JOURNAL OF FUNCTIONAL ANALYSIS 32, 353–380 (1979)

Eigenvalues of p-Summing and l_p-Type Operators in Banach Spaces

W. B. JOHNSON*

Ohio State University, Columbus, Ohio 43210, and
Institute for Advanced Studies of the Hebrew University of Jerusalem, Jerusalem, Israel

H. KÖNIG

Inst. f. Angewandte Mathematik, Universitat Bonn, Bonn, West Germany

B. MAUREY

Centre de Math. de l'École Polytechnique, Paris, and
Institute for Advanced Studies of the Hebrew University of Jerusalem, Jerusalem, Israel

AND

J. R. RETHERFORD†

Louisiana State University, Baton Rouge, Louisiana 70803

Received November 15, 1977

This paper is a study of the distribution of eigenvalues of various classes of operators. In Section 1 we prove that the eigenvalues $(\lambda_n(T))$ of a p-absolutely summing operator, $p \geqslant 2$, satisfy

$$\left(\sum_{n\in N} |\lambda_n(T)|^p\right)^{1/p} \leqslant \pi_p(T).$$

This solves a problem of A. Pietsch. We give applications of this to integral operators in L_p-spaces, weakly singular operators, and matrix inequalities.

In Section 2 we introduce the quasinormed ideal $\Pi_2^{(n)}$, $P = (p_1, \ldots, p_n)$ and show that for $T \in \Pi_2^{(n)}$, $2 = (2, \ldots, 2) \in N^n$, the eigenvalues of T satisfy

$$\left(\sum_{i\in N} |\lambda_i(T)|^{2/n}\right)^{n/2} \leqslant \pi_2^{(n)}(T).$$

More generally, we show that for $T \in \Pi_P^{(n)}$, $P = (p_1, \ldots, p_n)$, $p_i \geqslant 2$, the eigenvalues are absolutely p-summable,

$$\frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i} \qquad \text{and} \qquad \left(\sum_{n\in N} |\lambda_n(T)|^p\right)^{1/p} \leqslant c_p \pi_P^{(n)}(T).$$

* Partially supported by the National Science Foundation, USA, NSF-MCS 76-06565.
† Partially supported by the National Science Foundation, USA, NSF-MCS 77-01684 and the SFB72 at the University of Bonn.

0022-1236/79/060353-28$02.00/0

We also consider the distribution of eigenvalues of p-nuclear operators on L_r-spaces.

In Section 3 we prove the Banach space analog of the classical Weyl inequality, namely

$$\sum_{n\in N} |\lambda_n(T)|^p \leqslant c_p \sum_{n\in N} \alpha_n(T)^p,$$

$0 < p < \infty$, where α_n denotes the Kolmogoroff, Gelfand of approximation numbers of the operator T. This solves a problem of Markus-Macaev.

Finally we prove that Hilbert space is (isomorphically) the only Banach space X with the property that nuclear operators on X have absolutely summable eigenvalues. Using this result we show that if the nuclear operators on X are of type l_1 then X must be a Hilbert space.

Notation and Terminology

In this paper we assume all Banach spaces to be complex. The space of all continuous linear operators from a Banach space X into a Banach space Y is written $\mathscr{L}(X, Y)$, for $X = Y$ simply $\mathscr{L}(X)$. When necessary, we will denote the norm of $T \in \mathscr{L}(X, Y)$ by $\| T : X \to Y \|$. Given a map $T \in \mathscr{L}(X)$ the spectrum of which consists of eigenvalues only, we mean by $(\lambda_n(T))$ the sequence of eigenvalue of T ordered according to their magnitude in absolute value and counted according to their multiplicity.

For $1 \leqslant p < \infty$, we denote the Banach ideal of the absolutely p-summing operators by (Π_p, π_p). For the appropriate definitions see [5] and [20]. For a positive integer n and $P = (p_1, ..., p_n)$ with $1 \leqslant p_i < \infty$, $\Pi_P^{(n)}(X, Y)$ is the class of all operators $T \in \mathscr{L}(X, Y)$ for which there are Banach spaces $X_1, ..., X_{n-1}$, depending on T, and operators $T_i \in \Pi_{p_i}(X_{i-1}, X_i)$, $i = 1, ..., n$ with $X_0 = X$ and $X_n = Y$ such that $T = T_n \cdots T_1$.

$$\pi_P^{(n)}(T) = \inf \prod_{i=1}^{n} \pi_{p_i}(T_i)$$

is a quasinorm on $\Pi_P^{(n)}(X, Y)$ (the infimum taken over all possible factorizations over Banach spaces $X_1, ..., X_{n-1}$) which makes $(\Pi_P^{(n)}, \pi_P^{(n)})$ into a complete quasinormed operator ideal.

For $0 < p \leqslant 1$, we call those operators $T \in \mathscr{L}(X, Y)$ which admit a representation $T = \sum_{i\in N} f_i \otimes x_i$, $f_i \in X'$, $x_i \in Y$ with $\sum_{i\in N} \|f_i\|^p \| x_i \|^p < \infty$, p-nuclear and write $T \in \mathscr{N}_p(X, Y)$.

Given any operator $T \in \mathscr{L}(X, Y)$, one defines the approximation numbers of T by

$$\alpha_n(T) = \inf\{\| T - T_n \|: T_n \in \mathscr{L}(X, Y), \text{rank } T_n < n\},$$

the Kolmogorov numbers of T

$$\delta_n(T) = \inf[\sup(\inf\{\| Tx - y \|: y \in Z \subset Y\}: \| x \| = 1): \dim Z < n]$$

and the Gelfand numbers of T by

$$\gamma_n(T) = \inf\{\| T|_Z \| : Z \subset X, \operatorname{codim} Z < n\}$$

for any $n \in N$. For properties of these s-numbers we refer to Pietsch [22]. We mention here only

$$\max(\gamma_n(T), \delta_n(T)) \leqslant \alpha_n(T), \qquad \gamma_n(T) = \delta_n(T')$$

and for compact T, $\delta_n(T) = \gamma_n(T')$. For any of these s-numbers $s_n \in \{\alpha_n, \gamma_n, \delta_n\}$ and $0 < p < \infty$, we define a complete quasinormed operator ideal

$$S_p^s = \left\{ T \in \mathscr{L} : \sigma_p^s(T) = \left(\sum_{n \in N} s_n(T)^p \right)^{1/p} < \infty \right\}$$

with quasinorm σ_p^s. In Hilbert spaces, the $s_n(T)$ are just the singular numbers of T, i.e. the eigenvalues of $(T^*T)^{1/2}$. The above s-numbers are multiplicative,

$$s_{m+n-1}(TS) \leqslant s_m(S) \cdot s_n(T)$$

for $m, n \in N$ and $S \in \mathscr{L}(X, Y)$, $T \in \mathscr{L}(Y, Z)$.

Finally, we will have occasion to use the following standard notation. Given a sequence of Banach spaces (X_n), we denote their l_p-direct sum by $(\bigoplus_n X_n)_p$. The Banach-Mazur distance between Banach spaces X and Y is defined by

$$d(X, Y) = \inf\{\| T \| \, \| T^{-1} \| : T: X \to Y \text{ is an isomorphism}\},$$

with $d(X, Y) = \infty$ if no such isomorphism exists.

1. Eigenvalues of p-Absolutely Summing Operators

In [18] Pietsch showed that for $1 \leqslant p \leqslant 2$, $T \in \Pi_p(x)$ implies

$$\sum_{n \in N} |\lambda_n(T)|^2 < +\infty \tag{1.0}$$

This generalized, and simplified earlier work of Grothendieck [6] and Saphar [24]. Examples of Grothendieck [6] of nuclear convolution operators on L_1 and L_∞ whose eigenvalues are not p-summable for $p < 2$ show that, for this range of p, the Pietsch result is the best possible.

In this chapter we prove the following result.

Theorem 1.1. *The eigenvalues $(\lambda_n(T))$ of any absolutely p-summing operator $T \in \Pi_p(x)$ for $p \geqslant 2$ are absolutely p-summable and satisfy*

$$\left(\sum_{n \in N} |\lambda_n(T)|^p \right)^{1/p} \leqslant \pi_p(T) \tag{1.1}$$

This answers affirmatively a question raised by Pietsch [18], [21]. To prove (1.1) observe that it is enough to consider only operators on l_∞^n. Indeed if $T \in \Pi_p(X)$ and $x_1, \dots, x_n$ are the first n eigenvectors of T, $X_n = [x_1, \dots, x_n]$ and $T_n = T|_{X_n}$ then $\pi_p(T_n) \leqslant \pi_p(T)$ and we have the factorization

$$\begin{array}{ccccc} X_n & \xrightarrow{T_n} & X_n & \xrightarrow{A} & L_\infty(\mu) \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle B} & & \\ L_\infty(\mu) & \xrightarrow{j} & Y & \hookrightarrow & L_p(\mu) \end{array} \tag{1.2}$$

Choosing m large enough so that AX_n embeds $(1+\epsilon)$ isomorphically in l_∞^m, we have that $S = ABj|_{l_\infty^m}$ has the same first n eigenvalues as T_n and $\pi_p(S) \leqslant \pi_p(T)$. Now the idea of the proof of (1.1) is simple:

If $T \in \mathscr{L}(l_\infty^n)$ and $\pi_p(T) \leqslant 1$ we show that there is an invertible diagonal operator δ so that

$$\delta^{-1}T\delta \in \mathscr{L}(l_2^n) \qquad \text{and} \qquad \sigma_p(\delta^{-1}T\delta) \leqslant 1.$$

The proof of this result is a bit involved. Our starting point is well known.

LEMMA 1.2. *Let X be a Banach space and $T \in \mathscr{L}(l_\infty^n, X)$. Then for $1 \leqslant p < \infty$*

$$\pi_p(T) = \inf \| T: l_p^n(m_j) \to X \| \tag{1.3}$$

where $l_p^n(m_j)$ denotes $\mathbb{C}^n$ under the norm

$$\|(x_j)\| = \left(\sum_{j=1} |x_j|^p m_j^p \right)^{1/p}, \qquad m_j > 0$$

and the infimum is taken over all non-negative numbers m_j with $\sum_{j=1}^n m_j^p \leqslant 1$.

Indeed $\pi_p(T)$ is the infimum over all constants $c > 0$ such that there is a probability measure μ on the extreme points of the dual ball of l_∞^n, i.e. on $K = \{\pm e_i\}$ in l_1^n such that

$$\| Tx \| \leqslant c \left(\int_K |\langle x, a \rangle|^p \, d\mu(a) \right)^{1/p} \tag{1.4}$$

If μ is given, let $m_i = (\mu(e_i) + \mu(-e_i))^{1/p}$ and if (m_i), $m_i \geqslant 0$, $\sum_{i=1}^n m_i^p = 1$, let $\mu(\pm e_i) = \frac{1}{2} m_i^{1/p}$. Then clearly (1.3) is a restatement of (1.4).

Remark. For $p = 1$, we define the *canonical measure* by $m_j = \| Te_j \|$.

For our main results we introduce the following notation. For $\alpha \geqslant 0$ and $z \in \mathbb{C}$ let

$$\alpha^z = \begin{cases} e^{z \log \alpha} & \alpha > 0 \\ 0 & \alpha = 0 \end{cases} \tag{1.5}$$

If X is an n-dimensional Banach space with basis $(e_i)_{i=1}^n$ and $\beta = (\beta_i)_{i=1}^n \in \mathbb{C}^n$, the diagonal operator β (with respect to (e_i)) is defined by $\beta(e_i) = \beta_i e_i$. If $(\alpha_i)_{i=1}^n$, $\alpha_i \geqslant 0$, $z \in \mathbb{C}$, the diagonal operator α^z is given by (α_i^z). Clearly $\alpha^z \alpha^{z'} = \alpha^{z+z'}$. Again, our next lemma is essentially known.

LEMMA 1.3. *Let* $1 \leqslant p \leqslant \infty$, $1/p + 1/p' = 1$, $A \in \mathscr{L}(\mathbb{C}^n)$, *and* $\alpha, \beta \in \mathbb{C}^n$ *with* $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ *for* $i = 1,..., n$. *Suppose*

$$\| A : l_p{}^n \to l_\infty{}^n \| \leqslant 1,$$

and

$$\| \beta^{1/p'} A \alpha^{-1/p'} : l_1{}^n \to l_{p'}^n \| \leqslant 1.$$

Then for any v *satisfying* $p' \leqslant v \leqslant \infty$

$$\| \beta^{1/v} A \alpha^{-1/v} : l_u{}^n \to l_v{}^n \| \leqslant 1$$

where $1/u - 1/v = 1/p$.

To prove lemma 1.3, apply the complex interpolation method to the analytic family of operators $T_z = \beta^{z/p'} A \alpha^{-z/p'}$. One must observe that $T_0 = \beta^0 A \alpha^0$ is not, in general, the same as A, but $\| T_0 : l_p{}^n \to l_\infty{}^n \| \leqslant 1$ and so the interpolation still applies.

The next lemma is quite similar.

LEMMA 1.4. *Let H be a positive Hermitian operator on $l_2{}^n$ and δ a non-negative diagonal operator on $l_2{}^n$ with*

$$\| \delta H : l_2{}^n \to l_u{}^n \| \leqslant 1 \qquad 1 \leqslant u \leqslant \infty.$$

For $\theta \in (0, 1)$ *let* $1/v = (1 - \theta)/2 + \theta/u$. *Then*

$$\| \delta^\theta H^\theta : l_2{}^n \to l_v{}^n \| \leqslant 1.$$

Proof. For $z \in \mathbb{C}$ let $T_z = \delta^z H^z$. For $\alpha \in l_2{}^n$, $\| \alpha \|_2 = 1$, the mapping $z \to T_z \alpha$ is analytic and bounded in a neighborhood of the strip $\Delta = \{z : 0 \leqslant \operatorname{Re} z \leqslant 1\}$. Since $l_v{}^n$ is a complex interpolation space between $l_2{}^n$ and $l_u{}^n$,

$$\| T_\theta \alpha \|_v \geqslant \sup_y \{\| T_{iy} \alpha \|_2, \| T_{1+iy} \alpha \|_u\}.$$

Since H^0 and δ^0 are orthogonal projections,

$$T_{iy} = \delta^{iy} \delta^0 H^0 H^{iy} \qquad \text{and} \qquad T_{1+iy} = \delta^{iy} \delta H H^{iy}$$

both have norm no bigger than one (as operators from $l_2{}^n$ to $l_2{}^n$ and from $l_2{}^n$ to $l_u{}^n$, respectively.) Thus $\| T_\theta \alpha \|_v \leqslant 1$.

We can now prove the main theorem.

THEOREM 1.5. *Let $p > 2$ and $A \in \mathscr{L}(l_\infty^n)$ with $\pi_p(A) \leqslant 1$. Assume $Ae_i \neq 0$ for $i \leqslant n$. Then there is an invertible diagonal operator δ so that $\sigma_p(\delta^{-1}A\delta : l_2^n \to l_2^n) \leqslant 1$.*

Proof. Since $\pi_p(A) \leqslant 1$, there are, by lemma 1.2, numbers $\alpha_1, \ldots, \alpha_n \geqslant 0$, $\sum_{i=1}^n \alpha_i = 1$ such that

$$\| Ax \|_\infty \leqslant \left(\sum_{i=1}^n \alpha_i \mid x_i \mid^p \right)^{1/p} \qquad \text{for} \quad x \in l_\infty^n.$$

Since $Ae_i \neq 0$, $\alpha_i > 0$ and so

$$\| A\alpha^{-1/p} : l_p^n \to l_\infty^n \| \leqslant 1. \tag{1.6}$$

Let $K = \{\gamma = (\gamma_i)_{i=1}^n : \gamma_i \geqslant 0,\ \sum_{i=1}^n \gamma_i = 1\}$, and let $\theta \in (0, 1)$. For $\gamma \in K$, $\sum_{i=1}^n \alpha_i^\theta \gamma_i^{1-\theta} \leqslant 1$, and letting $\delta = (\alpha_i^\theta \gamma_i^{1-\theta})_{i=1}^n$, the diagonal operator $\delta^{1/p'}$ satisfies

$$\pi_{p'}(\delta^{1/p'} : l_\infty^n \to l_{p'}^n) \leqslant 1.$$

In particular,

$$\sum_{j=1}^n \| \delta^{1/p'} A\alpha^{-1/p} e_j \|_{p'}^{p'} \leqslant 1$$

or equivalently

$$\sum_{j=1}^n \sum_{i=1}^n \delta_i \mid a_{ij} \mid^{p'} \alpha_j^{-p'/p} \leqslant 1, \tag{1.7}$$

where $A = (a_{ij})$.

Define $\varphi: K \to K$ by

$$\varphi(\gamma) = \left(\sum_{i=1}^n \gamma_i^{1-\theta} \alpha_i^\theta \mid a_{ij} \mid^{p'} \alpha_j^{p'/p} + \frac{1}{n} \left(1 - \sum_{i,l=1}^n \gamma_i^{1-\theta} \alpha_i^\theta \mid a_{il} \mid^{p'} \alpha_l^{-p'/p} \right) \right)_{j=1}^n \tag{1.8}$$

Since φ is continuous, it has a fixed point, i.e. there is some $\gamma \in K$ such that $\varphi(\gamma) = \gamma = (\gamma_j)_{j=1}^n$, i.e. γ_j is given by (1.8) for each j.

If $\gamma_j = 0$ for some j, then

$$\sum_{i,l=1}^n \gamma_i^{1-\theta} \alpha_i^\theta \mid a_{il} \mid^{p'} \alpha_l^{-p'/p} = 1$$

and so $\sum_{i=1}^n \gamma_i^{1-\theta} \alpha_i^\theta = 1$ (since

$$\| A\alpha^{-1/p} : l_p^n \to l_\infty^n \| = \max_{1 \leqslant i \leqslant n} \sum_{l=1}^n \mid a_{il} \mid^{p'} \cdot \alpha_l^{-p'/p} \leqslant 1).$$

An application of Hölder's inequality to this equality yields a contradiction since $\alpha_j > 0$ for each j; thus $\gamma_j > 0$ for $j = 1,\dots, n$.

Setting $\delta = \alpha^\theta\gamma^{1-\theta}$ for this γ, we also have $\sum_{i=1}^n \delta_i \mid a_{ij} \mid^{p'} \alpha_j^{p'/p} \leqslant \gamma_j$ (by (1.8)) and so

$$\| \delta^{1/p'} A\alpha^{-1/p}\gamma^{-1/p'}\colon l_1{}^n \to l_{p'}^n \| \leqslant 1 \tag{1.9}$$

We introduce the operator $B = \delta^{1/2}A\alpha^{-1/p}\gamma^{-1/p-1/2}$ on $l_2{}^n$. Setting $\theta = 2/p$, $\delta = \alpha^{2/p}\gamma^{-1-2/p}$ and so

$$B = \delta^{1/2}A\delta^{-1/2}. \tag{1.10}$$

The remainder of the proof is to show that $\sigma_p(B) \leqslant 1$. To this end we first observe that if $p' \leqslant v \leqslant \infty$ and $1/v = 1/u - 1/p$ then

$$\| \delta^{1/v-1/2}B\gamma^{-1/2-1/u}\colon l_u{}^n \to l_v{}^n \| \leqslant 1. \tag{1.11}$$

Indeed, by (1.6), (1.9) and lemma 1.3, we obtain

$$\| \delta^{1/v}A\alpha^{-1/p}\gamma^{-1/v}\colon l_u{}^n \to l_v{}^n \| \leqslant 1.$$

But $\delta^{1/v}A\alpha^{-1/p}\gamma^{-1/v} = \delta^{1/v-1/2}B\gamma^{1/2-1/u}$. By duality we also have

$$\| \gamma^{1/v-1/2}B^*\delta^{1/2-1/u}\colon l_u{}^n \to l_v{}^n \| \leqslant 1. \tag{1.12}$$

Now define a sequence $(r_k)_{k=0}^m$ by $1/r_k = 1/2 - k/p$, $k \leqslant m = [p/2]$ and let C be the positive Hermitian operator $(B^*B)^{1/2}$. We show now that if $2 \leqslant v \leqslant r_m$ and $1/v = 1/2 - s/p$, then

$$\| \gamma^{-s/p}C^s\colon l_2{}^n \to l_v{}^n \| \leqslant 1. \tag{1.13}$$

To see this, first suppose that $m = 1$, i.e. $2 < p < 4$. From (1.12) we have

$$\| \gamma^{-1/p}B^*\colon l_2{}^n \to l_{r_1}^n \| \leqslant 1,$$

since $v \leqslant r_1$. And, since $C = B^*U$, U an isometry on $l_2{}^n$, we have

$$\| \gamma^{-1/p}C\colon l_2{}^n \to l_{r_1}^n \| \leqslant 1. \tag{1.14}$$

For $\sigma \in (0, 1)$ write $(1 - \sigma)/2 + \sigma/r_1 = 1/2 - s/p$. Then by lemma 1.4.

$$\| \gamma^{-\sigma/p}C^\sigma\colon l_2{}^n \to l_v{}^n \| \leqslant 1. \tag{1.15}$$

For the general case we have from (1.11) and (1.12), with $u = r_k$.

$$\| \delta^{-(k+1)/p}B\gamma^{k/p}\colon l_{r_k}^n \to l_{r_{k+1}}^n \| \leqslant 1$$

and

$$\| \gamma^{-(k+1)/p}B^*\delta^{k/p}\colon l_{r_k}^n \to l_{r_{k+1}}^n \| \leqslant 1 \tag{1.16}$$

If m is even we have

$$(\gamma^{-m/p}B^*\delta^{(m-1)/p})(\delta^{-(m-1)/p}B\gamma^{(m-2)/p}) \cdots (\gamma^{-2/p}B^*\delta^{1/p})(\delta^{-1/p}B) \\ = \gamma^{m/p}(B^*B)^{m/2} = \gamma^{-m/p}C^m;$$

and if m is odd

$$(\gamma^{-m/p}B^*\delta^{(m-1)/p}) \cdots (\gamma^{-3/p}B^*\delta^{2/p})(\delta^{-2/p}B\gamma^{1/p})(\gamma^{-1/p}C) = \gamma^{-m/p}C^m.$$

Thus in every case

$$\| \gamma^{-m/p}C^m\colon l_2{}^n \to l_{r_m}^n \| \leqslant 1 \tag{1.17}$$

Thus if $2 \leqslant v \leqslant r_m$, again let $1/v = (1-\sigma)/2 + \sigma/r_m = 1/2 - m\sigma/2 = 1/2 - s/p$. Then by lemma 1.4,

$$\| \gamma^{-s/p}C^s\colon l_2{}^n \to l_v{}^n \| \leqslant 1.$$

Since $1/r_m = 1/2 - m/2$ and $m = [p/2]$ we have $m + 1 > p/2$ or $p < r_m$. Thus applying the above with $s = p/2 - 1$, we obtain

$$\| \gamma^{-s/p}C^s\colon l_2{}^n \to l_p{}^n \| \leqslant 1. \tag{1.18}$$

We now put the pieces together. Let D be the mapping defined by the diagram

$$l_2{}^n \xrightarrow{\gamma^{-s/p}C^s} l_p{}^n \xrightarrow{A\alpha^{-1/p}} l_\infty{}^n \xrightarrow{\delta^{1/2}} l_2{}^n.$$

Then we have

$$\sigma_2(D) \leqslant \pi_2(\delta^{1/2}) \| A\alpha^{-1/p} \| \| \delta^{-s/p}C^s \| \leqslant 1,$$

by the formulas displayed above and

$$D = \delta^{1/2}A\alpha^{-1/p}\gamma^{1/p-1/2}C^s = BC^s.$$

Since D has Hilbert-Schmidt norm no larger than one,

$$D^*D = C^sB^*BC^s = C^{2s+2} = C^p$$

has trace class norm no larger than one, i.e. $\sigma_p(B) \leqslant 1$. This proves theorem 1.5.

If $Ae_i = 0$ for some i, a perturbation argument yields an invertible δ with $\sigma_p(\delta^{-1}A\delta) \leqslant 1 + \epsilon$. This proves theorem 1.1.

We now consider a few applications of theorem 1.1.

COROLLARY 1.6. *Let $p \geqslant 2$ and μ be a probability measure on a measure space Ω. Then any operator $T\colon L_p(\mu) \to L_p(\mu)$ whose image is contained in $L_\infty(\mu)$, has absolutely p-summable eigenvalues with*

$$\left(\sum_{n\in N} |\lambda_n(T)|^p\right)^{1/p} \leqslant \| T\colon L_p(\mu) \to L_\infty(\mu)\|.$$

Proof. The natural inclusion $I: L_\infty(\mu) \to L_p(\mu)$ has π_p-norm one.

EXAMPLE. Let $0 < \alpha < N$, $\Omega \subset \mathbb{R}^N$ be a bounded region, $\Delta = \{(x, x): x \in \Omega\}$ and $K: \Omega^2 - \Delta \to \mathbb{C}$ a measurable kernel with

$$| K(x, y)| \leqslant c/| x - y |^\alpha.$$

Then the weakly singular integral operator

$$Tf(x) = \int_\Omega K(x, y) f(y)\, dy \tag{1.19}$$

defines a continuous linear map $T: L_\infty(\Omega) \to L_\infty(\Omega)$ with $\| T: L_p(\Omega) \to L_\infty(\Omega)\| < \infty$ for any $p > N/(N - \alpha)$. Hence the eigenvalues of T are absolutely p-summable for any $p \geqslant 2$ with $p > N/(N - \alpha)$. This improves a result of P. Saphar [24].

PROPOSITION 1.7. *Let (Ω, μ) be a measure space and $K: \Omega^2 \to \mathbb{C}$ a measurable kernel with*

$$\int_\Omega \left(\int_\Omega | K(x, y)|^{p'} d\mu(y)\right)^{p/p'} d\mu(x) < \infty$$

where $p > 2$. Then (1.19) *defines a continuous linear operator $T: L_p(\Omega) \to L_p(\Omega)$ with absolutely p-summable eigenvalues,*

$$\left(\sum_{n\in N} | \lambda_n(T)|^p\right)^{1/p} \leqslant \left(\int_\Omega \left(\int_\Omega | K(x, y)|^{p'} d\mu(y)\right)^{p/p'} d\mu(x)\right)^{1/p}. \tag{1.20}$$

Proof. It is easy to see that $\pi_p(T)^p$ is smaller than or equal to the double integral of (1.20). Apply theorem 1.1.

COROLLARY 1.8 (Hausdorff-Young inequality). *Let $p \geqslant 2$ and $f \in L_{p'}(0, 1)$. If $\hat{f}(n)$ denote the Fourier coefficients of f,*

$$\left(\sum_{n\in N} |\hat{f}(n)|^p\right)^{1/p} \leqslant \|f\|_{L_{p'}}.$$

Proof. The Fourier coefficients are eigenvalues of the operator $F_f: L_p \to L_p$ of convolution with f, with eigenvectors $\exp(2\pi in\ \cdot)$. The double integral in proposition 1.7 for $K(x, y) = f(x - y)$ reduces to $\|f\|_{L_{p'}}$.

In the case of square matrices of complex numbers, proposition 1.7 simply reads ($p \geqslant 2$)

$$\left(\sum_i | \lambda_i(T)|^p\right)^{1/p} \leqslant \left(\sum_j \left(\sum_k | t_{jk} |^{p'}\right)^{p/p'}\right)^{1/p} \tag{1.21}$$

We do not know a simpler proof of this inequality except for $p = 2$. The case $p = 2$ is a classical result of I. Schur [25]. If there is an earier proof, the proof of theorem 1.1 could be simplified:

Remark 1.9. *Assume* (1.21) *has been shown for any matrix operator* $T \simeq (t_{jk})$ *and* $p \geqslant 2$. *Then the absolute p-summability of the eigenvalues of absolutely p-summing operators is an easy consequence with*

$$\left(\sum_{j\in N} |\lambda_j(T)|^p\right)^{1/p} \leqslant \pi_p(T). \tag{1.22}$$

Proof. It is enough to derive (1.22) for operators $T: l_\infty^n \to l_\infty^n$. Let $\delta: l_\infty^n \to l_\infty^n$ be a diagonal map with $\| \delta \|_p = 1$ and $\delta_i \neq 0$. Applying (1.21) to $S = \delta T \delta^{-1}$, we get using lemma 1.2.

$$\begin{aligned}\left(\sum_i |\lambda_i(T)|^p\right)^{1/p} &\leqslant \inf\left\{\left(\sum_j\left(\sum_k |s_{jk}|^{p'}\right)^{p/p'}\right)^{1/p} : \|\delta\|_p = 1\right\} \\ &\leqslant \inf\left\{\sup_j\left(\sum_k |t_{jk}/\delta_k|^{p'}\right)^{1/p'} : \|\delta\|_p = 1\right\} \\ &= \inf\{\| T: l_p^n(\delta) \to l_\infty^n \| : \|\delta\|_p = 1\} = \pi_p(T).\end{aligned}$$

EXAMPLE. Let (Ω, μ) be a measure space and $p > 2$. If $K: \Omega^2 \to \mathbb{C}$ does not fulfill the integrability condition

$$J_p = \left(\int_\Omega \left(\int_\Omega |K(x,y)|^{p'} \, d\mu(y)\right)^{p/p'} d\mu(x)\right)^{1/p} < \infty$$

of proposition 1.7, but only the weaker (but similar looking)

$$I_p = \left(\int_\Omega \int_\Omega |K(x,y)|^{p'} \, d\mu(y) \, d\mu(x)\right)^{1/p'} < \infty,$$

the eigenvalue distribution may change drastically. The following example of Hille-Tamarkin [8] shows that the spectrum then may consist of an arbitrary sequence of complex numbers (λ_n). Let $b_0 = 0$ and b_n be a positive monotone increasing sequence converging to one and define $K: [0, 1]^2 \to \mathbb{C}$ by

$$K(x,y) = \begin{cases}\lambda_n/(b_n - b_{n-1}) & b_{n-1} < x, \quad y < b_n \\ 0 & \text{otherwise}\end{cases}$$

If (λ_n) is bounded, the operator T defined by K will be bounded $T: L_p \to L_p$. Otherwise T may be unbounded. In any case T has the (λ_n) as eigenvalues, since the characteristic functions χ_n of the intervals (b_{n-1}, b_n) are eigenvectors. Nevertheless

$$\int_\Omega \int_\Omega |K(x,y)|^{p'} \, dy \, dx = \sum_{n\in N} |\lambda_n|^{p'} (b_n - b_{n-1})^{2-p'}$$

will be finite, only if the b_n tend fast enough to one, even in the case that the (λ_n) are unbounded.

2. Eigenvalues of Operators in Class $\Pi_p^{(n)}$

In [18] Pietsch made the following observation:

Lemma 2.1. *Let $P \in \mathscr{L}(X, Y)$ and $Q \in \mathscr{L}(Y, X)$. Then the spectra of PQ and QP are identical and the multiplicities of non-zero eigenvalues coincide.*

Using this lemma, Pietsch gave a remarkably simply proof of (1.0). We extend this result of Pietsch as follows.

Proposition 2.2. *Let $n \in N$ and $T \in \Pi_2^{(n)}(X)$, X a Banach space and $2 = (2,\dots, 2)$. Then the eigenvalues $\lambda_j(T)$ of T are $2/n$-summable with*

$$\left(\sum_{j\in N} |\lambda_j(T)|^{2/n}\right)^{n/2} \leqslant \pi_2^{(n)}(T).$$

Proof. Let $\epsilon > 0$. Choose $\delta > 0$ and a factorization for T, $T = T_n ,\dots, T_1$ such that

$$\prod_{i=1}^{n} (\pi_2(T_i) + \delta) \leqslant \pi_2^{(n)}(T) + \epsilon. \tag{2.1}$$

Each T_i can be decomposed as $T_i = Q_i P_i$, $P_i \in \Pi_2(X_{i-1}, l_2)$, $Q_i \in \mathscr{L}(l_2, X_i)$ with

$$\pi_2(P_i)\, \| Q_i \| \leqslant \pi_2(T_i) + \delta. \tag{2.2}$$

Consider $S = (P_n Q_{n-1} P_{n-1} ,\dots, Q_1 P_1) Q_n \in \mathscr{L}(l_2, l_2)$. Since $T = Q_n(P_n ,\dots, Q_1 P_1)$, the eigenvalues of S and T coincide by lemma 2.1. The factorization of S contains n absolutely 2-summing operators from l_2 to l_2, namely $P_n Q_{n-1} ,\dots, P_2 Q_1, P_1 Q_n$. Since $\Pi_2(l_2) = S_2^{\alpha}(l_2)$, we have by the S_p^{α}-composition formula $S \in S_{2/n}^{\alpha}(l_2)$, and the $\sigma_{2/n}^{\alpha}$-norm of S in l_2 is smaller than the product of the σ_2^{α}-norms of all factors. Therefore by (2.1), (2.2) and Weyl's inequality in Hilbert spaces, cf. [26],

$$\begin{aligned}
\left(\sum_{j\in N} |\lambda_j(T)|^{2/n}\right)^{n/2} &= \left(\sum_{j\in N} |\lambda_j(S)|^{2/n}\right)^{n/2} \\
&\leqslant \sigma_{2/n}^{\alpha}(S\colon l_2 \to l_2) \\
&\leqslant \sigma_2^{\alpha}(P_n Q_{n-1}),\dots, \sigma_2^{\alpha}(P_2 Q_1)\, \sigma_2^{\alpha}(P_1 Q_n) \\
&= \pi_2(P_n Q_{n-1}),\dots, \pi_2(P_2 Q_1)\, \pi_2(P_1 Q_n) \\
&\leqslant \prod_{i=1}^{n} \pi_2(P_i)\, \| Q_i \| \\
&\leqslant \prod_{i=1}^{n} (\pi_2(T_i) + \delta) \\
&\leqslant \pi_2^{(n)}(T) + \epsilon.
\end{aligned}$$

Remark. Proposition 2.2 shows (without interpolation theory) that if $p = 2n$, $n \in N$ and $T \in \Pi_p(X)$ then

$$\left(\sum_{j\in N} |\lambda_j(T)|^p\right)^{1/p} \leqslant \pi_p(T).$$

We can now generalize proposition 2.2 to operators of class $\Pi_P^{(n)}$.

THEOREM 2.3. *Let $0 < p < \infty$. Then there exists a constant d_p such that if $p_i \geqslant 2$, $1 \leqslant i \leqslant n$, $1/p = \sum_{i=1}^n 1/p_i$ and $P = (p_1, ..., p_n)$, then any operator $T \in \Pi_P^{(n)}(X)$ has absolutely p-summable eigenvalues with*

$$\left(\sum_{i\in N} |\lambda_i(T)|^p\right)^{1/p} \leqslant d_p \pi_P^{(n)}(T). \tag{2.3}$$

A major tool in the proof is the following lemma which is interesting in its own right.

LEMMA 2.4. *Let K be a compact Hausdorff space and $T \in \mathscr{L}(C(K))$. Assume $p > 2$ and μ is a probability measure on K such that*

$$\begin{aligned} \| T: L_p(K, \mu) \to C(K)\| &\leqslant 1 \\ \| T: L_1(K, \mu) \to L_{p'}(K, \mu)\| &\leqslant 1 \end{aligned} \tag{2.4}$$

Then $\sigma_p(T: L_2(K, \mu) \to L_2(K, \mu)) \leqslant 1$.

Conditions (2.4) of course imply $\pi_p(T: C(K) \to C(K)) \leqslant 1$ and $\pi_p(T^*: L_\infty(\mu) \to L_\infty(\mu)) \leqslant 1$, where T^* is the $L_2(K, \mu)$-adjoint of T. Lemma 2.4 is a continuous phrasing of interpolation results essentially contained in section 1. Therefore we will omit the proof of lemma 2.4 which comes out of the proofs of lemmas 1.3, 1.4 and (1.13) as well as the last part of the proof of theorem 1.5.

The second main step in the proof of theorem 2.3 is provided by the next lemma which gives the idea of an alternate proof of theorem 1.5 which may be more comprehensible.

LEMMA 2.5. *Assume K is a finite set and $T_i \in \Pi_{p_i}(C(K))$, $1 \leqslant i \leqslant n$, where $p_i \geqslant 2$. Then there exists a positive probability measure μ on K, i.e. $\mu(x) > 0$ for every $x \in K$, such that for all i, $1 \leqslant i \leqslant n$,*

$$\| T_i: L_{p_i}(K, \mu) \to C(K)\| \leqslant 2n\pi_{p_i}(T_i)$$

and

$$\| T_i: L_1(K, \mu) \to L_{p_i'}(K, \mu)\| \leqslant 2n\pi_{p_i}(T_i).$$

Proof. Let λ_i be a probability measure on K such that

$$\pi_{p_i}(T_i) = \| T_i : L_{p_i}(K, \lambda_i) \to C(K) \|$$

and set $\lambda = 1/n(\sum_{i=1}^n \lambda_i)$. Then for $1 \leqslant i \leqslant n$

$$\| T_i : L_{p_i}(K, \lambda) \to C(K) \| \leqslant n^{1/p_i} \cdot \pi_{p_i}(T_i) \tag{2.5}$$

If λ is not strictly positive, we change λ a bit to make $\lambda > 0$, replacing n^{1/p_i} by n in (2.5). Therefore we may assume $\lambda > 0$. Let ρ be a probability measure on K. Then

$$\pi_1(C(K) \xrightarrow{T_i} C(K) \xrightarrow{j} L_{p_i'}(K, \rho)) \leqslant \pi_{p_i}(T_i) \tag{2.6}$$

since $\pi_{p_i'}(j) = 1$. Let $\nu_i(\rho)$ be the canonical measure given by the remark after lemma 1.2. Then,

$$\pi_1(C(K) \xrightarrow{jT_i} L_{p_i'}(K, \rho)) = \| T_i : L_1(K, \nu_i(\rho)) \to L_{p_i'}(K, \rho) \| \tag{2.7}$$

Let $\nu(\rho) = 1/2n \sum_{i=1}^n \nu_i(\rho) + \frac{1}{2}\lambda$. Thus for any ρ, $\nu(\rho)$ is a probability measure on K with $\nu(\rho) \geqslant \frac{1}{2}\lambda$. For all $1 \leqslant i \leqslant n$, by (2.6), (2.7)

$$\begin{aligned} \| T_i : L_1(k, \nu(\rho)) \to L_{p_i'}(K, \rho) \| &\leqslant 2n \| T_i : L_1(K, \nu_i(\rho)) \to L_{p_i'}(K, \rho) \| \\ &\leqslant 2n\pi_{p_i}(T_i). \end{aligned}$$

The map $\rho \to \nu(\rho)$ is continuous on the compact convex set

$$\{\rho : \rho \text{ is a probability measure on } K \text{ and } \rho \geqslant \tfrac{1}{2} \cdot \lambda\}.$$

By Brouwer's theorem, it has a fixed point μ. For this probability measure μ, $\mu \geqslant \frac{1}{2} \cdot \lambda$ and

$$\| T_i : L_1(K, \mu) \to L_{p_i'}(K, \mu) \| \leqslant 2n\pi_{p_i}(T_i)$$

Using (2.5) and $\mu \geqslant \frac{1}{2} \cdot \lambda$, we also have

$$\| T_i : L_{p_i}(K, \mu) \to C(K) \| \leqslant 2n\pi_{p_i}(T_i), \qquad 1 \leqslant i \leqslant n.$$

Proof of theorem 2.3. As in Section 1, it is enough to show (2.3) for operators on $l_\infty^m = C(K)$, where the cardinality of K is m. More exactly, we can assume T to be of the form $T = T_n, \ldots, T_1$, $T_i \in \Pi_{p_i}(C(K))$, $i = 1, \ldots, n$. By lemmas 2.5 and 2.4 there exists a probability measure μ on K such that

$$\prod_{i=1}^n \sigma_{p_i}(T_i : L_2(K, \mu) \to L_2(K, \mu)) \leqslant (2n)^n \prod_{i=1}^n \pi_{p_i}(T_i)$$

Since $1/p = \sum_{i=1}^{n} 1/p_i$, we get using Weyl's inequality

$$\left(\sum_{i\in N} |\lambda_i(T)|^p\right)^{1/p} \leqslant \sigma_p(T: L_2(K,\mu) \to L_2(K,\mu))$$
$$\leqslant \prod_{i=1}^{n} \sigma_{p_i}(T_i: L_2(K,\mu) \to L_2(K,\mu))$$

Hence theorem 2.3 is clear except that $d_p = (2n)^n$ seems to depend on n. This is an illusion:

Indeed, if $p \geqslant 2$, we can take $d_p = 1$ by Theorem 1.1 and the composition formula for Π_p-operators [19]. If $p > 2$ and $T = T_n ,..., T_1$, $T_i \in \Pi_{p_i}$, $p_i \geqslant 2$ we can compose the T_i's together to write $T = S_m ,..., S_1$, where $S_j \in \Pi_{q_j}$, $q_j \geqslant 2$, $1/q + 1/q_{j+1} \geqslant \frac{1}{2}$ $j = 1,..., m-1$ and $\sum_{i=1}^{m} 1/q_i = \sum_{i=1}^{n} 1/p_i = 1/p$. Again by the Π_p-composition formula we will have $\prod_{j=1}^{m} \pi_{q_j}(S_j) \leqslant \prod_{i=1}^{n} \pi_{p_i}(T_i)$. It follows that $m \leqslant 4/p + 1$. Repeating the above argument with m replacing n we obtain $d_p \leqslant [2(4/p + 1)]^{4/p+1}$.

A weak form of theorem 2.3 yielding only that the eigenvalues $\lambda_n(T)$ are of order $n^{-1/p}$ can be proved easily using results of D. R. Lewis [12].

We now give an application of theorem 2.3 to p-nuclear operators. For $0 < p \leqslant 1$ p-nuclear operators on Banach spaces in general only have absolutely q-summable eigenvalues, where $1/q = 1/p - 1/2$. But on Hilbert space such operators have p-summable eigenvalues. We "interpolate" between this nice case (L_2) and the worst cases (L_1 and L_∞ of Grothendieck's examples) to obtain the following result.

THEOREM 2.6. *Let $0 < p \leqslant 1$ and $1 \leqslant r \leqslant \infty$, $1/q = 1/p - |1/2 - 1/r|$. Assume (Ω, μ) is a measure space and $T \in \mathcal{N}_p(L_r(\mu))$. Then the eigenvalues of T satisfy*

$$\left(\sum_{j\in N} |\lambda_j(T)|^q\right)^{1/q} \leqslant c_p \nu_p(T),$$

where c_p is a constant depending only on p

Proof. By duality, we may assume $r \geqslant 2$. If $T \in \mathcal{N}_p(L_r(\mu))$ then for $\epsilon > 0$ choose elements $x_i' \in L_r(\mu)'$ and $y_i \in L_r(\mu)$, $\|x_i'\| \leqslant 1$, $\|y_i\| \leqslant 1$, and $(\alpha_i) \in l_p$, $\alpha_i \geqslant 0$ such that

$$T = \sum_{i\in N} \alpha_i x_i' \otimes y_i \qquad \text{and} \qquad \left(\sum_{i\in N} \alpha_i{}^p\right)^{1/p} \leqslant \nu_p(T) + \epsilon.$$

Let $n := [2(1/p - 1)]$, $\delta := 1 - (n+2)p/2$ and $u := p/\delta$. Clearly $n \geqslant 0$, $\delta \geqslant 0$ and $2 \leqslant u \leqslant \infty$. Define operators

$$\begin{aligned} &O\colon L_r(\mu) \to l_u, && Ox = (\alpha_i^{\delta} x_i'(X))_i \\ &P\colon l_u \to l_2, && P(\xi_i)_i = (\alpha_i^{p/2}\xi_i)_i \\ &Q\colon l_2 \to l_2, && Q(\xi_i)_i = (\alpha_i^{p/2}\xi_i)_i \\ &R\colon l_2 \to l_1, && R(\xi_i)_i = (\alpha_i^{p/2}\xi_i)_i \\ &S\colon l_1 \to L_r(\mu), && S(\xi_i)_i = \sum_{i\in N} \xi_i y_i . \end{aligned}$$

These operators are continuous and $T = SRQ^nPO$ since $\delta + (n+1)p/2 + p/2 = 1$. Routine calculations show

$$\pi_u(O) \leqslant \left(\sum_{i\in N} \alpha_i^p\right)^{\delta/p}, \qquad \pi_2(P) = \left(\sum_{i\in N} \alpha_i^p\right)^{1/2}$$

and

$$\pi_2(Q) \leqslant \left(\sum_{i\in N} \alpha_i^p\right)^{1/2}.$$

Using Hölder's inequality we obtain

$$\| R \| \leqslant \left(\sum_{i\in N} \alpha_i^p\right)^{1/2}$$

and, finally, $\| S \| \leqslant 1$. The dual operator $R'\colon l_\infty \to l_2$ is in Π_2. Hence

$$(SR)' = R'S' \in \Pi_2(L_{r'}(\mu), l_2) \subset \Pi_r(L_{r'}(\mu), l_2).$$

Lemma 4 of Pietsch [20] implies

$$SR \in \Pi_r(l_2, L_r(\mu)) \qquad \text{with} \qquad \pi_r(SR) \leqslant \Pi_r((SR)'),$$

$$\begin{aligned} \pi_r(SR) &\leqslant \pi_r(R'S') \leqslant \pi_2(R'S') \leqslant \pi_2(R')\, \| S' \| \\ &= \pi_2(R') \leqslant \left(\sum_{i\in N} \alpha_i^p\right)^{1/2}, \end{aligned}$$

since R' is just multiplication by the diagonal sequence $\alpha_i^{p/2}$. Theorem 2.3. yields, for $T = (SR)Q^nPO$,

$$\left(\sum_{i\in N} |\lambda_i(T)|^q\right)^{1/q} \leqslant c_p(\nu_p(T) + \epsilon)$$

with $1/q = 1/u + (n+1)/2 + 1/r = 1/p - |1/2 - 1/r|$. This proves theorem 2.6.

Remark. Using well-known facts about $\mathscr{L}_r$-spaces, cf. [13], [14], theorem 2.6 can be easily generalized to $\mathscr{L}_r$-spaces; in this case

$$\left(\sum_{j\in N} |\lambda_j(T)|^q\right)^{1/q} \leqslant c_{\lambda,q}\nu_p(T\colon \mathscr{L}_r \to \mathscr{L}_r)$$

with some constant $c_{\lambda,p}$ depending on p and λ, if the space is a $\mathscr{L}_{r,\lambda}$-space.

EXAMPLE 2.7. The summability order q of the eigenvalues of T in theorem 2.6 cannot be improved (for fixed r), as the following example shows. Define the Littlewood matrices A_{2^n} inductively

$$A_{2^0} = (1), \qquad A_{2^{n+1}} = \begin{pmatrix} A_{2^n} & A_{2^n} \\ A_{2^n} & -A_{2^n} \end{pmatrix}, \qquad n \in N.$$

Then $A_{2^n}{}^2 = 2^n \,\mathrm{Id}$ and hence the spectrum of A_{2^n} consists of $\{\pm 2^{n/2}\}$. Both eigenvalues have multiplicity 2^{n-1}. Define, say for $r < 2 < r'$

$$A = \sum_{n\in N} \oplus\, n^{-2/p}(2^n)^{-(1/p+1/r')} A_{2^n}\colon \left(\bigoplus_n l_r^{2^n}\right)_r \to \left(\bigoplus_n l_r^{2^n}\right)_r,$$

i.e. $A\colon l_r \to l_r$ is a blockwise sum of multiples of matrices A_{2^n}. Let $a_j{}^n$ denote the j'th row of A_{2^n}, which consists of ± 1's. Then

$$\nu_p(A_{2^n})^p \leqslant \sum_j \| a_j{}^n \|_{r'}^p = 2^{n(1+p/r')},$$

which implies

$$\nu_p(A)^p \leqslant \sum_{n\in N} n^{-2} < \infty. \qquad \text{Hence } A \in \mathscr{N}_p(l_r).$$

The eigenvalues of A are just all eigenvalues of the multiples of the A_{2^n}'s, counted with the right multiplicity. Thus,

$$\left(\sum_{j\in N} |\lambda_j(T)|^q\right)^{1/q} = \left(\sum_{n\in N} n^{-2q/p}(2^n)^{-q(1/p+1/r')}(2^{n/2})^q\, 2^n\right)^{1/q}.$$

This expression if finite if and only if $1/q \leqslant 1/p - |1/2 - 1/r|$.

For $p = r = 1$, the example considered as an operator on the space $(\bigoplus_{n\in N} l_1^{2^n})_2$ also shows that there are nuclear operators on reflexive spaces whose eigenvalues are not q-summable for any $q < 2$.

In view of theorem 2.6 and this example, for any $q \in (1, 2)$ there are Banach spaces X_q such that $(\sum_j |\lambda_j(T)|^q)^{1/q} < \infty$ for all nuclear operators, but $(\sum_j |\lambda_j(T)|^r)^{1/r} = \infty$ for some nuclear operator T on X_q and any $r < q$. In fact, take X_q to be any $\mathscr{L}_{r(q)}$-space, where $r(q)$ fulfills $1/q = 1 - |1/2 - 1/r(q)|$.

Remark 2.8. Recently B. Carl [3] has obtained results on the distribution of eigenvalues of operators of type $\mathcal{N}_{p,r,s}$ [21] in arbitrary Banach spaces. These results complement the results given here.

3. Eigenvalues of Operators of Type l_p

A classical inequality of H. Weyl [26] used in the previous sections states that the eigenvalues of operators of class $S_p(H)$ in Hilbert spaces H are absolutely p-summable with

$$\sum_{n\in N} |\lambda_n(T)|^p \leqslant \sum_{n\in N} s_n(T)^p, \qquad T\in S_p(H)$$

for any $0 < p < \infty$. Here $s_n(T)$ stands for the singular numbers of T which coincide with the approximation-, Gelfand- or Kolmogorov-numbers of T in Hilbert spaces. A. S. Markus and V. I. Macaev [17] showed, by methods of analytic function theory, a weak extension of Weyl's inequality to operators in Banach spaces X, namely

$$\sum_{n\in N} |\lambda_n(T)|^p \leqslant c_p \sum_{n\in N} \alpha_n(T)^p \ln\left(1+\frac{\|T\|}{\alpha_n(T)}\right),$$

where α_n denotes the approximation numbers. That is, the eigenvalues of a compact operator $T\in\mathcal{K}(X)$ are p-summable provided the right side of the above expression is finite. We improve this result and generalize Weyl's inequality to Banach spaces as follows:

$$\sum_{n\in N} |\lambda_n(T)|^p \leqslant c_p \sum_{n\in N} s_n(T)^p, \qquad T\in S_p\mathrm{t}(X)$$

where s_n refers to either the approximation numbers or the Gelfand- or the Kolmogorov-numbers of T and c_p depends only on p. This answers positively a question of Pietsch [21] and Markus-Macaev [17].

Theorem 3.1 (Weyl's inequality). *Let s_n denote either one of the following s-number sequences: approximation-, Gelfand- or Kolmogorov-numbers. Then there is an absolute constant $c > 1$ such that for any $0 < p < \infty$ and any Banach space X, the eigenvalues of an operator T of type l_p, $T\in S_p{}^s(X)$, are absolutely p-summable with*

$$\left(\sum_{n\in N} |\lambda_n(T)|^p\right)^{1/p} \leqslant c_p \left(\sum_{n\in N} s_n(T)^p\right)^{1/p} \tag{3.1}$$

where $c_p = \max(c, c^{1/p})$.

We need some lemmas for the proof. The idea is to factor an operator of type l_p into a product of operators for which the π_2-norms can be estimated on finite-dimensional restrictions, then apply proposition 2.2 and "improve" the result by using a classical inequality of Hardy.

LEMMA 3.2. *Let s_n be a theorem 3.1. Then, for any $0 < p < \infty$, any positive integer N and all operators $T \in S_p{}^s(X)$*

$$\sigma_{p/N}^s(T^N)^{1/N} \leqslant N^{1/p}\sigma_p{}^s(T).$$

This follows easily from the fact that the s_n are monotone decreasing and multiplicative.

The following lemma is contained implicitly in Pietsch's proof that operators of class S_p are p-nuclear if $0 < p \leqslant 1$, which is given in [19].

LEMMA 3.3. *Let $0 < p \leqslant 1$ and $T \in S_p{}^\alpha(X, Y)$. For any $\epsilon > 0$ there are operators $D_j \in \mathscr{L}(X, Y)$, $j \in N$ with rank $D_j \leqslant d_j := 4 \cdot 3^{j-2}$,*

$$T = \sum_{j \in N} D_j$$

and

$$\left(\sum_{j \in N} d_j \| D_j \|^p\right)^{1/p} \leqslant (1 + \epsilon)\, 8^{1/p}\sigma_p{}^\alpha(T).$$

The connection between the π_2- and $\sigma_p{}^\alpha$-(quasi)norms we want to use is given by the following result.

LEMMA 3.4. *For any $2 < p < \infty$ there is $c_p > 0$ such that for all Banach spaces X and Y with* $\dim X = n$ *and any $T \in \mathscr{L}(X, Y)$*

$$\pi_2(T) \leqslant c_p \cdot n^{1/2-1/p} \cdot \sigma_p{}^\alpha(T).$$

On any closed subinterval I of $(2, \infty)$ one can choose $\sup\{c_p : p \in I\} < \infty$.

Proof. Assume $\dim X = n$, $N = [\log_2 n]$ and let $\epsilon > 0$. By definition there are operators T_i: $X \to Y$ with rank $T_i < 2^i$ and

$$\| T - T_i \| \leqslant (1 + \epsilon)\, \alpha_{2^i}(T),$$

$i = 1,..., N$. Let $D_1 = T_1$, $D_j = T_j - T_{j-1}$ for $j = 2,..., N$ and $D_{N+1} = T - T_N$. Then $T = \sum_{j=1}^{N+1} D_j$. If I_j denotes the identity map on $Y_j = D_j(X) \subset Y$, we have by Garling-Gordon [4]

$$\pi_2(I_j) < (3 \cdot 2^{j+1})^{1/2},$$

since $\dim Y_j < 3 \cdot 2^{j-1}$. Using the injectivity of the π_2-norm, we get

$$\pi_2(T) \leqslant \sum_{j=1}^{N+1} \pi_2(D_j) \leqslant \sum_{j=1}^{N+1} \pi_2(I_j) \| D_j \|$$

$$\leqslant \sum_{j=1}^{N+1} (3 \cdot 2^{j+1})^{1/2} \| D_j \|.$$

An application of Hölder's inequality yields

$$\pi_2(T) \leqslant \left(\sum_{j=1}^{N+1} (3 \cdot 2^{j+1})^{(1/2-1/p)p'} \right)^{1/p'} \left(\sum_{j=1}^{N+1} (3 \cdot 2^{j+1}) \| D_j \|^p \right)^{1/p}$$

$$\leqslant a_p \cdot n^{1/2-1/p} \cdot \left(\sum_{j=1}^{N+1} (3 \cdot 2^{j+1}) \| D_j \|^p \right)^{1/p} \tag{3.2}$$

where the constant a_p depends only on p and is bounded as a function of p on any closed subinterval of $(2, \infty)$. To estimate the right side of (3.2), use

$$\| D_j \| < (1 + \epsilon)(\alpha_{2^i}(T) + \alpha_{2^{i-1}}(T)), \qquad j = 2, \ldots, N + 1$$

and the monotonicity of the approximation numbers to obtain, after reordering terms,

$$\left(\sum_{j=2}^{N+1} (3 \cdot 2^{j+1}) \| D_j \|^p \right)^{1/p} \leqslant (1 + \epsilon) \cdot 2 \cdot 3^{2/p} \left(\sum_{j=1}^{N} 2^{j-1} \alpha_{2^i}(T)^p \right)^{1/p}$$

$$\leqslant 6(1 + \epsilon)\, \sigma_p^\alpha(T).$$

This together with (3.2) implies the lemma.

We remark that by a similar argument one derives for $1 \leqslant p < q \leqslant 2$ and any pair of Banach spaces X and Y that

$$S_p^\alpha(X, Y) \subset \pi_q(X, Y).$$

Since lemma 3.4 assumes $p > 2$, we need a proposition on the decomposition of operators of type l_p.

PROPOSITION 3.5. *Let $0 < p \leqslant 1$, N a natural number larger than one and $T \in S_p^\alpha(X, Y)$. Then there are Banach spaces X_j, $j = 0, \ldots, N$ with $X_0 = X$ and $X_N = Y$ as well as operators $S_j \in S_{Np}^\alpha(X_{j-1}, X_j)$, $j = 1, \ldots, N$ such that $T = S_N \cdots S_1$ and*

$$\prod_{j=1}^{N} \sigma_{Np}^\alpha(S_j) \leqslant 2 \cdot 8^{1/p} \cdot \sigma_p^\alpha(T).$$

Proof. Let $\epsilon = 1/3$ and decompose $T \in S_p^\alpha(X, Y)$ according to lemma 3.3 as $T = \sum_{n\in N} D_n$ with

$$\left(\sum_{n\in N} d_n \| D_n \|^p\right)^{1/p} \leqslant (1+\epsilon)\cdot 8^{1/p}\cdot \sigma_p^\alpha(T), \qquad d_n = 4\cdot 3^{n-2} \tag{3.3}$$

Let $Y_k = D_k(X) \subset Y$ and define the Banach spaces X_j mentioned in the proposition as certain l_r-sums of the Y_k,

$$X_j = \left(\bigoplus_{k\in N} Y_k\right)_{N/j}, \qquad j = 1,\dots, N-1$$

and the operators S_j by

$$S_1\colon X_0 \to X_1, \qquad S_1 x = (\| D_k \|^{1/N-1} D_k x)_{k\in N}$$

$$S_j\colon X_{j-1} \to X_j, \qquad S_j(\xi_k)_{k\in N} = (\| D_k \|^{1/N} \xi_k)_{k\in N}, \qquad j = 2,\dots, N-1$$

$$S_N\colon X_{N-1} \to X_N, \qquad S_N(\xi_k)_{k\in N} = \sum_{k\in N} \| D_k \|^{1/N} \xi_k .$$

These operators are well-defined, continuous and

$$T = \sum_{n\in N} D_n = S_N ,\dots, S_1 .$$

To estimate the approximation numbers of the S_j, $j = 2,\dots, N-1$, consider the operators

$$P_j^n\colon X_{j-1} \to X_j, \qquad P_j^n(\xi_k)_{k\in N} = (\eta_k)_{k\in N}, \qquad \eta_k = \begin{cases} \| D_k \|^{1/N} \xi_k & k \leqslant n \\ 0 & k > n \end{cases}$$

which are of finite rank,

$$\operatorname{rank} P_j^n \leqslant \sum_{k=1}^{n} \operatorname{rank} D_k < 2\cdot 3^{n-1} =: m(n).$$

Therefore by definition of the α_j,

$$\alpha_{m(n)}(S_j) \leqslant \| S_j - P_j^n \| = \sup_{(\xi_j)\neq 0} \frac{(\sum_{k>n} \| D_k \|^{1/j} \| \xi_k \|^{N/j})^{j/N}}{(\sum_{k\in N} \| \xi_k \|^{N/(j-1)})^{(j-1)/N}}$$

An application of Hölder's inequality yields that this quantity is less than or equal

$$\leqslant \left(\sum_{k>n} \| D_k \|\right)^{1/N}, \qquad j = 2,\dots, N-1$$

Observe that a similar estimate holds for $j = 1$ and $j = N$ so that

$$\alpha_{m(n)}(S_j) \leqslant \left(\sum_{k>n} \| D_k \|\right)^{1/N}, \qquad j = 1,..., N$$

Using the fact that the l_1-norm is smaller that the l_p-quasinorm, we get

$$\begin{aligned}\sigma^{\alpha}_{Np}(S_j) &\leqslant \left(\| S_j \|^{Np} + \sum_{n\in N} 4 \cdot 3^{n-1}\alpha_{m(n)}(S_j)^{Np}\right)^{1/Np} \\ &\leqslant \left(\sum_{k\in N} \| D_k \|^p + \sum_{n\in N} 4 \cdot 3^{n-1} \cdot \sum_{k>n} \| D_k \|^p\right)^{1/Np} \\ &= \left(\sum_{k\in N} \| D_k \|^p + \sum_{k\geqslant 2} \left(\sum_{n<k} 4 \cdot 3^{n-1}\right) \| D_k \|^p\right)^{1/Np} \\ &\leqslant \left(\sum_{k\in N} 2 \cdot 3^{k-1} \| D_k \|^p\right)^{1/Np} \\ &\leqslant (2 \cdot 8^{1/p} \cdot \sigma_p^{\alpha}(T))^{1/N}.\end{aligned}$$

Hence the product of the N factors $\sigma^{\alpha}_{Np}(S_j)$ fulfills the inequality stated in the proposition.

We are now ready to prove the following weak form of Weyl's inequality in Banach spaces.

PROPOSITION 3.6. *There is $c > 0$ such that for all $0 < p \leqslant 1$, all Banach spaces X, all $T \in S_p{}^s(X)$ and any $n \in N$*

$$| \lambda_n(T)|^p \leqslant c\sigma_p{}^s(T)^p/n,$$

where s stands for either the approximation- or the Gelfand- or the Kolmogorov-numbers.

Proof. The equality $\delta_n(T) = \gamma_n(T')$ holds for compact operators T, in particular for $T \in S_p{}^s(X)$. Further, $\delta_n(T) \leqslant \alpha_n(T)$. Therefore it suffices to give the proof for the Gelfand numbers. If K is the unit ball of X' and $i: X \to C(K)$ the natural imbedding, one has by [22]

$$\gamma_n(T) = \alpha_n(iT), \qquad \sigma_p{}^{\gamma}(T) = \sigma_p{}^{\alpha}(iT).$$

Choose a natural number $N \geqslant 2$ such that $3 \leqslant Np \leqslant 4$. We decompose $iT \in S_p{}^{\alpha}(X, C(K))$ according to proposition 3.5 as $iT = S_N ,..., S_1$ with $S_j \in S^{\alpha}_{Np}(X_{j-1}, X_j)$ where $X_0 = X$, $X_N = C(K)$ and

$$\prod_{j=1}^{N} \sigma^{\alpha}_{Np}(S_j) \leqslant 2 \cdot 8^{1/p} \cdot \sigma_p{}^{\gamma}(T). \tag{3.4}$$

Let $Y_m{}^0$ be the range of the spectral projection of T relative to the first m eigenvalues of T, i.e. the span of (generalized) eigenvactors and $Y_m{}^j = S_j(Y_m^{j-1})$ for $j = 1,..., N$. We may assume without loss of generality that $\lambda_m(T) \neq 0$. Then $l := \dim Y_m{}^0 \geqslant m$, the case $l > m$ occurs only for multiple eigenvalues. But in any case $\lambda_m(T) = \lambda_1(T)$. Since

$$T \,|_{Y_m{}^0} = \prod_{j=1}^{N} S_j \,|_{Y_m^{j-1}}$$

has the same first l eigenvalues as T, proposition 2.2 yields

$$\left(\sum_{j=1}^{l} |\lambda_j(T)|^{2/N} \right)^{N/2} \leqslant \prod_{j=1}^{N} \pi_2(S_j\colon Y_m^{j-1} \to Y_m{}^j)$$

$$= \prod_{j=1}^{N} \pi_2(S_j\colon Y_m^{j-1} \to X_j).$$

We next apply lemma 3.4 N times to the right side, with Np instead of p. Using that the $|\lambda_j(T)|$ are monotone decreasing we get by (3.4)

$$l^{N/2} |\lambda_1(T)| \leqslant \left(\sum_{j=1}^{l} |\lambda_j(T)|^{N/2} \right)^{2/N}$$

$$\leqslant c_{Np}^N \cdot l^{N(1/2-1/Np)} \prod_{j=1}^{N} \sigma_{Np}^{\alpha}(S_j)$$

$$\leqslant 2 \cdot 8 \quad \cdot c_{Np}^N \cdot l^{N/2-1/p} \cdot \sigma_p{}^{\gamma}(T)$$

which implies

$$m \,|\lambda_m(T)|^p \leqslant d_p \cdot \sigma_p{}^{\gamma}(T)^p$$

where the constant $d_p = 2^p 8 c_{Np}^{Np}$ is bounded as a function of $p \in (0, 1]$.

Corollary 3.7. *There is $c > 0$ such that for all $0 < p \leqslant 1$, all Banach spaces X, all $T \in \mathscr{K}(X)$ and any $n \in N$*

$$|\lambda_n(T)|^p \leqslant c \sum_{j=1}^{n} s_j(T)^p / n.$$

Proof. Apply proposition 3.6 to the Gelfand number ideal and the restriction of T to the range X_n of the spectral projection of T relative to the first n eigenvalues. The corollary follows for the Gelfand numbers by using their injectivity $\gamma_j(T\colon X_n \to X_n) \leqslant \gamma_j(T\colon X \to X)$.

We need the following classical inequality of Hardy [7], chap. 9.

LEMMA 3.8. *Let* $0 < r < p < \infty$. *Then for any sequence* $\alpha \in l_p$

$$\left(\sum_{n \in N} \left(\frac{\sum_{i=1}^{n} \alpha_i^r}{n}\right)^{p/r}\right)^{1/p} \leqslant \left(\frac{p}{p-r}\right)^{1/r} \| \alpha \|_p .$$

Proof of theorem 3.1. If $0 < p \leqslant 2$, choose $r = p/2$. Then by corollary 3.7 and lemma 3.8.

$$\left(\sum_{n \in N} | \lambda_n(T)|^p\right)^{1/p} \leqslant c^{1/r} \left(\sum_{n \in N} \left(\frac{\sum_{j=1}^{n} s_j(T)^r}{n}\right)^{p/r}\right)^{1/p}$$

$$\leqslant (2c)^{2/p} \left(\sum_{n \in N} s_n(T)^p\right)^{1/p}$$

showing (3.1) for $p \leqslant 2$. The statement for $p > 2$ and the behaviour of the constant c_p follows by an application of the previous case to $T^N \in S^s_{p/N}(X)$ with $N = [2p]$ using lemma 3.2, and $\sup\{ p^{1/p} : p \geqslant 1\} < \infty$. This ends the proof of Weyl's generalized inequality.

COROLLARY 3.9. *Let* $T \in \mathscr{L}(X)$ *be compact. Then for any* $0 < p < \infty$ *and* $n \in N$

$$\left(\sum_{j=1}^{n} | \lambda_j(T)|^p\right)^{1/p} \leqslant c_p \left(\sum_{j=1}^{n} s_j(T)^p\right)^{1/p}.$$

The proof of theorem 3.1 could be simplified a bit by directly relating the $\Pi_2^{(n)}$-quasinorms to the operators D_n in a decomposition of $T = \sum_{n \in N} D_n \in S_p^{\alpha}(X)$. However, the proof which was given, also yields some information on the dependence of c_p on p, and the factorization theorem for operators of type l_p seems to have some interest in its own right, since similar statements are false e.g. for the absolutely p-summing operators.

As an application of corollary 3.7, we prove a fact on eigenvalues of operators in L_p.

PROPOSITION 3.10. *Let* $\Omega \subset \mathbb{R}^N$ *be sufficiently regular domain,* $1 \leqslant p \leqslant \infty$ *and* $T: L_p(\Omega) \to L_p(\Omega)$ *a continuous linear operator whose image is contained in a Sobolev space* $W_p^m(\Omega)$. *Then the eigenvalues* $\lambda_n(T)$ *of* T *decrease of order* $n^{-m/N}$ *and this is in general the best possible result.*

Proof. As a map from $L_p(\Omega)$ into $W_p^m(\Omega)$, T has closed graph and therefore is continuous also with respect to the Sobolev norm. It is well-known, cf. [1], that the approximation numbers of the Sobolev imbedding $W_p^m(\Omega) \to L^p(\Omega)$ are of order $n^{-m/N}$, therefore also

$$\alpha_n(T: L_p \to L_p) \leqslant \| T: L_p \to W_p^m \| \alpha_n(W_p^m \to L_p) = O(n^{-m/N}).$$

Choose $r < N/m$ and apply corollary 3.7 to get

$$| \lambda_n(T)| \leqslant c \left(\sum_{j=1}^{n} \alpha_j(T)^r \Big/ n \right)^{1/r}$$

$$\leqslant d \left(\sum_{j=1}^{n} j^{-rm/N} \Big/ n \right)^{1/r} = O(n^{-m/N}).$$

The operators of m-fold integration show in the case of $N = 1$ that the order given in the proposition is, in general, optimal.

If T is a continuous integral operator in $L_p(\Omega)$, a sufficient condition for $T(L_p(\Omega)) \subset W_p{}^m(\Omega)$ can be given easily in terms of m-fold differentiability of the kernel with respect to the first variable.

In Hilbert spaces more is known on the operators of type l_p so far considered: For $p = 1$, they coincide isometrically with the nuclear operators, for $p = 2$ with the absolutely 2-summing operators. Over general Banach spaces this is not true; in fact, if $\mathscr{N}_1(X) = S_1{}^\alpha(X)$ holds with equality of the nuclear and $\sigma_1{}^\alpha$-(quasi)norms, X is isometrically isomorphic to a Hilbert space, cf. [23]. The same idea also shows that $\Pi_2(X) = S_2{}^\alpha(X)$ isometrically implies $X = H$ isometrically. The isomorphic problem: is it true that the equality $\mathscr{N}_1(X) = S_1{}^\alpha(X)$ alone implies that X is isomorphic to a Hilbert space, turns out to be more complicated. We answer this question affirmatively for both real and complex Banach spaces. This will follow from Weyl's inequality and the following theorem which characterizes Hilbert spaces by the absolute summability of the eigenvalues of nuclear operators.

THEOREM 3.11. *Let X be a Banach space and suppose that each nuclear operator on X has absolutely summable eigenvalues Then X is isomorphic to a Hilbert space.*

To show this, we need some notation.

DEFINITION. For $T \in \mathscr{F}(X, Y)$, the finite rank operators between X and Y, the projective tensor norm is given by

$$\| T \|_\wedge = \inf \left\{ \sum_{i=1}^{n} \| f_i \| \, \| y_i \| : f_i \in X', y_i \in Y \right\},$$

where the infimum is taken over all representations

$$T = \sum_{i=1}^{n} f_i \otimes y_i , \qquad n \text{ finite}.$$

If E is a subspace of a Banach space Y with $\dim E < \infty$, i_E denotes the inclusion mapping from E into Y. We define the norms $\| \cdot \|_Y$ and $\| \cdot \|_{\wedge Y}$ on $\mathscr{L}(E)$, $E \subset Y$ by

$$\| T \|_Y = \inf\{\| \tilde{T} \| : T \in \mathscr{L}(Y, E),\ \tilde{T} \text{ extends } T\}$$

and

$$\| T \|_{\wedge Y} = \| i_E T \|_{\wedge} .$$

It is well-known that $\mathscr{L}(E)$ is algebraically self-dual under the duality $\langle T, S \rangle = \operatorname{trace}(ST)$.

LEMMA 3.12. *For the norms* $\| \cdot \|_Y$ *and* $\| \cdot \|_{\wedge Y}$ *the above is a norm duality,*

$$(\mathscr{L}(E), \| \cdot \|_Y)' = (\mathscr{L}(E), \| \cdot \|_{\wedge Y}).$$

Proof. We show the equivalent equality $(\mathscr{L}(E), \| \cdot \|_{\wedge Y})' = (\mathscr{L}(E), \| \cdot \|_Y)$. Given $S \in \mathscr{L}(E)$,

$$\| i_E S \|_{\wedge} = \sup\{| \operatorname{trace}(T i_E S)| : T \in \mathscr{L}(Y, E), \| T \| \leqslant 1\}.$$

Clearly this supremum is just

$$= \sup\{| \operatorname{trace}(TS)| : T \in \mathscr{L}(E), \| T \|_Y \leqslant 1\}.$$

THEOREM 3.13. *Let X be a Banach space. Suppose there is $c > 0$ such that*

$$\sum_{i \in N} | \lambda_i(T)| \leqslant c \| T \|_{\wedge}$$

for every finite rank operator on X. Then X is isomorphic to a Hilbert space.

Proof. Let E be a finite-dimensional subspace of X. By lemma 3.12 there is $S \in \mathscr{L}(E)$ with $\| S \|_{\wedge X} = 1$ and $\operatorname{trace} S = \| i_E \|_X$. By definition of $\| S \|_{\wedge X}$ and the Hahn-Banach theorem, given $\epsilon > 0$ we can extend S to a finite rank operator $\tilde{S}$ on X with $\| \tilde{S} \|_{\wedge} \leqslant 1 + \epsilon$. Then

$$| \operatorname{trace} S | = \Big| \sum_i \lambda_i(S) \Big| \leqslant \sum_i | \lambda_i(\tilde{S})|$$
$$\leqslant c \| \tilde{S} \|_{\wedge} \leqslant c(1 + \epsilon).$$

Letting $\epsilon \to 0$ yields $\| i_E \|_X \leqslant c$. By the Lindenstrauss-Tzafriri theorem [15], X is isomorphic to a Hilbert space.

If $c = 1$, it follows from theorems of Kakutani-Bohnenblust [2], [9] that X is isometrically isomorphic to a Hilbert space, the exact converse of Weyl's inequality for Hilbert space.

Proof of theorem 3.11. Let $\varnothing(X) = \inf\{k: \sum_i |\lambda_i(T)| \leqslant k \| T \|_\wedge, T \in \mathscr{F}(X)\}$. First observe that if $\varnothing(Y) < \infty$ for some subspace Y of finite codimension in X, then by theorem 3.13 X is isomorphic to a Hilbert space. Thus if $\varnothing(X) = \infty$, the same is true for each subspace of finite codimension in X.

Suppose $\varnothing(X) = \infty$. Choose T_1 of finite rank with $\sum |\lambda_i(T_1)| > 2^2 \| T_1 \|_\wedge$. Let $E_1 \subset X$, $\dim E_1 < \infty$ be such that $T_1(X) \subset E_1$ and $2^2 \| \bar{T}_1 \|_\wedge < \sum |\lambda_i(T)|$. Here the bar denotes restriction and astriction of T_1 to E_1, similarly for what follows. Choose $Y_1 \subset X$, codim $Y_1 < \infty$ with $\| e + y \| \geqslant \frac{1}{2} \| e \|$ for all $e \in E_1$, $y \in Y_1$. Since $\varnothing(Y_1) = \infty$, there is an $E_2 \subset Y_1$, $\dim E_2 < \infty$ and a $T_2 \in \mathscr{F}(Y_1)$ with

$$T_2(Y_1) \subset E_2 \qquad \text{and} \qquad \sum_i |\lambda_i(T_2)| > 2^4 \| \bar{T}_2 \|_\wedge .$$

In general choose $Y_{k+1} \subset X$, codim $Y_{k+1} < \infty$, $\| e + y \| \geqslant \frac{1}{2} \| e \|$ for $e \in E_1 \oplus \cdots \oplus E_k$ and $y \in Y_{k+1}$, and choose $E_{k+1} \subset Y_{k+1}$, $\dim E_{k+1} < \infty$, and $T_{k+1} \in \mathscr{F}(Y_k)$ with $T_{k+1}(Y_k) \subset E_{k+1}$ and $\sum_i |\lambda_i(T_{k+1})| > 2^{2k} \| \bar{T}_{k+1} \|_\wedge$. Without loss of generality suppose that for all k $\| \bar{T}_k \|_\wedge = 1$. Define T on $\bigoplus_{k \in N} E_k$ by

$$T\left(\sum_{k=1}^{\infty} e_k\right) = \sum_{k=1}^{\infty} 2^{-k} T_k(e_k).$$

Clearly T is nuclear and so has a nuclear extension $\tilde{T}$ to all of X. But the eigenvalues of $\tilde{T}$ contain those of $2^{-k}T_k$ for each k and so $\sum_i |\lambda_i(\tilde{T})| = \infty$.

COROLLARY 3.14. *X is isomorphic to a Hilbert space if and only if $\Pi_2^{(2)}(X)$ $(2 = (2, 2))$ is a Banach space, coinciding with the nuclear operators $\mathscr{N}_1(X)$.*

Proof. If X is isomorphic to a Hilbert space, $\Pi_2^{(2)}(X)$ is the same as the nuclear operators on X, since any such operator can be written as the product of two Hilbert-Schmidt operators. Therefore $\Pi_2^{(2)}(X)$ is a Banach space.

On the other hand, if $\Pi_2^{(2)}(X)$ is a Banach space, it must contain the nuclear operators $\mathscr{N}_1(X)$, since $\mathscr{N}_1$ is the smallest Banach ideal. Therefore any nuclear operator is in $\Pi_2^{(2)}$ and hence has absolutely summable eigenvalues, by proposition 2.2. By theorem 3.11, X is isomorphic to a Hilbert space.

As a corollary to 3.11 we obtain the isomorphic characterization of Hilbert spaces mentioned above.

THEOREM 3.15. *A complex Banach space X is isomorphic to a Hilbert space if and only if the nuclear operators on X coincide with the operators $S_1^\alpha(X)$ of type l_1.*

Proof. It is clear that in Hilbert spaces H, $\mathscr{N}_1(H) = S_1^\alpha(H)$ and that in general $S_1^\alpha(X) \subset \mathscr{N}_1(X)$, [19]. If $S_1^\alpha(X) = \mathscr{N}_1(X)$, the eigenvalues of any nuclear, i.e. S_1^α-operator are absolutely summable by Weyl's inequality (3.1). Hence X is isomorphic to a Hilbert space by theorem 3.11.

We remark that the real version of 3.15 follows from the complex case. Indeed if X is a real Banach space and $Y = X + iX$ its complexifixation, then any $T \in \mathcal{N}_1(Y)$ induces, by considering real and imaginary parts of a nuclear representation of T, eight nuclear operators on X or isometric copies of X. If $\mathcal{N}_1(X) = l_1^\alpha(X)$, these eight operators add to yield $T \in l_1^\alpha(Y)$. By 3.15, Y, hence X, is isomorphic to a Hilbert space.

For the finite rank operators on a Banach space X, the trace functional $\mathrm{tr}(\cdot)$ is well-defined and continuous with respect to the σ_1^α-quasinorm. It may therefore be extended to any operator of type l_1. In Hilbert spaces, with $S_1^\alpha(H) = \mathcal{N}_1(H)$, Lidskij's theorem [16] states that the so-defined trace of any $S_1^\alpha(H)$-operator is equal to the sum of its eigenvalues. As with Weyl's inequality, this trace formula can be extended to the class $S_1^\alpha(X)$ of operators of type l_1 on general Banach spaces, cf. [10].

Remark 3.16. If $\mathcal{K}(X)$ denotes the compact operators on X, Grothendieck has conjectured that if $\mathcal{K}(X) = \mathcal{N}_1(X)$, then $\dim X < \infty$. Theorem 3.11 yields some more information on this problem: Suppose X is a Banach space with the property that for every $T \in \mathcal{N}_1(X)$ there are $A, B \in \mathcal{K}(X)$ with $T = AB$.

For such a space the Grothendieck conjecture is true. In fact, if $\mathcal{K}(X) = \mathcal{N}_1(X)$, we have $\sum |\lambda_i(T)| < \infty$ for $T \in \mathcal{N}_1(X) = \mathcal{N}_1 \circ \mathcal{N}_1(X)$ by the above hypothesis and proposition 2.2. Hence X is isomorphic to a Hilbert space. This violates $\mathcal{K}(X) = \mathcal{N}_1(X)$ unless $\dim X < \infty$. The above hypothesis is met by any space X such that X is isomorphic to $(\bigoplus X)_p$ for some p, $1 \leqslant p \leqslant \infty$. In particular let (G_n) be a sequence of finite-dimensional spaces such that for every finite dimensional space F there is n with $d(G_n, F) \leqslant 2$, d denoting the Banach-Mazur distance. Let $G_n{}^i = G_n$ for all i and let

$$X = \Big(\bigoplus_{n \in N} \Big(\bigoplus_{i \in N} G_n{}^i\Big)_\infty\Big)_\infty .$$

Then X lacks the approximation property but has the above property.

Acknowledgment

The fourth author would like to thank the Johann Kröhle family of Bonn-Mehlem for their kindness and hospitality during his stay in Germany.

References

1. M. S. Birman and M. Z. Solomjak, Piecewise polynomial approximation of functions of class $W_p{}^s$, *Mat. Sb.* **73** (1967), 331–355 [in Russian].
2. H. F. Bohnonblust, A characterization of complex Hilbert spaces, *Portugal. Math.* **3** (1942), 103–109.

3. B. Carl, Distribution of eigenvalues of nuclear operators in Banach spaces, *Vestnik Moskov. Univ.* **2** (1976), 3–10 [in Russian].
4. D. J. H. Garling and Y. Gordon, Relations between some constants associated with finite dimensional Banach spaces, *Israel J. Math.* **9** (1971), 346–361.
5. Y. Gordon, D. R. Lewis, and J. R. Retherford, Banach ideals of operators with applications, *J. Functional Analysis* **14** (1973), 85–129.
6. A. Grothendieck, "Produits tensoriels topologiques et espaces nucléaires," Chap. II, Sect. 1, Amer. Math. Soc. Memoirs No. 16, Providence, R. I., 1955.
7. G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities," Cambridge Univ. Press, Cambridge, 1952.
8. E. Hille and J. D. Tamarkin, On the theory of linear integral equations, *Ann. of Math.* **31** (1930), 479–528.
9. S. Kakutani, Some characterizations of Euclidean spaces, *Japan. J. Math.* **16** (1939), 93–97.
10. H. König, s-numbers, eigenvalues and the trace formula in Banach spaces, *Studia Math.*, in press.
11. S. Kwapień, On operators factorizable through L_p-spaces, *Bull. Soc. Math. France, Mém.* **31–32** (1972), 215–225.
12. D. R. Lewis, Finite dimensional subspaces of L_p, *Studia Math.*, in press.
13. J. Lindenstrauss, A. Pelczynski, Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications, *Studia Math.* **29** (1968), 275–326.
14. J. Lindenstrauss and H. P. Rosenthal, The $\mathscr{L}_p$-spaces, *Israel J. Math.* **7** (1969), 325–349.
15. J. Lindenstrauss and L. Tzafriri, On the complemented subspaces problem, *Israel J. Math.* **9** (1971), 263–269.
16. V. B. Lidskij, Non-selfadjoint operators with a trace, *Dokl. Akad. Nauk SSSR* **125** (1959), 485–488 [in Russian].
17. A. S. Markus and V. I. Macaev, Analogs of Weyl inequalities and the trace theorem in Banach space, *Mat. Sb.* **86** (1971), 299–313.
18. A. Pietsch, Eigenwertverteilungen von Operatoren in Banach-räumen, *in* "Theory of Sets and Topology," pp. 391–402, Berlin, 1972.
19. A. Pietsch, "Nukleare lokalkonvexe Räume," Akademie-Verlag, Berlin, 1969.
20. A. Pietsch, Absolutely p-summing operators in $\mathscr{L}_r$-spaces, *Bull. Soc. Math. France, Mém.* **31–32** (1972), 285–315.
21. A. Pietsch, "Theorie der Operatorenideale," Jena, 1972.
22. A. Pietsch, s-numbers of operators in Banach spaces, *Studia Math.* **51** (1974), 201–223.
23. B. Rosenberger, Approximationszahlen, p-nukleare Operatoren und Hilbertraum-charakterisierungen, *Math. Ann.* **213** (1975), 211–221.
24. P. Saphar, Applications à puissance nucléaire et applications de Hilbert–Schmidt dans les espaces de Banach, *Ann. Sci. École Norm. Sup.* **83** (1966), 113–151.
25. I. Schur, Über die characteristischen Wurzlen einer linearen Substitutionen mit einer Anwendung auf die Theorie der Integralgleichungen, *Math. Ann.* **66** (1909), 488–451.
26. H. Weyl, Inequalities between the two kinds of eigenvalues of a linear transformation, *Proc. Nat. Acad. Sci. U.S.A.* 3L (1949), 408–411.