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## The Cardy–Verlinde formula and entropy of the charged rotating BTZ black hole

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## ABSTRACT

In this Letter we show that the entropy of black hole horizon in charged rotating BTZ space–time can be described by the Cardy–Verlinde formula, which is supposed to be an entropy formula of conformal field theory in any dimension.

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## 1. Introduction

The discovery of the existence of black hole solutions in three spacetime dimensions by Bañados, Teitelboim and Zanelli (BTZ) [1,2] (for a review see Ref. [3]) represented one of the main recent advances for low-dimensional gravity theories. Owing to its simplicity and to the fact that it can be formulated as a Chern–Simons theory, 3D gravity has become paradigmatic for understanding general features of gravity, and in particular its relationship with gauge field theories, in any spacetime dimensions.

The realization of the existence of three-dimensional (3D) black holes not only deepened our understanding of 3D gravity but also became a central key for recent developments in gravity, gauge and string theory.

The BTZ black hole continues to play a key role in recent investigations aiming to improve our understanding of 3D gravity and of general feature of the gravitational interaction [4].

A characterizing feature of the BTZ black hole (at least in its uncharged form) is the absence of curvature singularities. The scalar curvature is well-behaved (and constant) throughout the whole 3D spacetime. This feature is shared by other low-dimensional examples such as 2D AdS black holes (see e.g. Ref. [5]), for which also the microscopic entropy could be calculated [6,7] using the method proposed in Ref. [8].

The absence of curvature singularities makes the BTZ black hole very different from its higher-dimensional cousins such as the 4D Schwarzschild black hole. On the other hand, one can try to consider low-dimensional black holes with curvature singularities generated by matter sources. In this Letter we consider the alternative case in which the curvature singularity is not generated by mass sources but by charges of the matter fields. An example, which we

discuss in this Letter, is the electrically charged rotating BTZ (CR–BTZ) black hole.

One of the remarkable outcomes of the AdS/CFT correspondence has been the generalization of Cardy’s formula (Cardy–Verlinde formula) for arbitrary dimensionality, as well as a variety of AdS backgrounds. The Cardy–Verlinde formula proposed by Verlinde [9], relates the entropy of a certain CFT with its total energy and its Casimir energy in arbitrary dimensions. Quantum gravity in low-dimensional anti-de Sitter (AdS) spacetime has features that make it peculiar with respect to the higher-dimensional cases. For  $d = 2, 3$  the theory is a conformal field theory (CFT) describing (Brown–Henneaux-like) boundary deformations and has a central charge determined completely by Newton constant and the AdS length [6,10–12]. Conversely, in  $d > 4$ , quantum gravity in AdS space–times should admit a near-horizon description in terms of BPS solitons and D-brane excitations, whose low-energy limit is an  $U(N)$  gauge theory [13–15]. The difference between these two descriptions is particularly evident in their application for computing the entropy of non-perturbative gravitational configurations such as black holes, black branes and BPS states.

In the present Letter we would like to check the consistency of the Cardy–Verlinde formula, for the charged rotating BTZ black hole.

## 2. The charged rotating BTZ black hole

The BTZ black hole solutions [1,2] in  $(2 + 1)$  spacetime dimensions are derived from a three-dimensional theory of gravity

$$I = \frac{1}{16\pi G} \int d\chi^3 \sqrt{-g} (R + 2\Lambda), \quad (1)$$

where  $G$  is the three-dimensional Newton constant and  $\Lambda = \frac{1}{l^2} > 0$  is the cosmological constant. Often in the literature units are chosen such that  $G$  is dimensionless,  $8G = 1$ , here we use such units.

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The corresponding line element in Schwarzschild coordinates is

$$ds^2 = -f(r) dt^2 + f^{-1} dr^2 + r^2 \left( d\theta - \frac{J}{2r^2} dt \right)^2 \quad (2)$$

with metric function:

$$f(r) = \left( -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right), \quad (3)$$

where  $M$  is the Arnowitt–Deser–Misner (ADM) mass,  $J$  the angular momentum (spin) of the BTZ black hole and  $-\infty < t < +\infty$ ,  $0 \leq r < +\infty$ ,  $0 \leq \theta < 2\pi$ . The outer and inner horizons, i.e.  $r_+$  (henceforth simply black hole horizon) and  $r_-$  respectively, concerning the positive mass black hole spectrum with spin ( $J \neq 0$ ) of the line element (2) are given as

$$r_{\pm}^2 = \frac{l^2}{2} \left( M \pm \sqrt{M^2 - \frac{J^2}{l^2}} \right). \quad (4)$$

In addition to the BTZ solutions described above, it was also shown in [1,16] that charged black hole solutions similar to (2) exist. These are solutions following from the action [16,17]

$$I = \frac{1}{2\pi} \int dx^3 \sqrt{-g} \left( R + 2\Lambda - \frac{\pi}{2} F_{\mu\nu} F^{\mu\nu} \right). \quad (5)$$

The Einstein equations are given by

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \pi T_{\mu\nu}, \quad (6)$$

where  $T_{\mu\nu}$  is the energy–momentum tensor of the electromagnetic field:

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F^2. \quad (7)$$

Electric charged black hole solutions of Eqs. (6) takes the form (2), but with

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r, \quad (8)$$

whereas the  $U(1)$  Maxwell field is given by

$$F_{tr} = \frac{Q}{r}, \quad (9)$$

where  $Q$  is the electric charge. Although these solution for  $r \rightarrow \infty$  are asymptotically AdS, they have a power law curvature singularity at  $r = 0$ , where  $R \sim \frac{\pi Q^2}{r^2}$ . This  $r \rightarrow 0$  behavior of the charged BTZ black hole has to be compared with that of the uncharged one, for which  $r = 0$  represent just a singularity of the causal structure. For  $r > l$ , the charged black hole is described by the Penrose diagram as usual [18].

Horizons of the CR-BTZ metric are roots of the lapse function  $f$ . Depending on these roots there are three cases of the CR-BTZ black hole [19,20] (see also [21]): Two distinct horizons  $r_{\pm}$  exist where plus correspond to the event horizon while minus gives the Cauchy horizon (the usual CR-BTZ); black hole in case of two repeated real roots gives a single horizon (extreme case); and the case when no real root exists thus no horizon exists (naked singularity).

We shall assume the first case in this Letter. The black hole mass and the angular momentum are given respectively by

$$M = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2} - \frac{\pi}{2} Q^2 \ln r \quad (10)$$

and

$$J = 2r_+ \sqrt{M - \frac{r_+^2}{l^2} + \frac{\pi}{2} Q^2 \ln r_+} \quad (11)$$

with the corresponding angular velocity to be

$$\Omega_+ = \frac{J}{2r_+^2} = \frac{1}{r_+} \sqrt{M - \frac{r_+^2}{l^2} + \frac{\pi}{2} Q^2 \ln r_+}. \quad (12)$$

The Hawking temperature  $T_H$  of the black hole horizon is

$$T_H = \left. \frac{df}{dr} \right|_{r=r_+} = \frac{1}{4\pi} \left( \frac{2r_+}{l^2} - \frac{J^2}{2r_+^3} - \frac{\pi}{2} \frac{Q^2}{r_+} \right). \quad (13)$$

The entropy of the charged rotating BTZ black hole takes the form

$$S_{\text{BH}} = 4\pi r_+. \quad (14)$$

Also the electric potential of the black hole is

$$\Phi = \left. \frac{\partial M}{\partial Q} \right|_{r=r_+} = -\pi Q \ln r_+. \quad (15)$$

The generalized Cardy formula (hereafter named Cardy–Verlinde formula) is given by

$$S_{\text{SFT}} = \frac{2\pi R}{\sqrt{ab}} \sqrt{E_C(2E - E_C)}, \quad (16)$$

where  $E$  is the total energy and  $E_C$  is the Casimir energy. The definition of the Casimir energy is derived by the violation of the Euler relation

$$E_C = n(E + PV - TS - \Phi Q - \Omega_+ J), \quad (17)$$

where the pressure of the CFT is defined as  $P = E/nV$ . The total energy may be written as the sum of two terms

$$E = E_E + \frac{1}{2} E_C, \quad (18)$$

where  $E_E$  is the purely extensive part of the total energy  $E$ . The Casimir energy  $E_C$  as well as the purely extensive part of energy  $E_E$  expressed in terms of the radius  $R$  and the entropy  $S$  are written as

$$E_C = \frac{b}{2\pi R}, \quad (19)$$

$$E_E = \frac{a}{4\pi R} S^2. \quad (20)$$

### 3. Entropy of charged rotating BTZ black hole in Cardy–Verlinde formula

The Casimir energy  $E_C$ , defined as Eq. (17), and  $n = 1$  in this case, is found to be

$$E_C = \frac{1}{2} \left( \frac{J^2}{r_+^2} + \pi Q^2 \right). \quad (21)$$

Additionally, it is obvious that

$$2E - E_C = \frac{2r_+^2}{l^2} - \pi Q^2 \left( \ln r_+ + \frac{1}{2} \right). \quad (22)$$

The purely extensive part of the total energy  $E_E$  by substituting Eq. (22) in Eq. (18), is given as

$$E_E = \frac{r_+^2}{l^2} - \frac{\pi}{2} Q^2 \left( \ln r_+ + \frac{1}{2} \right) \quad (23)$$

whilst by substituting Eq. (14) in Eq. (20), it takes the form

$$E_E = \frac{4\pi a}{R} r_+^2. \quad (24)$$

Making use of expression (19), Casimir energy  $E_C$  can also be written as

$$E_C = \frac{b}{2\pi R}. \quad (25)$$

At this point it is useful to evaluate the radius  $R$ . By equating the right-hand sides of (21) and (25), the radius is written as

$$R = \frac{b}{\pi \left( \frac{l^2}{r_+^2} + \pi Q^2 \right)} \quad (26)$$

while by equating the right-hand sides (23) and (24) it can also be written as

$$R = \frac{4\pi a r_+^2 l^2}{r_+^2 - \frac{\pi}{2} Q^2 l^2 (\ln r_+ + \frac{1}{2})}. \quad (27)$$

Therefore, the radius expressed in terms of the arbitrary positive coefficients  $a$  and  $b$  is

$$R = \frac{2r_+ l \sqrt{ab}}{\sqrt{\left( \frac{l^2}{r_+^2} + \pi Q^2 \right) \left( r_+^2 - \frac{\pi}{2} Q^2 l^2 (\ln r_+ + \frac{1}{2}) \right)}}. \quad (28)$$

Finally, we substitute expressions (21), (22) and (28) which were derived in the context of thermodynamics of the charged rotating BTZ black hole, in the Cardy–Verlinde formula (16) which in turn was derived in the context of CFT, and we get

$$S_{\text{CFT}} = S_{\text{BH}}. \quad (29)$$

It has been proven that the entropy of the charged rotating BTZ black hole can be expressed in the form of Cardy–Verlinde formula.

#### 4. Conclusion

The Cardy–Verlinde formula proposed by Verlinde [9], relates the entropy of a certain CFT to its total energy and Casimir energy in arbitrary dimensions, which is shown to hold for topological Reissner–Nordstrom [22] and topological Kerr–Newman [23] black holes in de Sitter spaces, Taub–Bolt–AdS<sub>4</sub> [24], Kerr–(A)dS [25]. There are many other relevant papers on the subject [26–28]. Thus, one may naively expect that the entropy of all CFTs that have an AdS-dual description is given as the form (16). However, AdS black holes do not always satisfy the Cardy–Verlinde formula [29]. For systems that admit 2D CFTs as duals, the Cardy formula [30] can be

applied directly. This formula gives the entropy of a CFT in terms of the central charge  $c$  and the eigenvalue of the Virasoro operator  $l_0$ . However, it should be pointed out that this evaluation is possible as soon as one has explicitly shown (e.g. using the AdS<sub>d</sub>/CFT<sub>d-1</sub> correspondence) that the system under consideration is in correspondence with a 2D CFT [6,7]. The aim of this Letter is to further investigate the AdS/CFT correspondence in terms of Cardy–Verlinde entropy formula. In this Letter, we have shown that the entropy of the black hole horizon of charged rotating BTZ spacetime can also be rewritten in the form of Cardy–Verlinde formula.

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