Sharp condition of global existence for second-order derivative nonlinear Schrödinger equations in two space dimensions

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Abstract

This paper discusses a class of second-order derivative nonlinear Schrödinger equations which are used to describe the upper-hybrid oscillation propagation. By establishing a variational problem, applying the potential well argument and the concavity method, we prove that there exists a sharp condition for global existence and blow-up of the solutions to the nonlinear Schrödinger equation. In addition, we also answer the question: how small are the initial data, the global solutions exist?

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1. Introduction

In this paper, we study a class of second-order derivative nonlinear Schrödinger equations as follows:

\[
i\phi_t + \Delta \phi + \phi \Delta |\phi|^2 + \phi |\phi|^2 = 0, \quad t \geq 0, \quad x \in \mathbb{R}^2,
\]  

\[\tag{1}\]

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where \( \varphi = \varphi(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \to C, i = \sqrt{-1}, \Delta \) is Laplacian operator. In physics, Eq. (1) is used to describe the upper-hybrid oscillation propagation [1–3].

For Eq. (1) and its more general situations, Liapunov stability of soliton solutions had been considered in [4], local well-posedness and global well-posedness were discussed in \( H^\infty \) and \( H^k \) in [5,6], the blow-up property of solutions was studied in [7], the existence of soliton solutions was discussed in [8,9]. In this paper, we are interested in studying the sharp condition of global existence for the Cauchy problem of Eq. (1). Firstly, we construct a type of constrained variational problem and establish its property, then apply it to the second-order derivative nonlinear Schrödinger equation (1). By studying the corresponding invariant manifolds under the flow of Eq. (1), applying the potential well argument and the concavity method [10–14], we establish the sharp condition for global existence and blow-up of the solutions. At the same time, by using the scaling argument, we also can answer that how small the initial data are, the global solutions exist? Berestycki and Cazenave [15], Weinstein [16] as well as Zhang [17] have studied the similar problems.

This paper organizes as follows. In Section 2, we give some preliminaries of the second-order derivative nonlinear Schrödinger equation (1) and define a constrained variational problem. In Section 3, we derive some new theorems of blow-up and sharp sufficient conditions of global existence.

2. Preliminaries and the variational problem

For Eq. (1), we impose the initial data as follows:

\[
\varphi(0, x) = \varphi_0, \quad x \in \mathbb{R}^2,
\]

(2)

for simplicity, we denote \( \int_{\mathbb{R}^2} . dx \) by \( \int . dx \).

In this paper, as in Refs. [18,19], we do not study the local well-posedness of Eq. (1) with initial data (2).

Firstly give two lemmas [7,19,20] as follows:

**Lemma 2.1.** Let \( T > 0, \varphi_0 \in H^2 \cap H^1 \) and \( \varphi \) be a solution of the Cauchy problem (1)–(2) in \( C([0, T); H^2 \cap H^1) \). Put the energy functional

\[
E(t) = \int \left( |\nabla \varphi|^2 + \frac{1}{2} |\nabla |\varphi||^2 - \frac{1}{2} |\varphi|^4 \right) dx.
\]

Then one has

\[
\int |\varphi|^2 dx = \int |\varphi_0|^2 dx,
\]

(3)

and

\[
E(t) \equiv E(0).
\]

(4)

**Lemma 2.2.** Let \( \varphi_0 \in H^2 \cap H^1 \) and \( \varphi \) be a solution of the Cauchy problem (1)–(2) in \( C([0, T); H^2 \cap H^1) \), \( |x|\varphi_0 \in L^2(\mathbb{R}^2) \). Put \( J(t) = \int |x|^2 |\varphi|^2 dx \). Then one has

\[
J''(t) = 8 \int \left( |\nabla \varphi|^2 + |\nabla |\varphi||^2 - \frac{1}{2} |\varphi|^4 \right) dx.
\]

(5)

For \( u \in H^2 \cap H^1 \), we define the following functionals:
\( S(u) = \int \left( |\nabla u|^2 + \frac{1}{2} |\nabla |u|^2 \right) dx, \) \( (6) \)

\( Q(u) = \int \left( |u|^2 - \frac{1}{2} |u|^4 \right) dx, \) \( (7) \)

\( H(u) = \int \left( |\nabla u|^2 + \frac{1}{2} |\nabla |u|^2 + |u|^2 - \frac{1}{2} |u|^4 \right) dx. \) \( (8) \)

In addition, we define a manifold as follows:

\[ M := \{ u \in H^2 \cap H^1 \setminus \{0\}, \ Q(u) = 0 \}. \] \( (9) \)

Now we consider the following constrained-variational problem:

\[ d := \inf_{u \in M} S(u). \] \( (10) \)

Firstly, we have:

**Lemma 2.3.** \( d > 0. \)

**Proof.** From \( Q(u) = 0 \) and Gagliardo–Nirenberge inequality

\[ \int |u|^2 dx \leq c \int |\nabla u|^2 dx \int |u|^2 dx. \] \( (11) \)

Here and hereafter \( c \) denote various positive constants. Thus

\[ \int |\nabla u|^2 dx \geq c > 0. \] \( (12) \)

So

\[ S(u) = \int \left( |\nabla u|^2 + \frac{1}{2} |\nabla |u|^2 \right) dx \geq c > 0 \] \( (13) \)

it follows that \( d > 0. \) \( \Box \)

Now we give:

**Proposition 2.4.** Put

\[ K_1 = \{ v \in H^2 \cap H^1, \ Q(v) < 0, \ H(v) < d \}, \]

\[ K_2 = \{ v \in H^2 \cap H^1, \ Q(v) > 0, \ H(v) < d \} \]

then \( K_1 \) and \( K_2 \) are invariant under the flow generated by the Cauchy problem (1)–(2).

**Proof.** Let the initial data \( \varphi_0 \in K_1, \varphi(t) \) is the solution of the Cauchy problem (1)–(2). From (3), (4), we have

\[ H(\varphi_0) = H(\varphi(t)), \quad t \in [0, T). \] \( (14) \)

Because \( H(\varphi_0) < d \), so

\[ H(\varphi(t)) < d, \quad t \in [0, T). \] \( (15) \)
To check $\varphi(t) \in K_1$, need to prove that

$$Q(\varphi(t)) < 0, \quad t \in [0, T).$$

(16)

Now prove it by contradiction. Assume that (16) is not true, that is

$$Q(\varphi(t)) \geq 0, \quad \text{for some } t \in [0, T).$$

(17)

By continuity and $Q(\varphi_0) < 0$, there would exist a $\bar{t} > 0$, such that

$$Q(\varphi(\bar{t})) = 0.$$  

(18)

It follows that $\varphi(\bar{t}) \in M$. On the other hand, from (15), (18) and $H(\varphi(\bar{t})) = S(\varphi(\bar{t})) + Q(\varphi(\bar{t}))$, have

$$S(\varphi(\bar{t})) < d, \quad \varphi(\bar{t}) \in M.$$  

(19)

From (10), it is impossible, thus (16) is true for $t \in [0, T)$. So $K_1$ is invariant under the flow generated by the Cauchy problem (1)–(2). By the same argument as the above, we can prove that $K_2$ is invariant under the flow generated by the Cauchy problem (1)–(2).  

3. Sharp conditions for global existence

In this section, we will give main results of this paper.

**Theorem 3.1.** Let $\varphi_0 \in H^2 \cap H^1$ and satisfy the condition

$$\int \left( |\nabla \varphi_0|^2 + \frac{1}{2} |\nabla|\varphi_0|^2|^2 + |\varphi_0|^2 - \frac{1}{2} |\varphi_0|^4 \right) dx < d$$  

(20)

then

(1) if

$$\int |\varphi_0|^4 dx > 2 \int |\varphi_0|^2 dx$$  

(21)

then the solution of the Cauchy problem (1)–(2) will blow up in finite time $T < \infty$;

(2) if

$$\int |\varphi_0|^4 dx < 2 \int |\varphi_0|^2 dx$$  

(22)

then the solution $\varphi(t, x)$ of the Cauchy problem (1)–(2) will globally exist in $t \in [0, \infty)$. In addition, $\varphi(t, x)$ also satisfies that

$$\frac{1}{2} \| \nabla|\varphi|^2 \|_{L^2}^2 + \| \nabla|\varphi|^2 \|_{L^2}^2 < d.$$  

(23)

**Proof.** (1) From (20), (21), $\varphi_0 \in K_1$, so for $t \in [0, \infty)$, the solution $\varphi(t)$ of the Cauchy problem (1)–(2) $\in K_1$. Thus have

$$Q(\varphi(t)) < 0, \quad H(\varphi(t)) < d.$$  

(24)

Since $|x|\varphi_0 \in L^2(R^2)$, then $|x|\varphi(t) \in L^2(R^2)$, by Lemma 2.2, it follows that

$$\frac{d^2}{dt^2} \int |x\varphi(t)|^2 dx = 8 \int \left( |\nabla \varphi|^2 + |\nabla|\varphi|^2|^2 - \frac{1}{2} |\varphi|^4 \right) dx.$$  

(25)
For fixed \( t \in [0, T) \), there exists \( 0 < \lambda < 1 \) such that \( Q(\lambda \varphi_0) = 0 \), that is
\[
\int \left( |\varphi_0|^2 - \frac{1}{2} \lambda^2 |\varphi_0|^4 \right) dx = 0. \tag{26}
\]

From (10) and (15), \( S(\lambda \varphi_0) \geq d, H(\varphi_0) < S(\lambda \varphi_0) \), that is
\[
\int \left( |\nabla \varphi_0|^2 + \frac{1}{2} |\nabla |\varphi_0||^2 + |\varphi_0|^2 - \frac{1}{2} |\varphi_0|^4 \right) dx \\
< \lambda^2 \int \left( |\nabla \varphi_0|^2 + \frac{1}{2} \lambda^2 |\nabla |\varphi_0||^2 \right) dx. \tag{27}
\]

By (25)–(27) and \( 0 < \lambda < 1 \), we have
\[
\frac{d^2}{dt^2} \int |x\varphi(t)|^2 dx < 0. \tag{28}
\]

From Tsutsumi and Zhang [10], it implies that there exists a finite time \( T < \infty \) such that the solution \( \varphi(t, x) \) of the Cauchy problem (1)–(2) blows up, that is
\[
\lim_{t \to T} \| \nabla \varphi \|_{L^2} = \infty.
\]

(2) From (20), (22), \( \varphi_0 \in K_2 \), then for \( t \in [0, \infty) \), the solution \( \varphi(t) \) of the Cauchy problem (1)–(2) \( \in K_2 \). Thus
\[
Q(\varphi(t)) > 0, \quad H(\varphi(t)) < d. \tag{29}
\]
Notice that \( H(\varphi) = S(\varphi) + Q(\varphi) \). Therefore, for \( t \in [0, \infty) \), have
\[
\int \left( |\nabla \varphi|^2 + \frac{1}{2} |\nabla |\varphi||^2 \right) dx < d. \tag{30}
\]
It implies that \( \varphi(t, x) \) is bounded, so it must be \( T = \infty \), that is, the solution \( \varphi(t, x) \) of the Cauchy problem (1)–(2) will globally exist in \( t \in [0, \infty) \). In addition, from (30), \( \varphi(t, x) \) satisfies (23). \( \square \)

Now we will answer such a problem: how small are the initial data, the solution \( \varphi(t, x) \) of the Cauchy problem (1)–(2) globally exists?

**Theorem 3.2.** Let \( \varphi_0 \) satisfy
\[
\int \left( |\nabla \varphi_0|^2 + \frac{1}{2} |\nabla |\varphi_0||^2 + |\varphi_0|^2 \right) dx < d \tag{31}
\]
then the solution \( \varphi(t, x) \) of the Cauchy problem (1)–(2) will globally exist and satisfy
\[
\frac{1}{2} \| \nabla |\varphi|^2 \|_{L^2}^2 + \| \nabla \varphi \|_{L^2}^2 < d. \tag{32}
\]

**Proof.** By (31), \( \varphi_0 \) satisfies (20). Now we prove that \( \varphi_0 \) satisfies (22). We prove it by contradiction. Assume that (22) is not true, then \( Q(\varphi_0) \leq 0 \), thus exists \( \lambda \in (0, 1] \) such that \( Q(\lambda \varphi_0) = 0 \). On the other hand, for \( \lambda \varphi_0 \), we have
\[
H(\lambda \varphi_0) \leq \int \left( |\nabla \lambda \varphi_0|^2 + \frac{1}{2} |\nabla |\lambda \varphi_0||^2 + |\lambda \varphi_0|^2 \right) dx. \tag{33}
\]
From (31), then

\[ H(\lambda \varphi_0) < \lambda^2 d \leq d. \]  \hspace{1cm} (34)

Notice that \( H(\lambda \varphi_0) = S(\lambda \varphi_0) + Q(\lambda \varphi_0) \). Thus \( Q(\lambda \varphi_0) = 0, \lambda \varphi_0 \in M, S(\lambda \varphi_0) < d \), it is impossible by (10), so \( \varphi_0 \) satisfies (22). From (2) of Theorem 3.1, the solution \( \varphi(t, x) \) of the Cauchy problem (1)–(2) globally exists and on \( t \in [0, \infty) \), (32) is true. \( \square \)

References