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On the numerical solution of nonlinear Black–Scholes equations

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Abstract

Nonlinear Black–Scholes equations have been increasingly attracting interest over the last two decades, since they provide more accurate values by taking into account more realistic assumptions, such as transaction costs, risks from an unprotected portfolio, large investor's preferences or illiquid markets (which may have an impact on the stock price), the volatility, the drift and the option price itself.

In this paper we will focus on several models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a volatility depending on different factors, such as the stock price, the time, the option price and its derivatives due to transaction costs. We will analytically approach the option price by transforming the problem for a European Call option into a convection-diffusion equation with a nonlinear term and the free boundary problem for an American Call option into a fully nonlinear nonlocal parabolic equation defined on a fixed domain following Ševčovič's idea. Finally, we will present the results of different numerical discretization schemes for European options for various volatility models including the Leland model, the Barles and Soner model and the Risk adjusted pricing methodology model.

Keywords: Nonlinear Black–Scholes equation; American and European options; Transaction costs; Finite difference schemes

1. Introduction

The interest in pricing financial derivatives – including pricing options – arises from the fact that financial derivatives can be used to minimize losses caused by price fluctuations of the underlying assets. This process of protection is called *hedging*. There is a variety of financial products on the market, such as futures, forwards, swaps and options. In this paper we will concentrate on European and American Call and Put options.

We recall that a *European Call* option is a contract where at a prescribed time in the future, known as the *expiry date* T, the owner of the option, known as the *holder*, may purchase a prescribed asset, known as the *underlying asset* $S(t)$, for a prescribed amount, known as the exercise or *strike price* K. The opposite party, or the *writer*, has the obligation to sell the asset if the holder chooses to excercise his right. Therefore, the value of the option at expiry, known as the *pay-off function*, is $V(S,T) = (S-K)^+$. Reciprocally, a *European Put* option is the right to sell the asset with the

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pay-off function $V(S,T) = (K-S)^+$ (see e.g. [1]). While European options can only be exercised in T, *American options* can be exercised at any time until the expiration, which complicates their pricing process significantly.

Option pricing theory has made a great leap forward since the development of the Black–Scholes option pricing model by Fischer Black and Myron Scholes in [2] in 1973 and previously by Robert Merton in [3]. The solution of the famous *(linear) Black–Scholes equation*

$$0 = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV, \tag{1}$$

where $S := S(t) > 0$ and $t \in (0,T)$, provides both the price for a European option and a hedging portfolio that replicates the option assuming that (see [4]):

- The price of the asset price or underlying derivative $S(t)$ follows a Geometric Brownian motion $W(t)$, meaning that S satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t).$$

- The *trend* or *drift* μ (measures the average rate of growth of the asset price), the *volatility* σ (measures the standard deviation of the returns) and the riskless interest rate r are constant for $0 \le t \le T$ and no dividends are paid in that time period.
- The market is *frictionless*, thus there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and any size. That is, all variables are perfectly divisible and may take any real number. Moreover, individual trading will not influence the price.
- There are no *arbitrage* opportunities, meaning that there are no opportunities of instantly making a risk-free profit.

Under these assumptions the market is *complete*, which means that any asset can be replicated with a portfolio of other assets in the market (see [5]). Then, the linear Black–Scholes equation (1) can be transformed into the heat equation and analytically solved to price the option [1].

One can argue that these restrictive assumptions never occur in reality. Due to transaction costs (see [6–8]), large investor preferences (see [9–11]) and incomplete markets [12] they are likely to become unrealistic and the classical model results in strongly or fully nonlinear, possibly degenerate, parabolic diffusion-convection equations, where both the volatility σ and the drift μ can depend on the time t, the stock price S or the derivatives of the option price V itself. In this paper we will be concerned with several transaction cost models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a constant drift μ and a nonconstant volatility $\widetilde{\sigma}^2 := \widetilde{\sigma}^2(t, S, V_S, V_{SS})$. Under these circumstances (1) becomes the following *nonlinear Black–Scholes equation*, which we will consider for European options:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + rSV_S - rV, \tag{2}$$

where $S > 0$ and $t \in (0,T)$.

Studying (1) for an American Call option would be redundant, since the value of an American Call option equals the value of a European Call option if no dividends are paid and the volatility is constant (for details see [13]). In order to make the model more realistic, we will consider a modification of (2) for American options, where S pays out a dividend $qSdt$ in a time step dt:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + (r-q)SV_S - rV, \tag{3}$$

where $S > 0, t \in (0,T)$ and the dividend yield q is constant.

2. Volatility models with transaction costs

The Black–Scholes model requires a continuous portfolio adjustment in order to hedge the position without any risk. In the presence of transaction costs it is likely that this adjustment easily becomes expensive, since an infinite number of transactions is needed [14]. Thus, the hedger needs to find the balance between the transaction costs that

are required to rebalance the portfolio and the implied costs of hedging errors. As a result to this "imperfect" hedging, the option might be over- or under-priced up to the extent where the riskless profit obtained by the arbitrageur is offset by the transaction costs, so that there is no single equilibrium price but a range of feasible prices. It has been shown that in a market with transaction costs, there is no replicating portfolio for the European Call option and the portfolio is required to dominate rather than replicate the value of the option (see [8]). Soner, Shreve and Cvitanič prove in [15] that the minimal hedging portfolio that dominates a European Call is the trivial one (hence holding one share of the stock that the Call is written on), so that efforts have been made to find an alternate relaxation of the hedging conditions to better replicate the payoffs of derivative securities.

2.1. Leland's model

Leland's idea of relaxing the hedging conditions is to trade at discrete times [6], which promises to reduce the expenses of the portfolio adjustment. He assumes that the transaction cost, $\frac{\kappa}{2}|\Delta|S$, is proportional to the monetary value of the assets bought or sold. Here, κ denotes the round trip transaction cost per unit dollar of the transaction and Δ the number of assets bought ($\Delta > 0$) or sold ($\Delta < 0$) at price S. Leland then deduces that the option price is the solution of the nonlinear Black–Scholes equation (2) with the *modified volatility*

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + Le \operatorname{sign}(V_{SS})\right), \tag{4}$$

where σ represents the original volatility and Le the *Leland number* given by

$$Le = \sqrt{\frac{2}{\pi}} \frac{\kappa}{\sigma\sqrt{\delta t}},$$

where δt denotes the transaction frequency (interval between successive revisions of the portfolio). It follows from (4) are that the more frequent the rebalancing (δt smaller), the higher the transaction cost and the greater the value of V.

It is known that $V_{SS} > 0$ for European Puts and Calls in the absence of transaction costs. Assuming the same behaviour in the presence of transaction costs, the Eq. (2) becomes linear with an adjusted constant volatility $\widetilde{\sigma}^2 = \sigma^2(1 + Le) > \sigma^2$.

Several authors (e.g. Hoggard et al. in [16], Parás and Avellaneda in [17]) discuss Leland's model for general pay-off functions dropping the assumption of the convexity of the resulting option price. From the binomial model making use of the algorithm of Bensaid et al. (see [18]), Parás and Avellaneda derive the same volatility (4) as Leland, and state that in case $V_{SS} < 0$ and $Le > 1$ the problem (2) becomes mathematically ill-posed and has no solution for general pay-off functions. For the case $V_{SS} > 0$ and $Le > 1$ they propose several hedging strategies.

In [7] Boyle and Vorst derive from the binomial model that as the time step δt and the transaction cost κ tend to zero, the price of the discrete option converges to a Black–Scholes price with the modified volatility of the form

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + Le\sqrt{\frac{\pi}{2}} \operatorname{sign}(V_{SS})\right). \tag{5}$$

Just like Leland, Boyle and Vorst assume convexity of V, so that $\widetilde{\sigma}^2 = \sigma^2(1 + Le\sqrt{\pi/2})$ and (2) turns into a linear equation.

2.2. Barles' and Soner's model

Barles and Soner derived a more complicated model by following the utility function approach of Hodges and Neuberger [19], that was further developed by Davis et al. in [20]. They use an exponential utility function and prove – using the theory of stochastic optimal control [21] – that as ε and κ go to 0, V is the unique (viscosity) solution of (2) where

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + \Psi(\mathrm{e}^{r(T-t)} a^2 S^2 V_{SS})\right), \tag{6}$$

with $a = \kappa/\sqrt{\varepsilon}$ and $\Psi(x)$ denotes the solution to the following nonlinear ordinary differential equation

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\,\Psi(x)} - x}, \quad x \neq 0, \tag{7a}$$

with the initial condition

$$\Psi(0) = 0. \tag{7b}$$

The analysis of this ordinary differential equation by Barles and Soner in [8] implies that

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \Psi(x) = -1. \tag{8}$$

This property allows the treatement of the function $\Psi(\cdot)$ as the identity for large arguments and therefore the volatility becomes

$$\widetilde{\sigma}^2 = \sigma^2(1 + \mathrm{e}^{r(T-t)}a^2S^2V_{SS}). \tag{9}$$

2.3. Risk adjusted pricing methodology

In this model, proposed by Kratka and improved by Jandačka and Ševčovič in [22], the optimal time-lag δt between the transactions is found to minimize the sum of the rate of the transaction costs and the rate of the risk from an unprotected portfolio. That way the portfolio is still well protected and the modified volatility is now of the form

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + 3\left(\frac{C^2M}{2\pi}SV_{SS}\right)^{\frac{1}{3}}\right), \tag{10}$$

where $M \geq 0$ is the transaction cost measure and $C \geq 0$ the risk premium measure.

Note that these nonlinear models are all consistent with the linear model if the additional parameters for transaction costs vanish (Le, $\Psi(\cdot)$, M). We will study these models – more precisely Eqs. (2) and (3) where the volatility is given by the Eqs. (4), (6), (9) and (10) – for both European and American Call options. The European Call option is the solution to (2) on $0 \leq S < \infty$, $0 \leq t \leq T$ with the following terminal and boundary conditions:

$$\begin{aligned} &V(S,T) = (S-K)^+ \quad \text{for } 0 \leq S < \infty \\ &V(0,t) = 0 \quad \text{for } 0 \leq t \leq T \\ &V(S,t) \sim S - K\mathrm{e}^{-r(T-t)} \quad \text{as } S \to \infty. \end{aligned} \tag{11}$$

For the American Call option the 'spatial' domain is divided into two regions by the free boundary $S_f(t)$, the *stopping region* $S_f(t) < S < \infty$, $0 \leq t \leq T$, where the option is exercised or dead with $V(S,t) = S - K$ and the *continuation region* $0 \leq S \leq S_f(t)$, $0 \leq t \leq T$, where the option stays alive and (2) is valid under the following terminal and boundary conditions (see e.g. [13]):

$$\begin{aligned} &V(S,T) = (S-K)^+ \quad \text{for } 0 \leq S \leq S_f(T) \\ &V(0,t) = 0 \quad \text{for } 0 \leq t \leq T \\ &V(S_f(t),t) = S_f(t) - K \quad \text{for } 0 \leq t \leq T \\ &V_S(S_f(t),t) = 1 \quad \text{for } 0 \leq t \leq T \\ &S_f(T) = \max(K, rK/q). \end{aligned} \tag{12}$$

The existence of a viscosity solution to (2) for European options with the volatility given by (6) has been proved by Barles and Soner in [8]. However, in general an exact analytical solution leading to a closed expression is not known neither for European nor for American options in a market with transaction costs.

The focus of this paper is the numerical solution of the problem. This is achieved by initially analytically approaching the solution for the European Call by transforming (2) with (11) into a forward-in-time parabolic problem. In the section thereafter both a classical and a compact finite difference scheme will be specified and used to solve the transformed problem. Finally, different volatility models will be compared to each other.

The numerical solution and the comparison study for American options will be discussed in detail in the thesis of the first author, though in this work restricted to the transformation of the free boundary problem (3) with (12) into

a parabolic equation defined on a fixed spatial domain. This new problem will be numerically solved and evaluated in [23].

3. Analytical solution

3.1. Transformation of the European Call option

In order to be able to solve the problem (2) subject to (11) numerically, we perform a variable transformation (see e.g. [1,24]):

$$x = \ln\left(\frac{S}{K}\right), \qquad \tau = \frac{1}{2}\sigma^2(T-t) \qquad u(x,\tau) = \mathrm{e}^{-x}\frac{V(S,t)}{K}.$$

Differentiation yields:

$$\begin{aligned} V_t &= u_\tau \tau_t S = -\frac{1}{2}\sigma^2 S u_\tau, \\ V_S &= u_x x_S S + u = u_x + u, \\ V_{SS} &= u_{xx}x_S + u_x x_S = \frac{1}{S}(u_{xx} + u_x). \end{aligned}$$

Plugging these derivatives into (2) leads to

$$0 = -\frac{1}{2}\sigma^2 S u_\tau + \frac{1}{2}\widetilde{\sigma}^2 S(u_{xx} + u_x) + rS(u_x + u) - ruS,$$

and a final multiplication by $-\frac{2}{S\sigma^2}$ gives

$$0 = u_\tau - \frac{\widetilde{\sigma}^2}{\sigma^2}(u_{xx} + u_x) - Du_x, \tag{13}$$

where $D = \frac{2r}{\sigma^2}$ and $\widetilde{\sigma}^2$ depends on the volatility model, $x \in \mathbb{R}$ and $0 \le \tau \le \widetilde{T} = \frac{\sigma^2 T}{2}$. Model (4) becomes

$$\widetilde{\sigma}^2 = \sigma^2(1 + Le\,\mathrm{sign}(u_{xx} + u_x)), \tag{14a}$$

model (6)

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + \Psi\left(\mathrm{e}^{\frac{2r\tau}{\sigma^2}}a^2 K\mathrm{e}^x(u_{xx} + u_x)\right)\right), \tag{14b}$$

model (9)

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + \mathrm{e}^{\frac{2r\tau}{\sigma^2}}a^2 K\mathrm{e}^x(u_{xx} + u_x)\right) \tag{14c}$$

and model (10)

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + 3\left(\frac{C^2 M}{2\pi}(u_{xx} + u_x)\right)^{\frac{1}{3}}\right). \tag{14d}$$

Now $u(x,\tau)$ solves (13) on the transformed domain $x \in \mathbb{R}$, $0 \le \tau \le \widetilde{T}$ subject to the following initial and boundary conditions resulting from (11):

$$\begin{aligned} &u(x,0) = (1 - \mathrm{e}^{-x})^+ \quad \text{for } x \in \mathbb{R}, \\ &u(x,\tau) = 0 \quad \text{as } x \to -\infty, \\ &u(x,\tau) \sim 1 - \mathrm{e}^{-D\tau - x} \quad \text{as } x \to \infty. \end{aligned} \tag{15}$$

3.2. Transformation of the American Call option

The purpose of converting the free-boundary problem for the nonlinear Black–Scholes equation (3) subject to (12) into a quasilinear parabolic equation defined on a fixed domain is the minimization of the error resulting from the discontinuity of V_{SS}, which is achieved by only considering the domain where (3) holds. Following the idea of Ševčovič in [25] we change the variables to:

$$\tau = T - t, \quad x = \ln\left(\frac{\varrho(\tau)}{S}\right) \Leftrightarrow S = \mathrm{e}^{-x}\varrho(\tau), \quad \varrho(\tau) = S_f(T-\tau)$$

and construct a portfolio $\Pi(x,\tau) = V(S,t) - SV_S(S,t)$ by buying $\Delta = V_S$ shares S and selling an option V. Differentiating Π with respect to x and τ gives us

$$\Pi_x = V_S S_x - S_x V_S - SV_{SS}S_x = S^2 V_{SS}$$

and

$$\begin{aligned}\Pi_\tau &= V_S S_\tau + V_t t_\tau - S_\tau V_S - S(V_{SS}S_\tau + V_{St}t_\tau)\\ &= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)}S^2 V_{SS} + SV_{St}\\ &= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)}\Pi_x - S\partial_S(-V_t).\end{aligned} \tag{16}$$

Substituting

$$-V_t = \frac{\widetilde{\sigma}^2}{2}S^2 V_{SS} - r(V - SV_S) - qSV_S = \frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi - qSV_S$$

from (3) into (16) and using the fact that $-S\partial S = \partial x$, we get

$$\begin{aligned}\Pi_\tau &= \frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi - qSV_S - \frac{\varrho'(\tau)}{\varrho(\tau)}\Pi_x + \partial_x\left(\frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi\right) + S\partial_S(qSV_S)\\ &= \left(\frac{\widetilde{\sigma}^2}{2} - \frac{\varrho'(\tau)}{\varrho(\tau)} - r + q\right)\Pi_x - r\Pi + \frac{1}{2}\partial_x(\widetilde{\sigma}^2\Pi_x).\end{aligned}$$

We therefore obtain

$$0 = \Pi_\tau + \left(\frac{\varrho'}{\varrho}(\tau) + r - q - \frac{\widetilde{\sigma}^2}{2}\right)\Pi_x - \frac{1}{2}\partial_x(\sigma^2\Pi_x) + r\Pi, \tag{17}$$

defined on $x \in \mathbb{R}^+$, $0 \le \tau \le T$. The terminal condition from (12) in the original variables (S,T) becomes the intitial condition in the new variables $(x,0)$

$$\Pi(x,0) = V(S,T) - SV_S(S,T) = \begin{cases} -K & \text{for } S > K \Leftrightarrow x < \ln\dfrac{\varrho(0)}{K} \\ 0 & \text{otherwise}\end{cases} \tag{18a}$$

and the boundary conditions from (12) transform to

$$\Pi(x,\tau) = 0 \quad \text{as } x \to \infty, 0 \le \tau \le T \tag{18b}$$

$$\Pi(0,\tau) = -K \quad \text{for } 0 \le \tau \le T. \tag{18c}$$

To complete the system of equations that enables the computation of the portfolio Π we need to use the last two conditions of (12) to obtain an expression at the free boundary position $\varrho(\tau)$. Differentiating and evalutating V at the free boundary gives us

$$V_S(S_f(t),t)S_f'(t) + V_t(S_f(t),t) = S_f'(t).$$

Using (12), we conclude that

$$V_t(S_f(t), t) = 0 \quad \text{for } 0 \le \tau \le T.$$

Computing (3) at the point $(S_f(t), t)$ or at $(0, \tau)$ in the transformed variables yields:

$$\begin{aligned} 0 &= V_t(S_f(t), t) + \frac{1}{2}\widetilde{\sigma}^2 \Pi_x(0, \tau) + (r - q)S_f(t)V_S(S_f(t), t) - rV(S_f(t), t) \\ &= \frac{1}{2}\widetilde{\sigma}^2 \Pi_x(0, \tau) + rK - q\varrho(\tau). \end{aligned}$$

Assuming that $r \ge q$ for simplicity, we get the last condition:

$$\varrho(\tau) = \frac{1}{2q}\widetilde{\sigma}^2 \Pi_x(0, \tau) + \frac{rK}{q} \quad \text{with } \varrho(0) = \frac{rK}{q}, \tag{18d}$$

where $0 \le \tau \le T$ and $\widetilde{\sigma}^2$ depends on the volatility model we choose. The volatility (4) from the Leland model becomes

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + Le \operatorname{sign}(\Pi_x)\right), \tag{19a}$$

for (6) we get

$$\widetilde{\sigma}^2 = \sigma^2(1 + \Psi(\mathrm{e}^{r\tau} a^2 \Pi_x)), \tag{19b}$$

for (9) we obtain

$$\widetilde{\sigma}^2 = \sigma^2(1 + \mathrm{e}^{r\tau} a^2 \Pi_x) \tag{19c}$$

and for (10)

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + 3\left(\frac{C^2 M}{2\pi} \Pi_x \varrho(\tau) \mathrm{e}^{-x}\right)^{\frac{1}{3}}\right). \tag{19d}$$

This transformed problem (17) subject to (18) with the corresponding volatilities (19) is solved by the split step finite-difference method proposed by Ševčovič in [25] and elaborated on in [23]. The solution in the European case is specified below.

4. Numerical solution

4.1. Finite-difference schemes

There are several numerical methods of solving (13) subject to (15). This work's emphasis is on the finite-difference schemes, thus other methods will not be mentioned here.

To apply finite-difference schemes to the transformed problem (13) subject to the conditions (15) with the corresponding volatilities (14) we start by replacing $x \in \mathbb{R}$ and $\tau \in [0, \widetilde{T}]$ by a bounded inverval $x \in [-R, R]$, $R > 0$. We discretize the new computational domain by a uniform grid (x_i, τ_n) with $x_i = ih$ and $\tau_n = nk$, where h is the space step, k is the time step, $i \in [-N, N]$, $-R = -Nh$, $R = Nh$, $n \in [0, M]$ and $\widetilde{T} = Mk$. We denote the approximate solution of (13) in x_i at time τ_n by $U_i^n \approx u(x_i, \tau_n)$ and treat the initial and boundary conditions (15) in the following way:

$$\begin{aligned} &U_i^0 = (1 - \mathrm{e}^{-ih})^+, \\ &U_{-N}^n = 0, \\ &U_N^n = 1 - \mathrm{e}^{-Dnk - Nh}. \end{aligned} \tag{20}$$

For a more appropriate treatment of the boundary conditions so-called *artificial boundary conditions* [26] can be introduced to limit the unbounded spatial domain of (13). We bear in mind that we say a scheme is of order (m, n) if its truncation error is of order $\mathcal{O}(k^m + h^n)$.

To discretize (13) we introduce the following notation for the forward difference quotient with the spatial step size h:

$$D_h^+ U_i^n := \frac{U_{i+1}^n - U_i^n}{h} \approx u_x(x_i, \tau_n),$$

where we leave out the error term $\mathcal{O}(h)$. Similarly, the backward difference quotient with respect to the spatial variable is denoted as

$$D_h^- U_i^n := \frac{U_i^n - U_{i-1}^n}{h} \approx u_x(x_i, \tau_n)$$

and the central difference quotient as

$$D_h^0 U_i^n := \frac{U_{i+1}^n - U_{i-1}^n}{2h} \approx u_x(x_i, \tau_n).$$

For the second spatial derivative we introduce the standard difference quotient

$$D_h^2 U_i^n := \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} \approx u_{xx}(x_i, \tau_n)$$

with the error term $\mathcal{O}(h^2)$. Note that central differences in the time variable are never used in practice because they always lead to bad numerical schemes, that are inherently unstable (see [1]).

Most of the resulting schemes lead to systems of equations that can be written in matrix form:

$$A^n U^{n+1} = B^n U^n + d^n, \tag{21}$$

where

$$U^n = \begin{pmatrix} U_{-N+1}^n \\ \vdots \\ U_0^n \\ \vdots \\ U_{N-1}^n \end{pmatrix} \in \mathbb{R}^{2N-1}, \quad A^n = \begin{pmatrix} a_0 & a_1 & 0 & \cdots & 0 \\ a_{-1} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & a_{-1} & a_0 \end{pmatrix} \in \mathbb{R}^{(2N-1)\times(2N-1)},$$

$$B^n = \begin{pmatrix} b_0 & b_1 & 0 & \cdots & 0 \\ b_{-1} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_1 \\ 0 & \cdots & 0 & b_{-1} & b_0 \end{pmatrix} \in \mathbb{R}^{(2N-1)\times(2N-1)}$$

and

$$d^n = \begin{pmatrix} b_{-1}U_{-N}^n - a_{-1}U_{-N}^{n+1} \\ 0 \\ \vdots \\ 0 \\ b_1 U_N^n - a_1 U_N^{n+1} \end{pmatrix} \in \mathbb{R}^{2N-1}.$$

The vector d^n can be calculated with the boundary conditions (20) and the matrices A^n and B^n are triagonal, so that the resulting systems can be solved with linear effort $\mathcal{O}(N)$ using the Thomas algorithm [27]. We further suppose that

$$\sum_{i=-1}^{1} a_i = \sum_{i=-1}^{1} b_i = 1,$$

which is satisfied by any consistent scheme after normalization of the coefficients (cf. [28]).

There are different ways of treating the volatility. Düring suggests in [29] a smoother approximation of u_{xx} for the nonlinear part by choosing:

$$u_{xx}(x_i, \tau_n) \approx \frac{U_{i+2}^n - 2U_i^n + U_{i-2}^n}{4h^2} := D_{2h}^2 U_i^n$$

with the truncation error of $\mathcal{O}(h^2)$. We will treat the nonlinearity explicitly in all the schemes and denote the volatility correction for Leland's model with the volatility (14a) by

$$s_i^n = \sqrt{\frac{2}{\pi}} \frac{\kappa}{\sigma\sqrt{\delta t}} \operatorname{sign}\left(D_{2h}^2 U_i^n + D_h^0 U_i^n\right), \tag{22a}$$

the volatility correction for Barles' and Soner's model with the volatility (6) by

$$s_i^n = \Psi\left(\mathrm{e}^{D\tau_n + x_i} a^2 K (D_{2h}^2 U_i^n + D_h^0 U_i^n)\right), \tag{22b}$$

the volatility correction in case of treating $\Psi(\cdot)$ as the identity with the volatility (9) by

$$s_i^n = \mathrm{e}^{D\tau_n + x_i} a^2 K\left(D_{2h}^2 U_i^n + D_h^0 U_i^n\right), \tag{22c}$$

and the volatility correction for the Risk Adjusted Pricing Methodolody with the volatility (10) by

$$s_i^n = 3\left(\frac{C^2 M}{2\pi}(D_{2h}^2 U_i^n + D_h^0 U_i^n)\right)^{\frac{1}{3}}. \tag{22d}$$

One problem with s_i^n is the calculation at the boundary, since theoretically we need $U^n \in \mathbb{R}^{2N+3}$ to be able to calculate s_{N-1}^n and s_{-N+1}^n. This calculation involves U_{-N-1}^n and U_{N+1}^n, which are outside the computational domain. Düring states in [29] that the influence of the nonlinearity at the boundary is not significant and can be therefore neglected for large R. We will assume that

$$U_{-N-1}^n = 0 \text{ and } U_{N+1}^n = 1 - \mathrm{e}^{-Dnk-(N+1)h} \tag{23}$$

for these ghost or *auxiliary values* (see [30]) and hence denote

$$s^n = \left(s_{-N+1}^n, \quad \ldots, \quad s_0^n, \quad \ldots, \quad s_{N-1}^n\right)^\top \in \mathbb{R}^{2N-1}.$$

The ordinary differential Eq. (7) is solved with the `ode45` function in MATLAB, which is based on an explicit Runge–Kutta (4, 5) one-step solver, the Dormand–Prince pair [31]. The values between the calculated values for s^n are obtained by a cubic spline interpolation (see Fig. 1).

In the following we introduce both a *classical* and a *compact finite-difference scheme* and present the numerical results.

4.1.1. Crank–Nicolson method

This classical finite-difference scheme computes the solution better than the forward and backward difference methods due to its superior order of (2, 2) (cf. [4,30]). We approximate the second spatial derivative by $D_h^2 U_i^n$ and $D_h^2 U_i^{n+1}$ except in the nonlinear volatility term s_i^n. Bringing (13) into the form of a convection-diffusion equation with a nonlinear term

$$u_\tau = s_i^n(u_{xx} + u_x) + (1 + D)u_x + u_{xx}, \tag{24}$$

where s_i^n is (22) depending on the model, and replacing all the derivatives in (24) by their corresponding finite-difference quotients we get:

$$\begin{aligned} D_k^+ U_i^n + D_k^- U_i^{n+1} = {} & s_i^n\left(D_h^2 U_i^n + D_h^0 U_i^n\right) + s_i^n\left(D_h^2 U_i^{n+1} + D_h^0 U_i^{n+1}\right) \\ & + (1 + D)\left(D_h^0 U_i^n + D_h^0 U_i^{n+1}\right) + D_h^2 U_i^n + D_h^2 U_i^{n+1}. \end{aligned} \tag{25}$$

This is equivalent to

$$\frac{U_i^{n+1}-U_i^n}{k} = \frac{s_i^n}{2}\left(\frac{U_{i+1}^n-2U_i^n+U_{i-1}^n}{h^2}+\frac{U_{i+1}^n-U_{i-1}^n}{2h}\right) + \frac{s_i^n}{2}\left(\frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{h^2}+\frac{U_{i+1}^{n+1}-U_{i-1}^{n+1}}{2h}\right)+(1+D)\frac{U_{i+1}^n-U_{i-1}^n+U_{i+1}^{n+1}-U_{i-1}^{n+1}}{4h} + \frac{U_{i+1}^n-2U_i^n+U_{i-1}^n+U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{2h^2}.$$

Rearranging leads to the linear system (21) with the following coefficients:

$$\begin{aligned}
a_{-1} &= s_i^n\left(-\frac{r}{2}+\frac{\mu}{4}\right)-\frac{r}{2}-\frac{\lambda\mu}{4},\\
a_0 &= 1+r(1+s_i^n),\\
a_1 &= s_i^n\left(-\frac{r}{2}-\frac{\mu}{4}\right)-\frac{r}{2}+\frac{\lambda\mu}{4},\\
b_{-1} &= s_i^n\left(\frac{r}{2}-\frac{\mu}{4}\right)+\frac{r}{2}+\frac{\lambda\mu}{4},\\
b_0 &= 1-r(1+s_i^n),\\
b_1 &= s_i^n\left(\frac{r}{2}+\frac{\mu}{4}\right)+\frac{r}{2}-\frac{\lambda\mu}{4},
\end{aligned}$$

where

$$\lambda=-(1+D),\qquad \alpha=\frac{\lambda h}{2},\qquad r=\frac{k}{h^2},\qquad \mu=\frac{k}{h}.$$

The Crank–Nicolson scheme is unconditionally stable in the linear case [30].

4.1.2. Rigal compact schemes

In [28] Rigal develops two-level three-point finite difference schemes of order $(2,4)$ that are stable and non-oscillatory and give more efficient and accurate results than implicit fourth-order schemes. Düring follows Rigal's ideas and generalizes his results for a nonlinear Black–Scholes equation in [29]. A general two-level three-point scheme for the problem (24) can be written as:

$$D_k^+U_i^n = (1+s_i^n)\left(\left(\frac{1}{2}+A_1\right)D_h^2U_i^n+\left(\frac{1}{2}+B_1\right)D_h^0U_i^n\right)+(1+s_i^n)\left(\left(\frac{1}{2}+A_2\right)D_h^2U_i^{n+1} + \left(\frac{1}{2}+B_2\right)D_h^0U_i^{n+1}\right)+D\left(\frac{1}{2}+B_1\right)D_h^0U_i^n+D\left(\frac{1}{2}+B_2\right)D_h^0U_i^{n+1}, \tag{26}$$

where A_1, A_2, B_1 and B_2 are real constants which should be chosen in such a way that they eliminate the lower order terms in the truncation error. Note, that if these constants are equal to zero, then (26) reduces to the classical Crank–Nicolson scheme (25) of order (2, 2). If we choose

$$\begin{aligned}
B_1 &= \frac{1+4r^2\alpha^2}{12\beta r},\\
B_2 &= -\frac{1+4r^2\alpha^2}{12\beta r},\\
A_1 &= -\frac{1}{12k\beta}(-2h^2+6\widetilde{\lambda}^2k^2B_2-k^2\widetilde{\lambda}^2-12k\beta^2B_2),\\
A_2 &= -\frac{1}{12k\beta}(2h^2+6\widetilde{\lambda}^2k^2B_2+k^2\widetilde{\lambda}^2+12k\beta^2B_2),
\end{aligned}$$

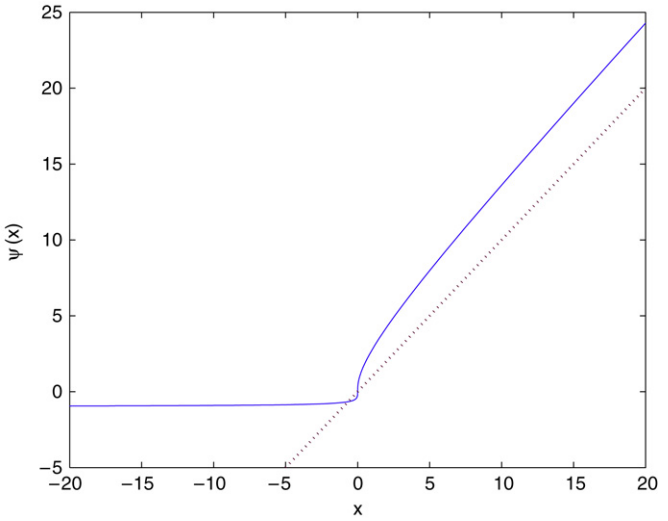


Fig. 1. Solution to Eq. (7) using `ode45` (solid line) and the identity function (dotted line).

with $\beta := 1 + s_i^n$ and $\widetilde{\lambda} := -(1 + s_i^n + D)$, plug into the Eq. (26) and rearrange the U_i^ns, then our coefficients become

$$
\begin{aligned}
a_{-1} &= -\frac{12r\beta^2 - 2\beta + r\widetilde{\lambda}^2h^2 + r^3\widetilde{\lambda}^4h^4 + 6r\widetilde{\lambda}h\beta - \widetilde{\lambda}h - r^2\widetilde{\lambda}^3h^3}{24\beta},\\
a_0 &= \frac{10\beta + 12r\beta^2 + r\widetilde{\lambda}^2h^2 + r^3\widetilde{\lambda}^4h^4}{12\beta},\\
a_1 &= -\frac{12r\beta^2 - 2\beta + r\widetilde{\lambda}^2h^2 + r^3\widetilde{\lambda}^4h^4 - 6r\widetilde{\lambda}h\beta + \widetilde{\lambda}h + r^2\widetilde{\lambda}^3h^3}{24\beta},\\
b_{-1} &= \frac{12r\beta^2 + 2\beta + r\widetilde{\lambda}^2h^2 + r^3\widetilde{\lambda}^4h^4 + 6r\widetilde{\lambda}h\beta + \widetilde{\lambda}h + r^2\widetilde{\lambda}^3h^3}{24\beta},\\
b_0 &= \frac{-10\beta + 12r\beta^2 + r\widetilde{\lambda}^2h^2 + r^3\widetilde{\lambda}^4h^4}{12\beta},\\
b_1 &= \frac{12r\beta^2 + 2\beta + r\widetilde{\lambda}^2h^2 + r^3\widetilde{\lambda}^4h^4 - 6r\widetilde{\lambda}h\beta - \widetilde{\lambda}h - r^2\widetilde{\lambda}^3h^3}{24\beta}.
\end{aligned}
$$

This scheme is known as the R3C scheme [29]. Note that if $\beta = 1$ or $s_i^n = 0$ this scheme reduces to the R3B scheme developed by Rigal [28], which is also unconditionally stable and non-oscillatory in the linear case.

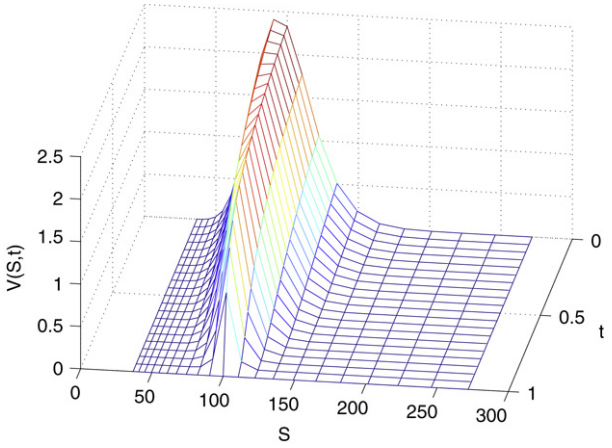
4.2. Comparison study

In this part we compare the different transaction cost models to the model without transaction costs and to each other. The influence of transaction costs modelled by the volatilities (4), (6), (9) and (10) and computed with the Crank–Nicolson finite-difference scheme can be seen in Fig. 2. We plot the difference $V_{\text{nonlinear}}(S,t) - V_{\text{linear}}(S,t)$ between the price of the European Call option with transaction costs and the price of the European Call without transaction costs. As expected the numerical results indicate an economically significant price deviation between the standard (linear) Black–Scholes model and the nonlinear models.

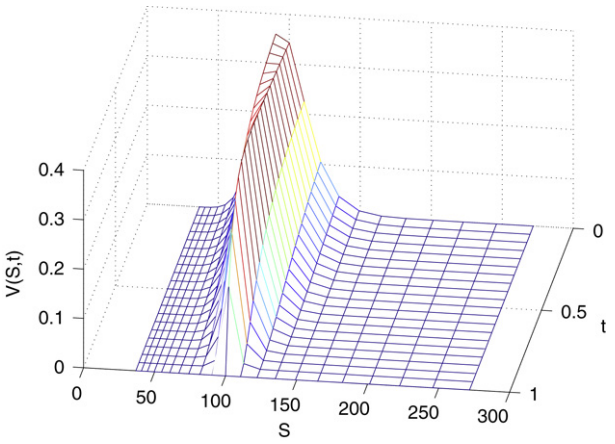
For all calculations we used the following parameters:

$$
\begin{aligned}
&r = 0.1, \qquad \sigma = 0.2, \qquad K = 100, \qquad T = 1 \text{ (one year)},\\
&R = 1, \qquad k = 0.001 \qquad h = 0.1.
\end{aligned}
$$

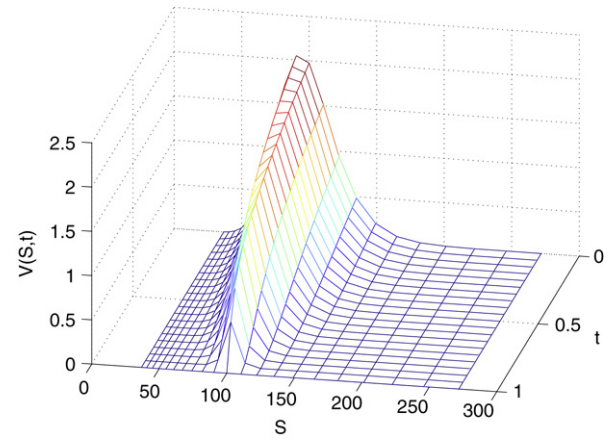
In all these models the difference is not symmetric, but decreases closer to the expiry date. This is an expected consequence of the decreasing necessity of portfolio adjustment and hence lower transaction costs closer to the expiry.



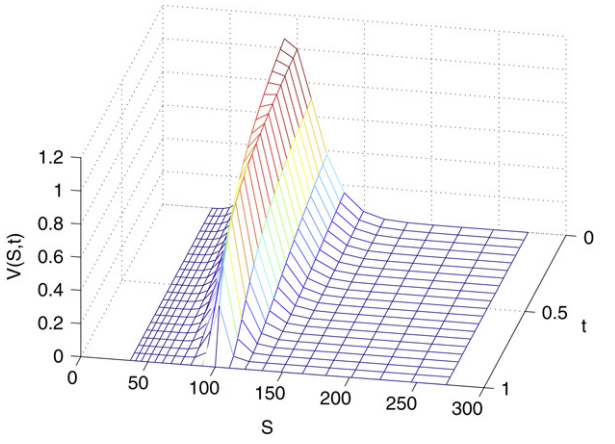
(a) Barles' and Soner's model ($a = 0.02$) vs. linear model.



(b) $\Psi(x) := x$ chosen as the identity ($a = 0.02$) vs. linear model.



(c) Leland's model ($\delta t = 0.01$, $\kappa = 0.05$) vs. linear model.



(d) RAPM model ($M = 0.01$, $C = 30$) vs. linear model.

Fig. 2. The influence of transaction costs (linear vs. nonlinear model).

The difference is maximal at one year to expiry at $S \approx 95$, where the nonlinear price is significantly higher than the linear price. At this point with the given parameters Barles' and Soner's model provides the highest price (≈12.4), followed by Leland's model (≈11.9), RAPM (≈11.0), the identity (≈10.0) and finally the linear price (≈9.9) (see Fig. 3).

For each volatility model and each difference scheme we compare the error of accuracy of the above computation one year to expiry, that is at $t = 0$ or $\tau = \widetilde{T} = Mk$, and denote this ℓ^2-error by

$$\mathrm{err}_2(Mk) = \left(h \sum_{i=-N}^{N} |u(x_i, \widetilde{T}) - U_i^M|^2 \right)^{\frac{1}{2}}.$$

For the reference solution $u(x_i, \widetilde{T})$ we compute a solution for each volatility model with the Crank–Nicolson and the R3C scheme on a very fine grid with $k = 0.001$ and $h = 0.01$. For U_i^M we use the parameters as indicated above.

We see that in the linear case the compact R3C scheme yields better results than the Crank–Nicolson scheme in terms of accuracy, even though the error resulting from the Crank–Nicolson scheme is only slightly bigger (see Table 1). Reducing the spatial step size to $h = 0.001$ improves the accuracy considerably, though it does increase the computational time tremendously.

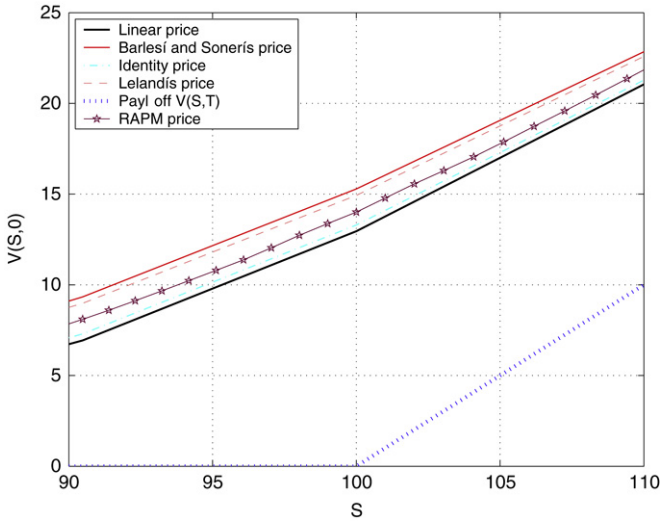


Fig. 3. Price of the European Call option.

Table 1
ℓ^2-error for different models and schemes

	Linear	Barles and Soner	Identity	Leland	RAPM
$\mathrm{err}_2(Mk)$ with CN	0.0016	0.0006	0.0031	0.0047	0.0006
$\mathrm{err}_2(Mk)$ with R3C	0.0009	0.0009	0.0024	0.0056	0.0005

5. Conclusions and outlook

We have compared several transaction cost models and used two difference schemes for the numerical computation of the option prices. Both the Crank–Nicolson and the R3C scheme provided accurate approximations to the European Call option price. They are unconditionally stable, non-oscillatory and excellent for the computation – in case of European options – due to their superiority to standard difference schemes. For the future work another two compact schemes, known as the *Numerov-type* (see [32,33]) and the *Crandall–Douglas scheme* (see [34]), will be generalized and analyzed for nonlinear Black–Scholes equations. For the computation of the option prices for American options in an market with transaction costs we refer the reader to [25,23].

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References

[1] P. Wilmott, S. Howison, J. Dewynne, The Mathematics of Financial Derivatives, Cambridge University Press, New York, 1995.
[2] F. Black, M. Scholes, The pricing of options and corporate liabilities, J. Polit. Econ. 81 (3) (1973) 637–654.
[3] R.C. Merton, Theory of rational option pricing, Bell J. Econ. 4 (1) (1973) 141–183.
[4] R. Seydel, Tools for Computational Finance, 2nd edition, Springer, Berlin, 2004.
[5] D. Tavella, C. Randall, Pricing Financial Instruments — The Finite Difference Method, John Wiley & Sons, Inc., New York, 2000.
[6] H.E. Leland, Option pricing and replication with transactions costs, J. Finance 40 (1985) 1283–1301.
[7] P. Boyle, T. Vorst, Option replication in discrete time with transaction costs, J. Finance 47 (1) (1992) 271–293.
[8] G. Barles, H.M. Soner, Option pricing with transaction costs and a nonlinear Black–Scholes equation, Finance Stoch. 2 (1998) 369–397.
[9] R. Frey, P. Patie, Risk Management for Derivatives in Illiquid Markets: A Simulation Study, Springer, Berlin, 2002.
[10] R. Frey, A. Stremme, Market volatility and feedback effects from dynamic hedging, Math. Finance 4 (1997) 351–374.

[11] P. Schönbucher, P. Wilmott, The feedback–effect of hedging in illiquid markets, SIAM J. Appl. Math. 61 (2000) 232–272.
[12] H.M. Soner, N. Touzi, Superreplication under gamma constraints, SIAM J. Control Optim. 39 (1) (2001) 73–96.
[13] Y. Zhu, X. Wu, I. Chern, Derivative Securities and Difference Methods, Springer, Finance, New York, 2004.
[14] Y. Kwok, Mathematical Models of Financial Derivatives, Springer, Singapore, 1998.
[15] H.M. Soner, S.E. Shreve, J. Cvitanič, There is no nontrivial hedging portfolio for option pricing with transaction costs, Ann. Appl. Probab. 5 (2) (1995) 327–355.
[16] T. Hoggard, A.E. Whalley, P. Wilmott, Hedging option portfolios in the presence of transaction costs, Adv. Fut. Opt. Res. 7 (1994) 21–35.
[17] M. Avellaneda, A. Parás, Dynamic hedging portfolios for derivative securities in the presence of large transaction costs, Appl. Math. Finance 1 (1994) 165–193.
[18] B. Bensaid, J.P. Lesne, H. Pagès, J. Scheinkman, Derivative asset pricing with transaction costs, Math. Finance 2 (1992) 63–86.
[19] S. Hodges, A. Neuberger, Optimal replication of contingent claims under transaction costs, Rev. Futures Markets 8 (1989) 222–239.
[20] M.H.A. Davis, V.G. Panas, T. Zariphopoulou, European option pricing with transaction costs, SIAM J. Control. Optim. 31 (2) (1993) 470–493.
[21] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer, New York, 1993.
[22] M. Jandačka, D. Ševčovič, On the risk-adjusted pricing-methodology-based valuation of vanilla options and explanation of the volatility smile, J. Appl. Math. 2005 (3) (2005) 235–258.
[23] J. Ankudinova, The numerical solution of nonlinear Black–Scholes equations, Master's Thesis, Technische Universität Berlin, Berlin, in preparation.
[24] B. Düring, Black–Scholes type equations: Mathematical analysis, parameter identification & numerical solution, Dissertation, University Mainz, 2005.
[25] D. Ševčovič, An iterative algorithm for evaluating approximations to the optimal exercise boundary for a nonlinear Black–Scholes equation, Canad. Appl. Math. Quarterly 15 (1) (2007) 77–97.
[26] M. Ehrhardt, R.E. Mickens, Discrete artificial boundary conditions for the Black–Scholes equation of American options, Int. J. Appl. Theoret. Finance (in preparation).
[27] L.H. Thomas, Elliptic problems in linear difference equations over a network, Watson Sci. Comput. Lab. Rept., Columbia University, New York, 1949.
[28] A. Rigal, High order difference schemes for unsteady one-dimensional diffusion-convection problems, J. Comput. Phys. 114 (1) (1994) 59–76.
[29] B. Düring, M. Fournier, A. Jüngel, High order compact finite difference schemes for a nonlinear Black–Scholes equation, Int. J. Appl. Theoret. Finance 7 (2003) 767–789.
[30] J.C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, Wadsworth Publ. Co., Belmont, CA, USA, 1989.
[31] J.R. Dormand, P.J. Prince, A family of embedded Runge–Kutta formulae, J. Comput. Appl. Math. 6 (1980) 19–26.
[32] J. Zhao, M. Davison, R.M. Corless, Compact finite difference method for American option pricing, J. Comput. Appl. Math. 206 (1) (2007) 306–321.
[33] H. Sun, J. Zhang, A high order compact boundary value method for solving one dimensional heat equations, Tech. Rep. No. 333-02, University of Kentucky, 2002.
[34] B.J. McCartin, S.M. Labadie, Accurate and efficient pricing of vanilla stock options via the Crandall–Douglas scheme, Appl. Math. Comput. 143 (1) (2003) 39–60.