1} (i+1)_q \quad \text{if } \alpha_i : i \longleftarrow (i+1).$$

(These arrows represent the action on the basic vectors of the linear operator

$$\mathbb{C}i_1 \oplus \cdots \oplus \mathbb{C}i_{d_i} \;\text{——}\; \mathbb{C}(i+1)_1 \oplus \cdots \oplus \mathbb{C}(i+1)_{d_{i+1}}$$

directed as $\alpha_i : i$ —— $(i+1)$ and given by the matrix D_i.) Since in each row and in each column of D_i at most one entry is 1 and the others are 0, two arrows i_p —— $(i+1)_q$ and $i_{p'}$ —— $(i+1)_{q'}$ have no common vertices ($\mathscr{D}_i$ sends each basic vector to a basic vector or to 0 and cannot send two basic vectors to the same basic vector). Denote the obtained representation by

$$\mathbb{Q}. \tag{25}$$

The quiver representation $\mathbb{Q}$ is a union of nonintersecting chains; each of them determines a representation of the form $\mathbb{L}_{ij}$. Denote by $\oplus\mathbb{Q}$ the direct sum of the corresponding representations $\mathbb{L}_{ij}$.

Proposition 4.1. *The representation $\oplus\mathbb{Q}$ is the canonical form* (*see Theorem* 3.1) *of a matrix representation $\mathbb{A}$ of the quiver* (12).

Example 4.1. Suppose we apply the algorithm to a matrix representation

$$\mathbb{A} : 1 \xrightarrow{A_1} 2 \xrightarrow{A_2} 3 \xleftarrow{A_3} 4$$

of dimension (4, 5, 4, 5) and obtain

$$B_1 = \begin{bmatrix} 0_{31} & H_1 \\ 0_{21} & 0_{23} \end{bmatrix}, \quad B_2 = \left[\begin{array}{cc|cc} 0_{21} & H_2 & * & * \\ 0_{11} & 0_{12} & 0_{11} & H_3 \\ 0_{11} & 0_{12} & 0_{11} & 0_{11} \end{array}\right],$$

$$B_3 = \left[\begin{array}{ccc} 0_{22} & H_4 & * \\ \hline 0_{12} & 0_{12} & * \\ \hline 0_{12} & 0_{12} & H_5 \end{array}\right],$$

where $H_1, \ldots, H_5$ are nonsingular 3×3, 2×2, 1×1, 2×2, and 1×1 matrices. Then

$$\mathbb{D} : 1 \xrightarrow{\begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}} 2 \xrightarrow{\begin{bmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&0&0 \end{bmatrix}} 3 \xleftarrow{\begin{bmatrix} 0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&1 \end{bmatrix}} 4$$

and

$$\begin{tikzcd}[row sep=small]
 & 1_1 & 2_1 & 3_1 & 4_1 \\
 & 1_2 \arrow[ur] & 2_2 \arrow[ur] & 3_2 & 4_2 \\
\mathbb{Q}: & 1_3 \arrow[ur] & 2_3 \arrow[ur] & 3_3 & 4_3 \arrow[uul] \\
 & 1_4 \arrow[ur] & 2_4 & 3_4 & 4_4 \arrow[uul] \\
 & & 2_5 \arrow[uur] & & 4_5 \arrow[ul]
\end{tikzcd} \tag{26}$$

We have the canonical form of $\mathbb{A}$:

$$\mathbb{A} \simeq \oplus\mathbb{Q} = \mathbb{L}_{11} \oplus \mathbb{L}_{12} \oplus \mathbb{L}_{14} \oplus \mathbb{L}_{14} \oplus \mathbb{L}_{22} \oplus \mathbb{L}_{23} \oplus \mathbb{L}_{34} \oplus \mathbb{L}_{44} \oplus \mathbb{L}_{44}.$$

5. Cycles of linear mappings

In this section, we give an algorithm for constructing a regularizing decomposition (6) that involves only unitary transformations. In the same way, one may construct a regularizing decomposition over an arbitrary field using elementary transformations.

By analogy with Section 1, we say that a matrix representation $\mathbb{A} = (A_1, \ldots, A_t)$ of a cycle $\mathscr{C}$ (see (13)) is *regular* if

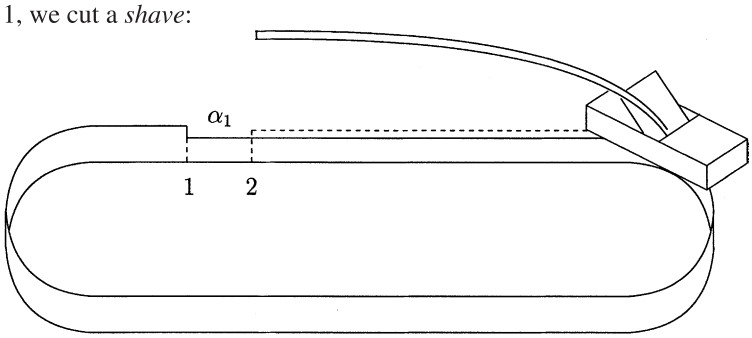
$$\dim_1 \mathbb{A} = \cdots = \dim_t \mathbb{A}$$

and all the matrices $A_1, \ldots, A_t$ are nonsingular; otherwise the representation is *singular*. A decomposition

$$\mathbb{A} \simeq \mathbb{D} \oplus \cdots \oplus \mathbb{G} \oplus \mathbb{P} \tag{27}$$

is a *regularizing decomposition* of $\mathbb{A}$ if $\mathbb{D}, \ldots, \mathbb{G}$ are matrix representations of the form $\mathbb{G}_{ij}$ (see Lemma 3.1) and $\mathbb{P}$ is a regular representation. By Theorem 3.2, the regularizing decomposition (27) is determined uniquely up to isomorphism of summands.

The algorithm works like a jack-plane in a woodworker's hands. Starting from the vertex 1, we cut a *shave*:



We make a full circle by the jack-plane and continue the process until the shave breaks away. Then we transpose all matrices of the remaining representation and repeat this process. The obtained representation $\mathbb{P}$ of $\mathscr{C}$ is regular, and the shaves split into a direct sum of matrix representations of the form $\mathbb{G}_{ij}$.

Note that this proves Theorem 3.2 since $\mathbb{P}$ is isomorphic to a matrix representation $(I_n, \ldots, I_n, J)$, where J is a nonsingular Jordan (or Frobenius) canonical matrix with respect to similarity; see the end of Section 1. Hence $\mathbb{A}$ is isomorphic to a direct sum of representations of the form (i)–(iii) from Theorem 3.2. The uniqueness of this decomposition follows from Theorem 2.1.

We will use the following notation. If all arrows in a matrix representation

$$\begin{tikzcd}[row sep=small] u_1 \arrow[dr, dash, "A_1"] & \\ u_2 \arrow[r, dash, "A_2"] & v \\ \cdots & \\ u_n \arrow[ur, dash, "A_n"'] & \end{tikzcd} \tag{28}$$

have the same orientation, then instead of (28) we will write

$$\begin{tikzcd}[row sep=small] u_1 \arrow[dr, dash, "\boxed{A}"] & \\ u_2 \arrow[r, dash] & v \\ \cdots & \\ u_n \arrow[ur, dash] & \end{tikzcd} \tag{29}$$

where

$$A = \begin{cases} \left[A_1 \mid \ldots \mid A_n\right] & \text{if } u_1 \longrightarrow v, u_2 \longrightarrow v, \ldots, u_n \longrightarrow v, \\ \begin{bmatrix} A_1 \\ \hline \cdots \\ \hline A_n \end{bmatrix} & \text{if } u_1 \longleftarrow v, u_2 \longleftarrow v, \ldots, u_n \longleftarrow v. \end{cases} \tag{30}$$

The partition of A into strips is fully determined by the dimensions of (28) at the vertices $u_1, \ldots, u_n$.

5.1. *The algorithm for cycles*

This algorithm for every matrix representation

$$\mathbb{A}: \begin{tikzcd} 1 \arrow[r, dash, "A_1"] \arrow[rrrr, dash, bend right, "A_t"'] & 2 \arrow[r, dash, "A_2"] & \cdots \arrow[r, dash, "A_{t-1}"] & t \end{tikzcd} \tag{31}$$

of a cycle $\mathscr{C}$ (see (13)) constructs a decomposition

$$\mathbb{A} \simeq \mathbb{P}(\mathbb{A}') \oplus \widetilde{\mathbb{A}}, \tag{32}$$

where $\mathbb{A}'$ is formed by the matrices of a chain of linear mappings, $\mathbb{P}$ sends $\mathbb{A}'$ to a representation of $\mathscr{C}$ that is isomorphic to a direct sum of representations of the form

$\mathbb{G}_{ij}$ (see (17) and compare with Example 3.1), and $\widetilde{\mathbb{A}}$ is a representation of $\mathscr{C}$ that satisfies the following condition for each arrow:

$$\text{If the arrow is oriented clockwise, then the matrix assigned to it has linearly independent rows.} \tag{33}$$

In steps $1, 2, \ldots$ of the algorithm we will construct quiver representations $\mathbb{A}^{(1)}$, $\mathbb{A}^{(2)}, \ldots$

Steps $1, 2, \ldots, l-1$*:* In step 1 of the algorithm, we check the condition (33) for the representation $\mathbb{A}$ and the arrow α_1. If this condition holds, we put $\mathbb{A}^{(1)} = \mathbb{A}$. If this condition holds for α_2 too, we put $\mathbb{A}^{(2)} = \mathbb{A}$, and so on.

If after t steps we found that this condition holds for all arrows of $\mathscr{C}$, then we put

$$l = t+1, \quad \mathbb{A}' = 0, \quad \widetilde{\mathbb{A}} = \mathbb{A} \tag{34}$$

and stop the algorithm. Otherwise, we set

$$l = \min\{i \in \{1, \ldots, t\} \mid \alpha_i \text{ in } \mathbb{A} \text{ does not satisfy (33)}\} \tag{35}$$

and continue the algorithm as follows:

Step l: By unitary transformations at the vertex $[l+1]$, we reduce the matrix A_l of $\mathbb{A}$ to a matrix

$$\begin{bmatrix} 0 \\ \hline A_l^{(l)} \end{bmatrix} \begin{matrix} \}d_{(l+1)'}^{(l)} \text{ rows,} \\ \}d_{[l+1]}^{(l)} \text{ rows,} \end{matrix} \qquad d_{(l+1)'}^{(l)} > 0,$$

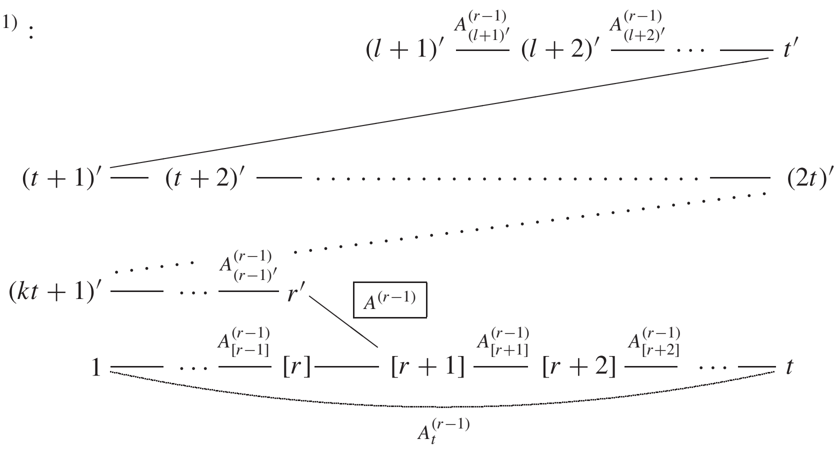
where the rows of $A_l^{(l)}$ are linearly independent. This changes A_{l+1}; we denote the obtained matrix by $A^{(l)}$ and construct the representation

$$\mathbb{A}^{(l)}: \quad \begin{tikzcd} & & & (l+1)' \arrow[dr, dash, "\boxed{A^{(l)}}"] & & & \\ 1 \arrow[r, dash] \arrow[rrrrrr, dotted, dash, bend right] & \cdots \arrow[r, dash, "A_{l-1}"] & l \arrow[r, "A_l^{(l)}"] & {[l+1]} \arrow[r, dash] & {[l+2]} \arrow[r, dash, "A_{[l+2]}"] & \cdots \arrow[r, dash] & t \end{tikzcd}$$

(the other matrices are the same as in (31)). Its dimensions at the vertices $(l+1)'$ and $[l+1]$ are $d_{(l+1)'}^{(l)}$ and $d_{[l+1]}^{(l)}$, and the arrow $(l+1)'$ — $[l+2]$ has the orientation of $[l+1]$ — $[l+2]$. The matrix $A^{(l)}$ is partitioned into the strips $A_{(l+1)'}^{(l)}$ and $A_{[l+1]}^{(l)}$, which are assigned to the arrows $(l+1)'$ — $[l+2]$ and $[l+1]$ — $[l+2]$ (see (29)).

Step r (r > l): Assume we have constructed in step $r-1$ a representation

$\mathbb{A}^{(r-1)}$:



where each arrow $\alpha_{i'} : i' — (i+1)'$ has the orientation of $\alpha_{[i]} : [i] — [i+1]$ in $\mathscr{C}$, and $\alpha_{r'} : r' — [r+1]$ has the orientation of $\alpha_{[r]} : [r] — [r+1]$.

We will reduce $\mathbb{A}^{(r-1)}$ by unitary transformations at the vertex $[r+1]$:

(i) If $\alpha_{[r]}$ is oriented clockwise, then $A^{(r-1)}$ consists of two vertical strips with $\dim_{r'} \mathbb{A}^{(r-1)}$ and $\dim_{[r]} \mathbb{A}^{(r-1)}$ columns (see (28)–(30)); we reduce it by unitary row-transformations as follows:

$$A^{(r-1)} = \left[A^{(r-1)}_{r'} \;\middle|\; A^{(r-1)}_{[r]} \right] \mapsto \left[\begin{array}{c|c} A^{(r)}_{r'} & 0 \\ * & A^{(r)}_{[r]} \end{array} \right], \tag{36}$$

where $A^{(r)}_{[r]}$ has linearly independent rows.

(ii) If $\alpha_{[r]}$ is oriented counterclockwise, then $A^{(r-1)}$ consists of two horizontal strips with $\dim_{r'} \mathbb{A}^{(r-1)}$ and $\dim_{[r]} \mathbb{A}^{(r-1)}$ rows; we reduce it by unitary column-transformations as follows:

$$A^{(r-1)} = \left[\begin{array}{c} A^{(r-1)}_{r'} \\ \hline A^{(r-1)}_{[r]} \end{array} \right] \mapsto \left[\begin{array}{cc} A^{(r)}_{r'} & 0 \\ \hline * & A^{(r)}_{[r]} \end{array} \right], \tag{37}$$

where $A^{(r)}_{r'}$ has linearly independent columns.

These unitary transformations at the vertex $[r+1]$ change the matrix $A^{(r-1)}_{[r+1]}$ too; we denote the obtained matrix by $A^{(r)}$ and construct the representation

$$
\begin{array}{l}
\mathbb{A}^{(r)}: \qquad\qquad (l+1)' \overset{A^{(r-1)}_{(l+1)'}}{\text{——}} (l+2)' \overset{A^{(r-1)}_{(l+2)'}}{\text{——}} \cdots \text{——}\ t' \\[1em]
(t+1)' \text{——} (t+2)' \text{——} \cdots\cdots\cdots\cdots \text{——} (2t)' \\[1em]
\cdots\cdots\cdots \\[1em]
(kt+1)' \text{——} \cdots \overset{A^{(r-1)}_{(r-1)'}}{\text{——}} r' \overset{A^{(r)}_{r'}}{\text{——}} (r+1)' \qquad \boxed{A^{(r)}} \\
1 \text{——} \cdots \overset{A^{(r-1)}_{[r-1]}}{\text{——}} [r] \overset{A^{(r)}_{[r]}}{\text{——}} [r+1] \text{——} [r+2] \overset{A^{(r-1)}_{[r+2]}}{\text{——}} \cdots \text{——}\ t
\end{array}
\tag{38}
$$

where $A^{(r)}$ is partitioned into two strips:

$$
A^{(r)} = \begin{cases} \left[\begin{array}{c|c} A^{(r)}_{(r+1)'} & A^{(r)}_{[r+1]} \end{array}\right] & \text{if } \alpha_{[r+1]} \text{ is oriented clockwise,} \\ \left[\begin{array}{c} A^{(r)}_{(r+1)'} \\ \hline A^{(r)}_{[r+1]} \end{array}\right] & \text{if } \alpha_{[r+1]} \text{ is oriented counterclockwise,} \end{cases}
\tag{39}
$$

and these strips are assigned to the arrows

$$(r+1)' \text{—} [r+2], \quad [r+1] \text{—} [r+2].$$

5.2. The result

We make at least t steps and stop at the first representation $\mathbb{A}^{(n)}$ with

$$n \geqslant t \ \text{ and } \ A^{(n)}_{(n+1)'} = 0. \tag{40}$$

The matrix $A^{(n)}_{(n+1)'}$ is assigned to the arrow $(n+1)' \text{—} [n+2]$. Deleting this arrow, we break $\mathbb{A}^{(n)}$ into two representations:

$$
\begin{array}{l}
\mathbb{A}': \qquad\qquad (l+1)' \overset{A^{(n)}_{(l+1)'}}{\text{——}} (l+2)' \overset{A^{(n)}_{(l+2)'}}{\text{——}} \cdots \text{——}\ t' \\[1em]
(t+1)' \text{——} (t+2)' \text{——} \cdots\cdots\cdots \text{——} (2t)' \\[1em]
\cdots\cdots\cdots \\[1em]
(kt+1)' \text{——} \cdots \overset{A^{(n)}_{n'}}{\text{——}} (n+1)'
\end{array}
\tag{41}
$$

and

$$\widetilde{\mathbb{A}}:\ 1 \,\rule{2em}{0.4pt}\ \cdots\ \rule{2em}{0.4pt}\,[n]\overset{A^{(n)}_{[n]}}{\rule{3em}{0.4pt}}[n+1]\overset{A^{(n)}_{[n+1]}}{\rule{3em}{0.4pt}}[n+2]\overset{A^{(n)}_{[n+2]}}{\rule{3em}{0.4pt}}\ \cdots\ \rule{2em}{0.4pt}\ t \tag{42}$$

The representation $\mathbb{A}'$ is a representation of the quiver

$$(l+1)'\overset{\alpha_{(l+1)'}}{\rule{4em}{0.4pt}}(l+2)'\overset{\alpha_{(l+2)'}}{\rule{4em}{0.4pt}}\cdots\overset{\alpha_{n'}}{\rule{3em}{0.4pt}}(n+1)', \tag{43}$$

whose arrows $i' — (i+1)'$ have the orientation of the arrows $\alpha_{[i]}: [i] — [i+1]$ in $\mathscr{C}$. By analogy with Example 3.1, we construct the mapping $\mathbb{P}$ that sends a representation $\mathbb{B}$ of the quiver (43) to a representation $\mathbb{D}$ of the cycle $\mathscr{C}$:

$$\begin{array}{ll}
 & (l+1)'\overset{B_{(l+1)'}}{\rule{3em}{0.4pt}}(l+2)'\overset{B_{(l+2)'}}{\rule{3em}{0.4pt}}\cdots\overset{B_{(t-1)'}}{\rule{3em}{0.4pt}}t' \\
 & \qquad\qquad B_{t'} \\
\mathbb{B}: & (t+1)'\underset{B_{(t+1)'}}{\rule{3em}{0.4pt}}(t+2)'\overset{B_{(t+2)'}}{\rule{3em}{0.4pt}}\cdots\cdots\overset{B_{(2t-1)'}}{\rule{3em}{0.4pt}}(2t)' \\
\Big\downarrow \mathbb{P} & \cdots\cdots\cdots\cdots \\
 & (kt+1)'\underset{B_{(kt+1)'}}{\rule{3em}{0.4pt}}(kt+2)'\overset{B_{(kt+2)'}}{\rule{3em}{0.4pt}}\cdots\overset{B_{n'}}{\rule{3em}{0.4pt}}(n+1)' \\
 & \\
\mathbb{D}: & 1\overset{D_1}{\rule{3em}{0.4pt}}2\overset{D_2}{\rule{3em}{0.4pt}}\cdots\cdots\overset{D_{t-1}}{\rule{3em}{0.4pt}}t \\
 & \qquad\qquad D_t
\end{array} \tag{44}$$

This mapping is known in representation theory as a *push-down functor* (see [7, Section 14.3]) and is determined as follows:

$$D_i = \bigoplus_{\substack{[j]=i\\ l\leqslant j\leqslant n+1}} B_{j'}, \quad i = 1, 2, \ldots, t, \tag{45}$$

(i.e., D_i is the direct sum of all $B_{j'}$ disposed over it), where

$$B_{l'} = 0_{p0} \quad \text{with } p = \dim_{(l+1)'} \mathbb{B} \tag{46}$$

(recall that the arrow α_l is oriented clockwise, see step l of the algorithm), and

$$B_{(n+1)'} = \begin{cases} 0_{0q} & \text{if } \alpha_{[n+1]}: [n+1] \longrightarrow [n+2], \\ 0_{q0} & \text{if } \alpha_{[n+1]}: [n+1] \longleftarrow [n+2], \end{cases} \quad \text{with } q = \dim_{(n+1)'} \mathbb{B}.$$

(The definition of $\mathbb{P}: \mathbb{B} \mapsto \mathbb{D}$ becomes clearer if the representations $\mathbb{B}$ and $\mathbb{D}$ are given by vector spaces and linear mappings: each vector space of $\mathbb{D}$ is the direct sum of the vector spaces of $\mathbb{B}$ disposed over it, and each linear mapping of $\mathbb{D}$ is determined by the linear mappings of $\mathbb{B}$ disposed over it.)

The following proposition will be proved in Section 8.

Proposition 5.1. *Let the algorithm for circles transform a matrix representation $\mathbb{A}$ of a cycle $\mathscr{C}$ to $\mathbb{A}'$ and $\widetilde{\mathbb{A}}$. Then*

(a) *The condition* (33) *holds for $\widetilde{\mathbb{A}}$ and all arrows.*
(b) *If an arrow α_i is oriented counterclockwise and the columns of A_i are linearly independent*, *then the columns of $\widetilde{A}_i$ are linearly independent too.*
(c) $\mathbb{A} \simeq \mathbb{P}(\mathbb{A}') \oplus \widetilde{\mathbb{A}}$.

6. Main theorem

Theorem 6.1. *A regularizing decomposition* (27) *of a matrix representation $\mathbb{A}$ of a cycle $\mathscr{C}$ can be constructed in* 3 *steps using only unitary transformations*:

1. *Applying the algorithm for cycles to $\mathbb{A}$, we get $\mathbb{A} \simeq \mathbb{P}(\mathbb{A}') \oplus \widetilde{\mathbb{A}}$.*
2. *Applying the algorithm for cycles to the matrix representation $\mathbb{B} := \widetilde{\mathbb{A}}^{\mathrm{T}}$ of the cycle $\mathscr{C}^{\mathrm{T}}$* (*see* (10)), *we get $\widetilde{\mathbb{A}}^{\mathrm{T}} \simeq \mathbb{P}(\mathbb{B}') \oplus \widetilde{\mathbb{B}}$.*
3. *Applying the algorithm for chains to $\mathbb{A}'$ and ${\mathbb{B}'}^{\mathrm{T}}$, we get*

$$\mathbb{A}' \simeq \mathbb{L}_{i_1 j_1} \oplus \cdots \oplus \mathbb{L}_{i_p j_p}, \quad {\mathbb{B}'}^{\mathrm{T}} \simeq \mathbb{L}_{i_{p+1} j_{p+1}} \oplus \cdots \oplus \mathbb{L}_{i_q j_q}. \tag{47}$$

Then

$$\mathbb{A} \simeq \mathbb{G}_{i_1 j_1} \oplus \cdots \oplus \mathbb{G}_{i_q j_q} \oplus \widetilde{\mathbb{B}}^{\mathrm{T}} \tag{48}$$

(*see* (17)) *and the representation $\widetilde{\mathbb{B}}^{\mathrm{T}}$ is regular*.

Proof. By (11) and Proposition 5.1(c),

$$\mathbb{A} \simeq \mathbb{P}(\mathbb{A}') \oplus (\mathbb{P}(\mathbb{B}'))^{\mathrm{T}} \oplus \widetilde{\mathbb{B}}^{\mathrm{T}} = \mathbb{P}(\mathbb{A}') \oplus \mathbb{P}({\mathbb{B}'}^{\mathrm{T}}) \oplus \widetilde{\mathbb{B}}^{\mathrm{T}}.$$

Substituting (47), we obtain

$$\mathbb{A} \simeq \mathbb{P}(\mathbb{L}_{i_1 j_1}) \oplus \cdots \oplus \mathbb{P}(\mathbb{L}_{i_q j_q}) \oplus \widetilde{\mathbb{B}}^{\mathrm{T}}.$$

This proves (48) since $\mathbb{P}(\mathbb{L}_{ij}) = \mathbb{G}_{ij}$.

Let us prove that $\widetilde{\mathbb{B}}^{\mathrm{T}}$ is regular. By Proposition 5.1(a), every matrix of $\widetilde{\mathbb{A}}$ assigned to an arrow oriented clockwise has linearly independent rows. The matrix representation $\mathbb{B} = \widetilde{\mathbb{A}}^{\mathrm{T}}$ is constructed by transposing all matrices, and it is a representation of the cycle $\mathscr{C}^{\mathrm{T}}$ obtained from $\mathscr{C}$ by changing the direction of each arrow. Hence every matrix of $\mathbb{B}$ assigned to an arrow oriented counterclockwise has linearly independent columns; by Proposition 5.1(b) the same holds for the matrices of $\widetilde{\mathbb{B}}$. Moreover, by Proposition 5.1(a) every matrix of $\widetilde{\mathbb{B}}$ assigned to an arrow oriented clockwise has linearly independent rows. Hence,

$$\dim_{[i+1]} \widetilde{\mathbb{B}} = \operatorname{rank} \widetilde{B}_i \leqslant \dim_i \widetilde{\mathbb{B}}$$

for all vertices $i = 1, \dots, t$. We have

$$\dim_1 \widetilde{\mathbb{B}} \geqslant \dim_2 \widetilde{\mathbb{B}} \geqslant \cdots \geqslant \dim_t \widetilde{\mathbb{B}} \geqslant \dim_1 \widetilde{\mathbb{B}}.$$

Therefore, each matrix $\widetilde{B}_i$ is square and its rows or columns are linearly independent. So $\widetilde{B}_i$ is nonsingular and the representation $\widetilde{\mathbb{B}}$ is regular. Then $\widetilde{\mathbb{B}}^{\mathrm{T}}$ is regular too. □

7. Proof of Proposition 4.1

In each step $r \in \{1, \dots, t-1\}$ of the algorithm for chains (Section 4) we constructed the matrix representation $\mathbb{A}^{(r)}$ of the form (23). Replacing all B_i by D_i (see (24)), we construct the representation

$$\mathbb{D}_r : 1 \xrightarrow{\;D_1\;} \cdots \xrightarrow{\;D_r\;} (r+1) \xrightarrow{\;A'_{r+1}\;} (r+2) \xrightarrow{\;A_{r+2}\;} \cdots \xrightarrow{\;A_{t-1}\;} t\,. \tag{49}$$

Let us prove that $\mathbb{D}_r$ is isomorphic to the initial representation $\mathbb{A}$:

$$\mathbb{A} \simeq \mathbb{D}_r, \quad r = 1, \dots, t-1. \tag{50}$$

In step 1 we reduced $\mathbb{A}$ to

$$\mathbb{B}_1 : 1 \xrightarrow{\;B_1\;} 2 \xrightarrow{\;A'_2\;} 3 \xrightarrow{\;A_3\;} \cdots \xrightarrow{\;A_{t-1}\;} t$$

by unitary transformations at vertices 1 and 2 (see (19)). Using transformations at vertex 1, we reduce B_1 to

$$D_1 = \begin{bmatrix} 0 & I_k \\ 0 & 0 \end{bmatrix}, \tag{51}$$

and so $\mathbb{A}$ is isomorphic to

$$\mathbb{D}_1 : 1 \xrightarrow{\;D_1\;} 2 \xrightarrow{\;A'_2\;} 3 \xrightarrow{\;A_3\;} \cdots \xrightarrow{\;A_{t-1}\;} t.$$

We may produce at vertex 2 of $\mathbb{D}_1$ every transformation given by a nonsingular block-triangular matrix

$$S_2 = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix},$$

where S_{11} is k-by-k if $\alpha_1 : 1 \longrightarrow 2$, and S_{22} is k-by-k if $\alpha_1 : 1 \longleftarrow 2$. This transformation spoils the block I_k of D_1 but we recover it by transformations at vertex 1.

Reasoning by induction on r, we assume that $\mathbb{A}$ is isomorphic to

$$\mathbb{D}_{r-1} : 1 \xrightarrow{\;D_1\;} \cdots \xrightarrow{\;D_{r-1}\;} r \xrightarrow{\;A'_r\;} (r+1) \xrightarrow{\;A_{r+1}\;} \cdots \xrightarrow{\;A_{t-1}\;} t$$

and that transformations at vertices $1, \dots, r-1$ may recover the matrices $D_1, \dots, D_{r-1}$ of $\mathbb{D}_{r-1}$ after each transformation at vertex r given by a nonsingular block-triangular matrix

$$S_r = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1r} \\ & S_{22} & \cdots & S_{2r} \\ & & \ddots & \vdots \\ 0 & & & S_{rr} \end{bmatrix} \tag{52}$$

in which the sizes of diagonal blocks coincide with the sizes of horizontal strips of B_{r-1} if $\alpha_{r-1} : (r-1) \longrightarrow r$, or with the sizes of vertical strips of B_{r-1} if $\alpha_{r-1} : (r-1) \longleftarrow r$ (see (21) and (22)).

In step r of the algorithm, we reduced A'_r to B_r of the form (21) or (22) by unitary transformations at the vertices r and $r+1$; moreover, we used only those transformation at vertex r that were given by unitary block-diagonal matrices partitioned as (52). By the same transformations at the vertices r and $r+1$ of $\mathbb{D}_{r-1}$, we reduce its matrix A'_r to B_r. Then we reduce B_r to D_r by a transformation at vertex r given by a matrix of the form (52), and restore $D_1, \dots, D_{r-1}$ by transformations at vertices $1, \dots, r-1$. The obtained representation is $\mathbb{D}_r$, and so

$$\mathbb{A} \simeq \mathbb{D}_{r-1} \simeq \mathbb{D}_r.$$

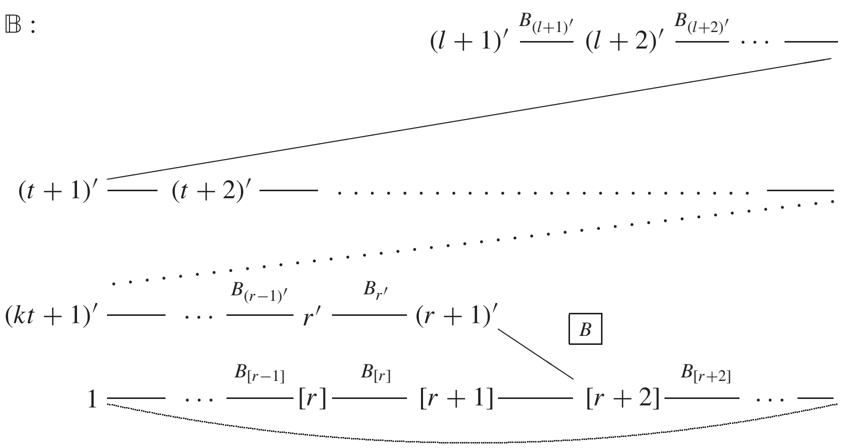
Moreover, we may produce at the vertex $r+1$ of $\mathbb{D}_r$ all transformations given by block-triangular matrices, restoring the matrix D_r by transformations at vertex r given by matrices of the form (52), and then restoring $D_1, \dots, D_{r-1}$ by transformations at the vertices $1, \dots, r-1$. This proves the isomorphism (50).

The representation $\mathbb{D}_{t-1}$, obtained in the last step of the algorithm, coincides with $\mathbb{D}$ and determines the representation $\mathbb{Q}$ (see (24) and (25)). Proposition 4.1 holds since

$$\mathbb{A} \simeq \mathbb{D}_{t-1} = \mathbb{D} \simeq \oplus\mathbb{Q}.$$

8. Proof of Proposition 5.1

The representation $\mathbb{A}^{(r)}$ (see (38)) is a representation of the quiver, which we will denote by $\mathcal{Q}^{(r)}$. For every representation



(53)

of this quiver, we define the representation

$$\mathbb{F}(\mathbb{B}):\quad \begin{tikzcd} 1 \arrow[r, dash, "D_1"] \arrow[rrr, dash, bend right, "D_t"'] & 2 \arrow[r, dash, "D_2"] & \cdots \arrow[r, dash, "D_{t-1}"] & t \end{tikzcd}$$

of the cycle $\mathscr{C}$ by "gluing down of the shave" (see the beginning of Section 5):

$$D_i = \left(\bigoplus_{\substack{[j]=i \\ l\leqslant j\leqslant r}} B_{j'} \right) \oplus \begin{cases} B_i & \text{if } i \neq [r+1], \\ B & \text{if } i = [r+1], \end{cases}$$

where $B_{l'}$ is defined by (46) (compare with (45)). The mapping $\mathbb{F}$ is analogous to the "push-down functor" (44). Moreover, for the representation $\mathbb{A}^{(n)}$, obtained in the last step of the algorithm for cycles, we have

$$\mathbb{F}(\mathbb{A}^{(n)}) = \mathbb{P}(\mathbb{A}') \oplus \widetilde{\mathbb{A}}, \tag{54}$$

where $\mathbb{A}'$ and $\widetilde{\mathbb{A}}$ are the representations (41) and (42).

By (30), the matrix B in (53) has the form

$$B = \begin{cases} \begin{bmatrix} B_{(r+1)'} & \big| & B_{[r+1]} \end{bmatrix} & \text{if } \alpha_{[r+1]} \text{ is oriented clockwise,} \\ \begin{bmatrix} B_{(r+1)'} \\ \hline B_{[r+1]} \end{bmatrix} & \text{if } \alpha_{[r+1]} \text{ is oriented counterclockwise.} \end{cases}$$

By *triangular transformations* with a representation $\mathbb{B}$ of the form (53), we mean the following transformations:

(i) additions of linear combinations of columns of $B_{[r+1]}$ to columns of $B_{(r+1)'}$ if $\alpha_{[r+1]}$ is oriented clockwise,
(ii) additions of linear combinations of rows of $B_{(r+1)'}$ to rows of $B_{[r+1]}$ if $\alpha_{[r+1]}$ is oriented counterclockwise.

We say that $\mathbb{B}$ is a *triangular representation* if

$$\mathbb{F}(\mathbb{B}) \simeq \mathbb{F}(\mathbb{B}^{\triangle})$$

for every representation $\mathbb{B}^{\triangle}$ obtained from $\mathbb{B}$ by triangular transformations.

Lemma 8.1. *Suppose* $\mathbb{D}$ *is obtained from a triangular representation* $\mathbb{B}$ *of* $\mathscr{Q}^{(r)}$ *by transformations at the vertex* $[r+2]$. *Then* $\mathbb{D}$ *is triangular too.*

Proof. Let

$$\mathscr{S} = (I, \dots, I, S_{[r+2]}, I, \dots, I) : \mathbb{B} \xrightarrow{\sim} \mathbb{D}$$

(see (9)). We must prove that $\mathbb{F}(\mathbb{D}) \simeq \mathbb{F}(\mathbb{D}^{\triangle})$ for every $\mathbb{D}^{\triangle}$ obtained from $\mathbb{D}$ by triangular transformations. Denote by $\mathbb{B}^{\triangle}$ the matrix representation obtained from $\mathbb{B}$ by the same triangular transformations. By (8) and the definition of triangular transformations, there is a block matrix

$$R = \begin{bmatrix} I & 0 \\ * & I \end{bmatrix}$$

such that

$$\left[D^{\triangle}_{(r+1)'} \,\middle|\, D^{\triangle}_{[r+1]}\right] = S_{[r+2]} \left[B_{(r+1)'} \,\middle|\, B_{[r+1]}\right] R \quad \text{if } \alpha_{[r+1]} \text{ is oriented clockwise,}$$

$$\begin{bmatrix} D^{\triangle}_{(r+1)'} \\ \hline D^{\triangle}_{[r+1]} \end{bmatrix} = R \begin{bmatrix} B_{(r+1)'} \\ \hline B_{[r+1]} \end{bmatrix} S^{-1}_{[r+2]} \quad \text{if } \alpha_{[r+1]} \text{ is oriented counterclockwise,}$$

$$D^{\triangle}_{[r+2]} = \begin{cases} B_{[r+2]} S^{-1}_{[r+2]} & \text{if } \alpha_{[r+2]} \text{ is oriented clockwise,} \\ S_{[r+2]} B_{[r+2]} & \text{if } \alpha_{[r+2]} \text{ is oriented counterclockwise.} \end{cases}$$

These equalities imply

$$\mathscr{S} = (I, \dots, I, S_{[r+2]}, I, \dots, I) : \mathbb{B}^{\triangle} \xrightarrow{\sim} \mathbb{D}^{\triangle}$$

and

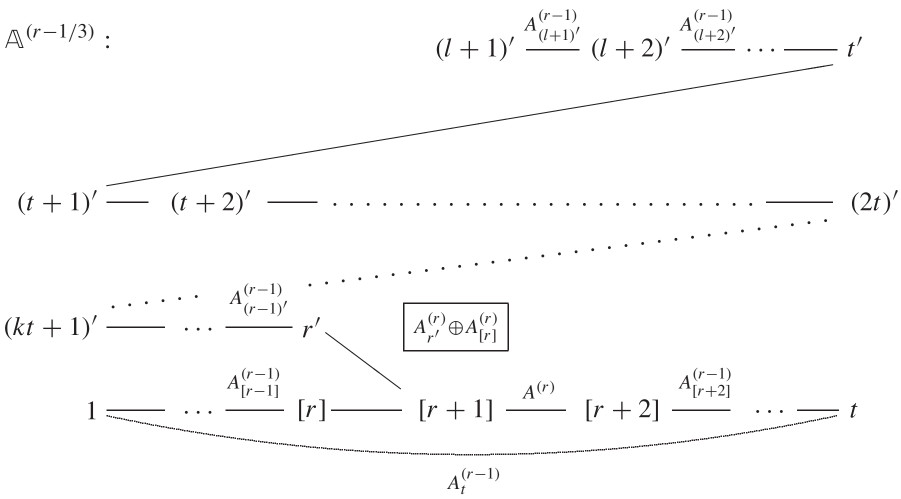
$$\mathbb{F}(\mathbb{D}) \simeq \mathbb{F}(\mathbb{B}) \simeq \mathbb{F}(\mathbb{B}^{\triangle}) \simeq \mathbb{F}(\mathbb{D}^{\triangle}). \qquad \square$$

Lemma 8.2. *Each representation* $\mathbb{A}^{(r)}$ (*obtained in step* r *of the algorithm for cycles*) *is triangular and* $\mathbb{F}(\mathbb{A}^{(r)}) \simeq \mathbb{A}$.

Proof. The lemma is obvious if $l = t + 1$ (see (34)). Suppose $l \leqslant t$. The statements hold for $\mathbb{A}^{(1)}, \dots, \mathbb{A}^{(l)}$. Reasoning by induction, we assume that they hold for $\mathbb{A}^{(r-1)}$ with $r - 1 \geqslant l$ and prove them for $\mathbb{A}^{(r)}$.

First we apply the unitary transformations at the vertex $[r+1]$ from step r of the algorithm for cycles to the representation $\mathbb{A}^{(r-1)}$ of the quiver $\mathscr{Q}^{(r-1)}$: we reduce the matrix $A^{(r-1)}$ to a block-triangular form by transformations (36) or (37) (depending on the orientation of $\alpha_{[r]}$), and the matrix $A^{(r-1)}_{[r+1]}$ to $A^{(r)}$. Denote the obtained representation by $\mathbb{A}^{(r-2/3)}$.

Then we make zero the block $*$ of (36) or (37) by triangular transformations and obtain the following representation $\mathbb{A}^{(r-1/3)}$ of the quiver $\mathcal{Q}^{(r-1)}$:



By the induction hypothesis, $\mathbb{A} \simeq \mathbb{F}(\mathbb{A}^{(r-1)})$ and $\mathbb{A}^{(r-1)}$ is triangular. By Lemma 8.1, $\mathbb{A}^{(r-2/3)}$ is triangular too, and so

$$\mathbb{F}(\mathbb{A}^{(r-2/3)}) \simeq \mathbb{F}(\mathbb{A}^{(r-1/3)}).$$

We have

$$\mathbb{A} \simeq \mathbb{F}(\mathbb{A}^{(r-1)}) \simeq \mathbb{F}(\mathbb{A}^{(r-2/3)}) \simeq \mathbb{F}(\mathbb{A}^{(r-1/3)}) = \mathbb{F}(\mathbb{A}^{(r)}).$$

Let $\mathbb{A}^{(r)\vartriangle}$ be obtained from $\mathbb{A}^{(r)}$ by triangular transformations. These transformations reduce $A^{(r)}$ (see (38)) to a new matrix $A^{(r)\vartriangle}$ and do not change the other matrices of $\mathbb{A}^{(r)}$. Since

$$A^{(r)} = A^{(r-1/3)}_{[r+1]},$$

these transformations with $A^{(r-1/3)}_{[r+1]}$ can be realized by transformations at the vertex $[r+1]$ of $\mathbb{A}^{(r-1/3)}$; denote the obtained representation by $\mathbb{A}^{(r-1/3)\vartriangle}$, it is triangular by Lemma 8.1. These transformations may spoil the subdiagonal block 0 of

$$A^{(r-1/3)} = A^{(r)}_{r'} \oplus A^{(r)}_{[r]},$$

but it is recovered by triangular transformations and so

$$\mathbb{F}(\mathbb{A}^{(r-1/3)\vartriangle}) \simeq \mathbb{F}(\mathbb{A}^{(r)\vartriangle}).$$

Since

$$\mathbb{F}(\mathbb{A}^{(r)}) = \mathbb{F}(\mathbb{A}^{(r-1/3)}) \simeq \mathbb{F}(\mathbb{A}^{(r-1/3)\vartriangle}) \simeq \mathbb{F}(\mathbb{A}^{(r)\vartriangle}),$$

the representation $\mathbb{F}(\mathbb{A}^{(r)})$ is triangular. $\square$

Lemma 8.3. *Let $\mathbb{A}^{(k)}$ be the representation obtained from a representation $\mathbb{A}$ in step k of the algorithm for cycles, and let $k \geqslant l$ (hence $l \leqslant t$ by* (34) *and* (35)). *Denote*

$$\widehat{A}_i^{(k)} = \begin{cases} A_i^{(k)} & \text{if } i \neq [k+1], \\ A^{(k)} & \text{if } i = [k+1], \end{cases}$$

where $i = 1, \dots, t$. Then

(i) *The rows of $\widehat{A}_i^{(k)}$ are linearly independent if α_i is oriented clockwise and $i \leqslant k$.*
(ii) *The columns of $\widehat{A}_i^{(k)}$ are linearly independent if α_i is oriented counterclockwise and the columns of A_i are linearly independent.*

Proof. We will prove the lemma by induction on k. Clearly, the statements (i) and (ii) hold for $k = l$. Assume they hold for $k = r - 1 \geqslant l$ and prove them for $k = r$. We need to check (i) and (ii) only for $i = [r]$ and $i = [r+1]$ since in step r of the algorithm we change $\widehat{A}_{[r]}^{(r-1)}$ and $\widehat{A}_{[r+1]}^{(r-1)}$.

By (36), the matrix $\widehat{A}_{[r]}^{(r)} = A_{[r]}^{(r)}$ has linearly independent rows if $\alpha_{[r]}$ is oriented clockwise. By (37), this matrix has linearly independent columns if both $\alpha_{[r]}$ is oriented counterclockwise and $\widehat{A}_{[r]}^{(r-1)} = A^{(r-1)}$ has linearly independent columns. Hence, (i) and (ii) hold for $i = [r]$.

The statements (i) and (ii) hold for $i = [r+1]$ by the induction hypothesis and since $\widehat{A}_{[r+1]}^{(r)} = A^{(r)}$ is obtained from $A_{[r+1]}^{(r-1)}$ by elementary transformations with its columns or rows. $\square$

Proof of Proposition 5.1. The statement (c) of Proposition 5.1 follows from (54) and Lemma 8.2, so we will prove (a) and (b).

If $l = t + 1$ (see (34)), then $\widetilde{\mathbb{A}} = \mathbb{A}$ satisfies (a) and (b).

Suppose $l \leqslant t$. Then $\widetilde{\mathbb{A}} = \mathbb{A}$ is the restriction of the representation $\mathbb{A}^{(n)}$ (obtained in the last step of the algorithm) to the cycle $\mathscr{C}$ and so $\widetilde{A}_i = A_i^{(n)}$ $(i = 1, 2, \dots, t)$.

Since

$$\widehat{A}_i^{(n)} = A_i^{(n)} = \widetilde{A}_i$$

if $i \neq [n+1]$,

$$\widehat{A}_{[n+1]}^{(n)} = A^{(n)} = \left[0 \;\middle|\; A_{[n+1]}^{(n)}\right] = \left[0 \;\middle|\; \widetilde{A}_{[n+1]}\right]$$

if $\alpha_{[n+1]}$ is oriented clockwise (see (39) and (40)), and

$$\widehat{A}_{[n+1]}^{(n)} = A^{(n)} = \begin{bmatrix} 0 \\ \hline A_{[n+1]}^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ \hline \widetilde{A}_{[n+1]} \end{bmatrix}$$

if $\alpha_{[n+1]}$ is oriented counterclockwise, the statements (i) and (ii) follow from Lemma 8.3, in which $k = n \geqslant t$. $\square$

9. Appendix: A diagrammatic form of the algorithm for chains

In this section we illustrate each step of the algorithm for chains, splitting sequentially a matrix representation (18) to the matrix representation $\mathbb{Q}$ (see (25)), which is a union of representations of the form $\mathbb{L}_{ij}$.

Step 1: We reduce A_1 to the form (19) by unitary transformations at vertices 1 and 2, and denote the "new" A_2 by A_2'.

Let us also denote by $\mathscr{P}_1$ the set consisting of $d_1 - k$ representations (where $d_1 = \dim_1 \mathbb{A}$ and $k \times k$ is the size of H in (19)) of the form $\mathbb{L}_{11}$ and transform the representation $\mathbb{A}$ into a representation $\mathbb{M}_1$ of a "split" quiver depending on the direction of α_1 in (12) as follows:

Case $\alpha_1 : 1 \longrightarrow 2$. Then

$$\mathbb{M}_1:\quad \left.\begin{array}{rr} & 1 \xrightarrow{I_1} 2 \\ (k \text{ copies}) & \cdots\cdots\cdots \\ & 1 \xrightarrow{I_1} 2 \\ & 2 \\ (d_2 - k \text{ copies}) & \cdots \\ & 2 \end{array}\right\rangle \boxed{A_2'}\ 3 \xrightarrow{\ A_3\ } 4 \xrightarrow{\ A_4\ } \cdots \xrightarrow{\ A_{t-1}\ } t \tag{55}$$

(see (29); there are k fragments of the form $1 \longrightarrow 2$ — 3 and $d_2 - k$ fragments of the form 2 — 3). The direction of the arrows is the same as in the quiver (12).

Case $\alpha_1 : 1 \longleftarrow 2$. Then

$$\mathbb{M}_1:\quad \left.\begin{array}{rr} & 2 \\ (d_2 - k \text{ copies}) & \cdots \\ & 2 \\ & 1 \xleftarrow{I_1} 2 \\ (k \text{ copies}) & \cdots\cdots\cdots \\ & 1 \xleftarrow{I_1} 2 \end{array}\right\rangle \boxed{A_2'}\ 3 \xrightarrow{\ A_3\ } 4 \xrightarrow{\ A_4\ } \cdots \xrightarrow{\ A_{t-1}\ } t \tag{56}$$

Step r $(1 < r < t)$: Assume we have constructed in step $r - 1$ the set $\mathscr{P}_{r-1}$ consisting of representations of the form $\mathbb{L}_{ij}$, $1 \leqslant i \leqslant j < r$, and a quiver representation $\mathbb{M}_{r-1}$:

$$\left.\begin{array}{ll} (k_1 \text{ copies}) & p_1 \xrightarrow{\ I_1\ } (p_1 + 1) \xrightarrow{\ I_1\ } \cdots \xrightarrow{\ I_1\ } r \\ (k_2 \text{ copies}) & p_2 \xrightarrow{\ I_1\ } (p_2 + 1) \xrightarrow{\ I_1\ } \cdots \xrightarrow{\ I_1\ } r \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (k_r \text{ copies}) & p_r \xrightarrow{\ I_1\ } (p_r + 1) \xrightarrow{\ I_1\ } \cdots \xrightarrow{\ I_1\ } r \end{array}\right\rangle \boxed{A_r'}\ (r + 1) \xrightarrow{\ A_{r+1}\ } \cdots \xrightarrow{\ A_{t-1}\ } t \tag{57}$$

in which every

$$p_i \xrightarrow{\ \ I_1\ \ } (p_i + 1) \xrightarrow{\ \ I_1\ \ } \cdots \xrightarrow{\ \ I_1\ \ } r$$

repeats k_i times, $k_1 + \cdots + k_r = d_r = \dim_r \mathbb{A}$, all $k_i \geqslant 0$, and

$$\{p_1, p_2, \ldots, p_r\} = \{1, 2, \ldots, r\}.$$

The direction of the arrows is the same as in the quiver (12).

Case $\alpha_r : r \longrightarrow r+1$ (see (12)). We divide A'_r into r vertical strips of sizes $k_1, k_2, \ldots, k_r$ and reduce A'_r to the form (21) starting from the first vertical strip by unitary column-transformations within vertical strips and by unitary row-transformations. Denote the "new" A_{r+1} by A'_{r+1}. Denote also by $\mathscr{P}_r$ the set obtained from $\mathscr{P}_{r-1}$ by including $k_i - l_i$ representations (where $l_i \times l_i$ is the size of H_i in (21)) of the form $\mathbb{L}_{p_i r}$ for all $i = 1, \ldots, r$. Construct the quiver representation

$$\mathbb{M}_r : \quad \begin{array}{lll} (l_1 \text{ copies}) & p_1 \overset{I_1}{\text{——}} \cdots \overset{I_1}{\text{——}} (r+1) & \\ \cdots\cdots\cdots & \cdots\cdots\cdots & (r+2) \overset{A_{r+2}}{\text{——}} \cdots \overset{A_{t-1}}{\text{——}} t \quad \boxed{A'_{r+1}} \\ (l_r \text{ copies}) & p_r \overset{I_1}{\text{——}} \cdots \overset{I_1}{\text{——}} (r+1) & \\ (d_{r+1} - l_1 - \ldots - l_r \text{ copies}) & (r+1) & \end{array}$$

(Hence, $k_i - l_i$ representations

$$\mathbb{L}_{p_i r} : p_i \overset{I_1}{\text{———}} (p_i + 1) \overset{I_1}{\text{———}} \cdots \overset{I_1}{\text{———}} r$$

for each $i = 1, \ldots, r$ "break away" from the representation (57) and join to the set $\mathscr{P}_{r-1}$.) In particular, if $r = t-1$, then $\mathbb{M}_r$ takes the form

$$\mathbb{M}_{t-1} : \quad \begin{array}{l} (l_1 \text{ copies}) \; p_1 \overset{I_1}{\text{———}} \cdots \overset{I_1}{\text{———}} t \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (l_{t-1} \text{ copies}) \; p_{t-1} \overset{I_1}{\text{———}} \cdots \overset{I_1}{\text{———}} t \\ (d_t - l_1 - \cdots - l_{t-1} \text{ copies}) \; t \end{array} \tag{58}$$

Case $\alpha_r : r \longleftarrow r+1$. We partition A'_r into r horizontal strips of sizes $k_1, k_2, \ldots, k_r$ and reduce A'_r to the form (22) starting from the lower strip, by unitary row-transformations within horizontal strips and by unitary column-transformations. Denote the "new" A_{r+1} matrix by A'_{r+1}. Denote also by $\mathscr{P}_r$ the set consisting of the elements of $\mathscr{P}_{r-1}$ and $k_i - l_i$ representations (where $l_i \times l_i$ is the size of H_i in (22)) of the form $\mathbb{L}_{p_i r}$ for all $i = 1, \ldots, r$. Construct the quiver representation

$$\mathbb{M}_r:\quad \left.\begin{array}{ll} (d_{r+1}-l_1-\ldots-l_r \text{ copies}) & \qquad\qquad\qquad (r+1) \overset{\boxed{A'_{r+1}}}{\overline{\qquad}} \\ (l_1 \text{ copies}) & p_1 \overset{I_1}{\overline{\qquad}} \cdots \overset{I_1}{\overline{\qquad}} (r+1) \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ (l_r \text{ copies}) & p_r \overset{I_1}{\overline{\qquad}} \cdots \overset{I_1}{\overline{\qquad}} (r+1) \end{array}\right\rangle (r+2) \overset{A_{r+2}}{\overline{\qquad}} \cdots \overset{A_{t-1}}{\overline{\qquad}} t$$

9.1. *The result*

After step $t-1$, we have obtained the set $\mathscr{P}_{t-1}$ consisting of representations of the form $\mathbb{L}_{ij}$, $j<t$, and the quiver representation $\mathbb{M}_{t-1}$ (see (58)), which may be considered as a set of representations of the form $\mathbb{L}_{it}$. Clearly,

$$\mathbb{Q} = \mathscr{P}_{t-1} \cup \mathbb{M}_{t-1}$$

(see (25)), and the representation $\oplus\mathbb{Q}$ is the canonical form of a matrix representation $\mathbb{A}$ of the quiver (12).

Note that the block-triangular form of S_r (see (52)) follows from the disposition of the chains

$$p_i \overset{I_1}{\overline{\qquad}} (p_i+1) \overset{I_1}{\overline{\qquad}} \cdots \overset{I_1}{\overline{\qquad}} r, \quad i=1,\ldots,r+1,$$

in the quiver representation $\mathbb{M}_{r-1}$ (see (57)): they represent the linear mappings and we may add these chains from the top down by changing bases in vector spaces; this is clear for the quiver representations (55), (56), and (26).

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