Generalized evolutes, vertices and conformal invariants of curves in $\mathbb{R}^{n+1}$

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ABSTRACT

We define the generalized evolute of a curve in $(n+1)$-space and find a duality relation between them. We also prove that the conformal torsion is a function of the speed of the generalized evolute and that the singular points of the generalized evolute (vertices) are conformal invariants.

INTRODUCTION

Conformal maps take hyperspheres of $\mathbb{R}^{n+1}(n \geq 2)$ into hyperspheres, and thus must preserve the contacts of the curves with them. The hyperspheres having higher order contact with a curve $\gamma$ are known as osculating hyperspheres of $\gamma$ and their centers form a curve in $\mathbb{R}^n$ that we call generalized evolute of the curve $\gamma$. Although generalized evolutes are not preserved by conformal maps (for they do not preserve the centers of the spheres!), we prove here that their speed, conveniently multiplied by the radii of curvature, provides a conformal invariant. We observe that for a generic curve the vanishing of this invariant implies that the considered curve lies on a hypersphere, so we can view it as a measure of how far the curve is from being 'flat' from the conformal viewpoint. This enables us to give it the name of conformal torsion. We must point out that A. Fialkow constructed in [5] a system of conformal Frenet equations (by using the notion of conformal derivative) and from these, he defined a complete set of conformal invariants, known as conformal curvatures. The calculation of ex-

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licit expressions for these curvatures in terms of the euclidean curvatures and its derivatives appears to be very complicated in general and, indeed, they were not given in Fialkow's paper. Nevertheless, the expression of the conformal torsion in the Fialkow's sense is known for curves in 3-space [3] and an easy calculation tells us that it coincides with ours. But, our expression, given in terms of the generalized evolutes, makes the results much easier to handle and generalize to higher dimensions.

The methods we use here rely on the idea that the conformal maps preserve, on one hand the contacts with spheres as we have pointed out before, and on the other hand the inversive distances between them. As a by-product we obtain a duality relation between a curve γ and its generalized evolute cγ in (n + 1)-space; the tangent indicatrices of γ and cγ are dual as curves in projective n-space.

We have included in Section 1 the proofs of some facts on curves in (n + 1)-space that although being classically accepted as a generalization of the well known situation of curves in R³, have no proofs readily found in the available literature. Section 2 contains the proof of the above mentioned duality result. And the Section 3 is devoted to the definition of the conformal torsion and the proof that it is invariant under the action of the conformal group on the curves in n-space. We must warn that, to be precise (and include the 2-dimensional case) we should use the word inversive instead of conformal. But, since the inversive and conformal groups coincide for n ≥ 2, the torsion has only meaning for n ≥ 2, and the word conformal appears to be more widely used, we have called the invariants after it, although we have used both terms, inversive and conformal, interchangeably throughout the text. We shall consider n ≥ 2 throughout the whole paper.

1. CLASSICAL PARAPHERNALIA FOR (n + 1)-SPACE CURVES

Given a curve γ : R → Rⁿ⁺¹ parametrized by arclength, consider the family of functions

\[ d_\gamma : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R} \]
\[ (t, x) \mapsto \frac{1}{2} \| \gamma(t) - x \|^2. \]

We shall assume in what follows that the set of vectors \{γ'(t), γ''(t), \ldots, γⁿ(t)\} is linearly independent at all the points. It is not difficult to prove that this is a generic condition, in the sense that it is satisfied for most curves.

The normal bundle of γ in Rⁿ⁺¹ is the submanifold (see [8]),

\[ N_\gamma = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n+1} : \frac{\partial d_\gamma}{\partial t} (t, x) = 0\}. \]

This is a trivial bundle over the curve, whose fibre N_γ at a point t is a hyperplane of Rⁿ⁺¹.

The focal set of γ is the bifurcation set of the family d_γ [8], that is the set of critical values of the projection \( \pi : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R} \) restricted to N_γ. Observe
that a hypersphere, $S(a, r)$, of centre $a \in \mathbb{R}^{n+1}$ and radius $r \in \mathbb{R}$, is tangent to $\gamma$ at $t_0$ if and only if $d_1(t_0, a) = r^2$ and $\frac{\partial d_1}{\partial t}(t_0, a) = 0$ and the point $a$ is in the focal set of $\gamma$ if and only if $d_1(t_0, a) = r^2$ and $\frac{\partial^2 d_1}{\partial t^2}(t_0, a) = 0$.

We say that $S(a, r)$ has contact of order $k$ with $\gamma$ at $t_0$ if and only if

$$d_1(t_0, a) = r^2, \frac{\partial d_1}{\partial t}(t_0, a) = \ldots = \frac{\partial^k d_1}{\partial t^k}(t_0, a) = 0$$

In what follows we shall denote $d_1(t_0, a) = \frac{\partial d_1}{\partial t}(t_0, a)$, dropping the suffix $\gamma$ when no confusion arises with respect to the curve.

A hypersphere of $\mathbb{R}^{n+1}$ whose contact with $\gamma$ at $t_0$ is of order $\geq n + 1$ is called osculating hypersphere of $\gamma$ at $t_0$.

Consider the Frenet frame of the curve $\gamma$, $\{T(t), N_1(t), \ldots, N_n(t)\}$ (see [6]) with the associate curvature functions $k_1(t), \ldots, k_n(t)$ and remember the Frenet formulas:

$$T'(t) = k_1N_1(t)$$
$$N_1'(t) = -k_1T(t) + k_2N_2(t)$$
$$\vdots$$
$$N_n'(t) = -k_nN_{n-1}(t).$$

The osculating hyperplane of $\gamma$ at $t$ is the subspace generated by $\{T(t), N_1(t), N_2(t), \ldots, N_n(t)\}$ that passes through $\gamma(t)$. The unit vector $N_n(t)$ is called the binormal vector of $\gamma$ at $t$. The normal hyperplane of $\gamma$ at $t$ is defined to be the one generated by $\{N_1(t), \ldots, N_{n-1}(t), N_n(t)\}$ passing through $\gamma(t)$.

**Lemma 1.** With the notation above,

(a) $N_1\gamma(t) = \gamma(t) + \langle N_1(t), \ldots, N_n(t) \rangle$.

(b) There is a unique osculating hypersphere $S(t)$ at each point of the curve for which the vectors $\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n+1)}(t)\}$ are linearly independent.

(c) The centers of all the osculating hyperspheres of $\gamma$ form a smooth curve $c_\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$.

**Proof.**

(a) $(t, x) \in N_1\gamma(t) \iff (\gamma(t) - x) \cdot \gamma'(t) = 0 \iff x = \gamma(t) + v$, with $v \cdot \gamma'(t) = 0$, but this means that $v$ is a linear combination of the vectors $N_i(t), i = 1, \ldots, n$.

(b) Observe that

$$d_j(t, x) = (\gamma(t) - x) \cdot \gamma^{(j)}(t) + F_j(\gamma'(t), \gamma''(t), \ldots, \gamma^{(n+1)}(t)),$$

where $F_j$ is a polynomial function. Now, $S(a, r)$ is the osculating hypersphere of $\gamma$ at $t$ if and only if $d_j(t, a) = r^2$ and $d_j(t, a) = 0$ for $j = 1, \ldots, n + 1$. So we have that $a = \gamma(t) + a_0T(t) + \sum_{i=1}^n a_iN_i(t)$ for some coefficients $a_i \in \mathbb{R}, i = 0, \ldots, n$. By substituting $a - \gamma(t) = a_0T(t) + \sum_{i=1}^n a_iN_i(t)$ in the set of $n + 1$ equations $d_1(t, a) = \ldots = d_{n+1}(t, a) = 0$, we obtain a system of $n + 1$ linear equations.
whose \( n + 1 \) variables are the coefficients \( a_0, \ldots, a_n \) of the centre \( a \). The solution of this system exists and is unique provided that the vectors \( \{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n+1)}(t)\} \) are linearly independent. (Observe that the coefficients \( a_0, \ldots, a_n \) are functions of \( t \) depending on the curvatures \( k_1(t), \ldots, k_n(t) \) and their derivatives respect the arclength of \( \gamma \)).

(c) It is easy to see that \( a_0 = 0 \) and thus we can write \( a_i = \mu_i, \ i = 1, \ldots, n \), where the \( \mu_i \) are rational functions of the curvatures \( k_j(t), \ j = 1, \ldots, n \) and their derivatives respect the arclength of \( \gamma \). So the expression \( c_\gamma(t) = \gamma(t) + \sum_{i=1}^n \mu_i N_i(t) \) represents a smooth curve in \( \mathbb{R}^{n+1} \).

**Definition.** The curve \( c_\gamma(t) \) is said to be the curve of spherical curvature centres of \( \gamma \) or generalized evolute of \( \gamma \).

**Remark.** The points \( t \) of \( \gamma \) for which the vectors \( \{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n+1)}(t)\} \) are linearly dependent are known as flattening points of \( \gamma \). At these, the osculating hyperplane can be seen to have contact of order at least \( n + 2 \) with \( \gamma \) and we can consider it as the (degenerate) osculating hypersphere. These points are isolated for a generic curve in \( \mathbb{R}^{n+1} \), and are also characterized as being the zeros of the \( n \)th (euclidean) curvature function \( k_n \).

**Lemma 2.** The speed of the curve \( c_\gamma(t) \) of spherical curvature centers of a curve \( \gamma \) at a point \( t \) has the direction of the binormal vector of \( \gamma \) at \( t \).

**Proof.** Consider \( d\gamma(t,x) = \frac{1}{2} (\gamma(t) - x) \cdot (\gamma(t) - x) \).

We have that \( x = c_\gamma(t) \) if and only if \( d_1(t, x) = \ldots = d_{n+1}(t, x) = 0 \). In particular, \( d_1(t, c_\gamma(t)) = \ldots = d_{n+1}(t, c_\gamma(t)) = 0 \).

Therefore, \( d_i(t, c_\gamma(t)) = (\gamma(t) - c_\gamma(t)) \cdot \gamma^{(i)}(t) + \sum F_i(\gamma'(t), \gamma''(t), \ldots, \gamma^{(n-1)}(t)) \) and differentiating here

\[
0 = d_i'(t, c_\gamma(t)) = (\gamma(t) - c_\gamma(t)) \cdot \gamma^{(i+1)}(t) + (\gamma'(t) - c_\gamma'(t)) \cdot \gamma^{(i)}(t) + \sum F_i(\gamma'(t), \gamma''(t), \ldots, \gamma^{(n-1)}(t))
\]

but \( d_{i+1}(t, c_\gamma(t)) = 0 \) for all \( i \leq n \), and hence \( c_\gamma'(t) \cdot \gamma^{(i)}(t) = 0 \) for all \( i \leq n \).

Which implies that \( c_\gamma'(t) \) is perpendicular to the osculating hyperplane

\[
\langle \gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t) \rangle = \langle T(t), N_1(t), \ldots, N_{n-1}(t) \rangle
\]

and hence it must have the direction of \( N_n(t) \). □

**Lemma 3.** The functions \( \{\mu_i\}_{i=1}^n \), defined in the proof of Lemma 1, satisfy the following relations:

\[
\mu_2(t)k_2(t) = \mu_1'(t)
\]

\[
\mu_i(t)k_i(t) = \mu_{i-1}'(t) + \mu_{i-2}(t)k_{i-1}(t), \ i = 3, \ldots, n.
\]
Proof. We have that
\[ c_\gamma(t) = \gamma(t) + \sum_{i=1}^{n} \mu_i(t)N_i(t), \]
and thus
\[ c_\gamma'(t) = \gamma'(t) + \sum_{i=1}^{n} \mu_i'(t)N_i(t) + \sum_{i=1}^{n} \mu_i(t)N'_i(t). \]
By applying the Frenet formulas, we obtain
\[ c_\gamma'(t) = (\mu_1'(t) - \mu_2(t)k_2(t))N_1(t) + (\mu_n'(t) + \mu_{n-1}(t)k_n(t))N_n(t) + \sum_{i=2}^{n-1} (\mu_i'(t) + \mu_{i-1}(t)k_i(t) - \mu_{i+1}(t)k_{i+1}(t))N_i(t) \]
and from the Lemma 2, it follows
\[ \mu_1'(t) - \mu_2(t)k_2(t) = 0 \quad \text{and} \quad \mu_i'(t) + \mu_{i-1}(t)k_i(t) - \mu_{i+1}(t)k_{i+1}(t) = 0, \quad i = 2, \ldots, n-1. \]

Lemma 4. The singular points of the curve \( c_\gamma(t) \) are those \( t \) for which the osculating hypersphere, \( S(t) \), of \( \gamma \) has contact of order at least \( n + 2 \) at \( t \).

Proof. From the Lemma 3 it follows
\[ c_\gamma'(t) = (\mu_1'(t) + \mu_{n-1}(t)k_n(t))N_n(t). \]
So \( c_\gamma'(t) = 0 \iff c_\gamma'(t) \cdot \gamma^{(i)}(t) = 0 \) for all \( i \leq n + 1 \). But this holds provided \( d_{i+1}(t, c_\gamma(t)) = 0 \), for all \( i \leq n + 1 \) and viceversa.

Definition. A point \( t \) such that \( c_\gamma'(t) \) is called a vertex of \( \gamma \).

This concept of vertex generalizes the classical one of a vertex of a plane curve, for these are the points at which the osculating hyperspheres has higher order of contact with the curve. Some results of global type, generalizing the 4-vertex theorem, can be seen in [10].

Remark. The osculating \( k \)-sphere of \( \gamma \) at a point \( t \) is defined as the intersection of the osculating sphere with the osculating \( k \)-plane of \( \gamma \) at \( t \).

The center and the radius of this \( k \)-sphere are given respectively by
\[ \gamma(t) + \sum_{i=1}^{k} \mu_i(t)N_i(t) \quad \text{and} \quad \sqrt{\sum_{i=1}^{k} \mu_i^2(t)}. \]
In particular, the radius of the osculating hypersphere is \( r(t) = \sqrt{\sum_{i=1}^{n} \mu_i^2(t)} \).

We observe that \( \mu_1(t) \neq 0 \) and none of these radii ever vanish.

2. DUALITY OF A CURVE AND ITS GENERALIZED EVOLUTE

Given a curve \( \gamma \) in projective \( n \)-space \( \mathbb{P}^n \), the dual of \( \gamma \) is the curve \( \gamma^* \) defined by the osculating projective hyperplanes of \( \gamma \) in \( \mathbb{P}^n \). By using the natural identifi-
cation of $\mathbb{P}^n$ and $\mathbb{P}^n$ (obtained by assigning its orthogonal directions to each projective hyperplane), we can view both $\gamma$ and $\gamma^*$ as curves in $\mathbb{P}^n$ related as follows: Considering $\mathbb{P}^n$ as the quotient of the unit sphere $S^n$ (in $\mathbb{R}^{n+1}$) through identification of antipodal points and choose a representative of $\gamma$ in $S^n$. Then the curve $\gamma^*$ is the one swept by the poles of the osculating $(n-1)$-spheres of $\gamma(t)$ as $t$ varies. Observe that these poles are given by the orthogonal direction to the hyperplane (in $\mathbb{R}^{n+1}$) spanned by the vectors $\{\gamma, \gamma'(t), \ldots, \gamma^{(n-1)}(t)\}$.

Given a curve $\gamma: \mathbb{R} \to \mathbb{R}^{n+1}$, we define the tangent (binormal) indicatrix of $\gamma$ as the curve $T_\gamma: \mathbb{R} \to \mathbb{P}^n, (B_\gamma: \mathbb{R} \to \mathbb{P}^n)$ that assigns to each $t$ the tangent (binormal) direction of $\gamma$ at the point $t$.

**Lemma 5.** Given any $\gamma: \mathbb{R} \to \mathbb{R}^{n+1}$

\[
T_\gamma^* = B_\gamma, \\
B_\gamma^* = T_\gamma.
\]

**Proof.** Suppose that $\gamma$ is parametrized by its arclength. Then $T_\gamma(t) = \gamma'(t) \in S^n$, for all $t$, and the poles of the osculating $(n-1)$-spheres of $\gamma'$ are given by the orthogonal directions to the hyperplanes spanned by $\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\}$. But these are the binormal directions of $B\gamma(t) = N_\gamma(t)$.

Analogously, it is not difficult to see from the Frenet formulas that the orthogonal direction to the hyperplane spanned by $\{N_n(t), N'_n(t), \ldots, N^{(n)}_n(t)\}$ is precisely $T_\gamma(t)$. □

**Theorem 1.** The tangent indicatrix of the curve $\gamma$ and of its generalized evolute $c_\gamma$ are mutually dual.

**Proof.** It follows immediately from Lemmas 2 and 5. □

### 3. Conformal Torsion and Conformal Flattenings

Consider the **inversive product** of two hyperspheres $S(a_i, r_i), i = 1, 2$ in $\mathbb{R}^{n+1}$, defined in [1],

\[
(S_1, S_2) = \left| \frac{r_1^2 + r_2^2 - (a_1 - a_2)^2}{2r_1r_2} \right|.
\]

When $S_1$ and $S_2$ intersect each other we have that $(S_1, S_2)$ is a function of their angle of intersection. Whereas, when they are disjoint, this inversive product is a function of the hyperbolic distance between them. In fact, this product is an inversive (conformal) invariant and by using an argument analogous to Coxeter's in [4] we can define the **inversive distance**, $\delta(S_1, S_2)$, between the hyperspheres $S_1$ and $S_2$ by means of the expression $\cosh\delta(S_1, S_2) = (S_1, S_2)$.
Given a curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ in the extended $(n+1)$-plane, an inversive map

$$\varphi : \mathbb{R}^{n+1} \cup \{\infty\} \to \mathbb{R}^{n+1} \cup \{\infty\}$$

transforms hyperspheres into hyperspheres in $\mathbb{R}^{n+1} \cup \{\infty\}$, and being a diffeomorphism, must also preserve the contact between $\gamma$ and the different hyperspheres of $\mathbb{R}^{n+1} \cup \{\infty\}$ (see [7]). Consequently it takes osculating hyperspheres of $\gamma$ into osculating hyperspheres of $\gamma = \varphi \circ \gamma$.

Now, the infinitesimal distance between two osculating hyperspheres of $\gamma$ must also be preserved, in the sense that

$$\delta(S_\gamma(t+h), S_\gamma(t)) = \delta(\varphi(S_\gamma(t+h)), \varphi(S_\gamma(t))) = \delta(S_\gamma(t+h), S_\gamma(t)),$$

for $h$ tending to 0.

We shall now use an argument analogous to that developed in [2] for plane curves in order to prove our main result:

**Theorem 2.** Given a curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ parametrized by arclength, the 1-form

$$\omega_\gamma = \frac{||c'_\gamma(t)||\dot{r}(t)}{r^2(t)} dt,$$

is an inversive invariant, where

- $||c'_\gamma(t)||$ is the speed of generalized evolute of $\gamma$,
- $r(t)$ is the radius of the osculating hypersphere of $\gamma$ and
- $\dot{r}(t)$ is the radius of the osculating $(n-1)$-sphere of $\gamma$.

**Proof.** Applying the formulas of the inversive product to the osculating hyperspheres of the curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ at the points $\gamma(t)$ and $\gamma(t+h)$, we obtain

$$\delta(S_{\gamma}(t+h), S_{\gamma}(t)) = \frac{r(t+h)^2 + r(t)^2 - (c_{\gamma}(t+h) - c_{\gamma}(t))^2}{2r(t+h)r(t)}.$$

Expanding in Taylor series in $h$

$$1 - \frac{h^2}{2} + \ldots = \cos(\delta(S_{\gamma}(t+h), S_{\gamma}(t)) = 1 - \frac{1}{2!} ||c'_\gamma(t)||^2 - \frac{r'(t)^2}{r(t)^2} h^2 + O(h^3).$$

We note in particular that if $||c'_\gamma(t)||^2 - r'(t)^2 \neq 0$ the right expression is smaller than 1, for $h$ small enough, proving that the osculating spheres of nearby points on a curve in $\mathbb{R}^{n+1}$, intersect each other in contrast to the case of plane curves in which the osculating circles of nearby points do not intersect, unless one of them is a vertex, (see [2]).

From the above Taylor series, we get:

$$\delta(t+h, t) = \frac{\sqrt{||c'_\gamma(t)||^2 - r'(t)^2}}{r(t)} h + O(h^2).$$

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It follows that:

\[ \omega_n = \sqrt{\frac{\|c'_n(t)\|^2 - r'(t)^2}{r(t)}} dt, \]

is also invariant under the corresponding inversive group. We shall prove that

\[ \|c'_n(t)\|^2 - r'(t)^2 = \frac{\|c'_n(t)\|^2 r^2(t)}{r^2(t)} \]

and apply this expression to obtain

\[ \omega_n = \frac{\|c'_n(t)\| r(t)}{r^2(t)} dt. \]

In order to prove the formula \([\ast]\), it is enough to verify the following equality:

\[
\sum_{i=1}^{n} \mu_i(t)^2 \left( \mu_n'(t) + \mu_{n-1}(t)k_n(t) \right)^2 - \left( \sum_{i=1}^{n} \mu_i(t)\mu'_n(t) \right)^2 = \sum_{i=1}^{n-1} \mu_i(t)^2 \left( \mu_n'(t) + \mu_{n-1}(t)k_n(t) \right)^2.
\]

But this is easily done by applying the formulas found in Lemma 3 and an induction argument. \(\square\)

**Corollary.** The number of vertices of a curve in is an inversive invariant.

**Remarks.** (1) Observe that the formula \([\ast]\) implies that a vertex of \(\gamma\) is always an extremal point of the radius, \(r(t)\), of the osculating hyperspheres. This has been observed in [10].

(2) The functional coefficient of the above 1-form measures, for all \(n\), how for the curve is from being locally contained in a hypersphere (i.e. of being flat in the conformal sense). In fact, we have that the curve \(\gamma\) is spherical if and only if it vanishes everywhere; we shall call it the conformal torsion of \(\gamma\).

In this sense, it seems quite appropriate to call the vertices conformal flattenings.

(3) Given a curve \(\gamma : \mathbb{R} \to \mathbb{R}^n\), let \(\tilde{\gamma} : \mathbb{R} \to S^n \to \mathbb{R}^{n+1}\) be its image under the inverse, \(\xi\), of the stereographic projection from \(S^n - \{(0, \ldots, 0, 1)\}\) to \(\mathbb{R}^n\), considered as a curve in \((n + 1)\)-space. It can be seen that \(\xi\) ‘transports’ contacts of \(\gamma\) with hyperspheres in \(\mathbb{R}^n\) into contacts of \(\tilde{\gamma}\) with hyperplanes in \(\mathbb{R}^{n+1}\), (see [9]). As a consequence of this it follows that \(t_0\) is a vertex of \(\gamma\) if and only if \(t_0\) is a zero of the \(n\)th metric curvature function of \(\tilde{\gamma}\) (see also [10]). So in view of the above comments, we can write:
$t_0$ is a conformal flattening of $\gamma$ if and only if $t_0$ is a flattening of $\tilde{\gamma}$.

REFERENCES